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FAST COMPUTATION FOR LARGE MAGNETOSTATIC SYSTEMS ADAPTED FOR MICROMAGNETISM*

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Abstract. In this paper, an efficient method is developed for computing the magnetostatic field for ferromagnetic materials on large structured meshes. The problem is discretized using a finite volume approximation. The discrete operator is proved to preserve the main properties of the continuous model, and a lower estimate of its lower eigenvalue is given. Using the fact that the discrete operator has a block-Toeplitz structure for cubic meshes in parallelepipedic domains, a fast solving method is built. Based upon the use of fast Fourier transform, this method allows one to reduce the computational cost from n^2 to $O(n\log(n))$ but also to reduce the storage to $O(n)$ instead of n^2, where n is the number of cells in the mesh.

Key words. finite volume method, magnetostatics, Maxwell equations, block-Toeplitz matrices

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1. Introduction.

When computing the magnetization of ferromagnetic materials, the theory of micromagnetism uses a nonlinear evolution equation, the Landau–Lipschitz equation, relating the magnetization field to the excitation field (see [3]). The excitation originates from various physical phenomena; one of them is induced by the stray field $\mathbf{H}$ that appears in the Maxwell equations. In the case where the wavelengths are large compared to the size of the material, the Maxwell system is usually replaced by the so-called quasi-static approximation. If the material fills a domain Ω in $\mathbb{R}^3$, the equation of magnetostatics relates the magnetization field $\mathbf{u}$, the magnetic field $\mathbf{B}$, and the stray field $\mathbf{H}$ in the following way:

$$\left\{\begin{array}{l} \mathbf{rot}\ \mathbf{H} = \mathbf{0}, \\ \text{div}\ \mathbf{B} = \mathbf{0}, \\ \mathbf{B} = \mu_0(\mathbf{H}+\mathbf{u}), \end{array}\right. \tag{1.1}$$

where μ_0 is the permeability of the vacuum and $\mathbf{u}$ vanishes outside Ω.

The numerical resolution of this system amounts to solving the Poisson equation, namely,

$$-\Delta\Phi = \text{div}\ \mathbf{u}, \qquad \mathbf{grad}\ \Phi = \mathbf{H},$$

where $\mathbf{u}$ is now considered as a datum. The problem has to be solved in the whole space $\mathbb{R}^3$ and thus requires very efficient solvers. The purpose of this paper is to propose such a solver adapted to the micromagnetism computations.

In the micromagnetism context, we seek equilibrium states whose characterization is Find $\mathbf{u}$ in $(\mathrm{L}^2(\Omega))^3$, $|\mathbf{u}| = 1$, for almost every point of Ω and such that $\mathbf{u}$ minimizes the energy $e(\mathbf{u}) = -\int_\Omega \mathbf{u}\cdot\mathbf{H}_T(\mathbf{u})\,dx$. Such minimizers verify

$$\|\mathbf{u}\wedge\mathbf{H}_T(\mathbf{u})\|_{0,\Omega} = 0,$$

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where $\mathbf{H}_T(\mathbf{u})$ is typically equal to $\mathbf{H} + A \triangle\mathbf{u}$ and $A > 0$ is called the exchange constant. We will see in section 2 that $\mathbf{u} \mapsto \mathbf{H}$ defines a linear negative operator. At this stage, we warn the reader that preserving this property with the discretized operator $\mathbf{H}_h$ might be crucial; otherwise, there may exist regions ω, included in Ω, in which $\mathbf{H}_h \cdot \mathbf{u}_h > 0$ almost everywhere in ω, whereas $\mathbf{H} \cdot \mathbf{u} \leq 0$ for the continuous case. In such regions, it is expected that the discretized solution would be in the opposite direction of the continuous one. Therefore, in this article, we will focus particularly on discretization methods preserving the negativity of the magnetostatic operator.

The first numerical method used to compute the magnetization field (see [18]) was based on the dipolar approximation. Its main drawback was producing negative eigenvalues of the approximation of a positive operator. Furthermore, the cost would be prohibitive for the applications that we have in mind. Improvements using finite volumes were made by Nakatami, Uesaka, and Hayashi [15], but the two main drawbacks remained.

On the other hand, efficient finite element methods have been used to compute the equilibrium states by minimization of the energy (see [20], [1]). However, in view of further use in computations of susceptibility, we really need to couple (1.1) to the time dependent Landau–Lipschitz equation.

In this paper we introduce a finite volume approximation, which preserves the main properties of the operator given in the continuous model: positivity and symmetry. Furthermore, the resulting system has a block-Toeplitz structure, which allows efficient and fast solvers.

In section 2 we introduce the continuous problem and the notation. In section 3, we define the discretization method. It is based on solving the exact problem for piecewise constant functions and then projecting the solution onto the space of piecewise constant functions. Actually the method used is based upon a semianalytical integration, as is often the case when solving integral equations. We prove that the properties of the continuous operator are preserved, and we are able to give a lower bound on the eigenvalues.

In section 4, we present a fast solver using a multilevel block-Toeplitz construction. The notion of Toeplitz matrices was introduced by Strang (see [19]). Thereafter, Tyrtyshnikov studied the spectrum of block-Toeplitz matrices and, together with Ivakhnenko, applied them to solving electromagnetic scattering problems (see [10]). We prove that this method, when applied to our problem, reduces the storage from $O(n^2)$ to $O(n)$ elements and the computation complexity from n^2 to $n \log(n)$, where n is the number of cells in the mesh. At the end of the paper (in section 5), we show some numerical experiments in order to illustrate the efficiency of the method.

2. The magnetostatic equations. First, we recall some notation used in Sobolev spaces. For any three dimensional domain Ω, $\mathrm{L}^2(\Omega)$ is the Hilbert space of square-integrable functions, furnished with the inner product

$$(v, w) = \int_\Omega v(x)w(x)\, dx,$$

and the corresponding norm is denoted by $|| \, ||_{0,\Omega}$.

$\mathcal{D}(\Omega)$ is the space of functions which are infinitely differentiable and compactly supported in Ω. Its dual is the space of distributions, denoted $\mathcal{D}'(\Omega)$.

For any positive integer m, $\mathrm{H}^m(\Omega)$ is the Sobolev space of distributions defined in

Ω, whose derivatives up to order m belong to $\mathrm{L}^2(\Omega)$, furnished with the inner product

$$(u,w)_{m,\Omega} = \sum_{|k|\leq m} (D^k v, D^k w)_{0,\Omega},$$

and the corresponding norm is denoted $||\ ||_{m,\Omega}$ (as usual, $\mathrm{H}^0(\Omega)$ is identical to $\mathrm{L}^2(\Omega)$). Furthermore, we define

$$|u|_{m,\Omega} = \sum_{|k|=m} ||D^k v||_{0,\Omega}.$$

The notations $(\ ,\ )_{m,\Omega}$ and $||\ ||_{m,\Omega}$ will be applied to $\mathrm{H}^m(\Omega)$ or $(\mathrm{H}^m(\Omega))^3$.

It is well known (see [5]) that, for $\mathbf{H}$ in $(\mathrm{L}^2(\Omega))^3$ satisfying $\mathbf{rot}\ \mathbf{H} = 0$, there exist a unique ϕ in the weighted Sobolev space $\mathrm{W}^1(\mathbb{R}^3)$ such that

$$\mathbf{H} = \mathbf{grad}\ \phi \ \text{ in } \mathbb{R}^3, \tag{2.1}$$

with

$$\mathrm{W}^1(\mathbb{R}^3) = \left\{ \varphi \in \mathcal{D}'(\mathbb{R}^3), \mathbf{grad}\ \varphi \in \mathrm{L}^2(\mathbb{R}^3), \frac{\varphi}{\sqrt{1+r^2}} \in \mathrm{L}^2(\mathbb{R}^3) \right\}.$$

By (1.1) we derive an equation for $\phi \in \mathrm{W}^1(\mathbb{R}^3)$:

$$\triangle\phi(\mathbf{u}) = -\mathrm{div}\ \mathbf{u} \ \text{ in } \mathbb{R}^3. \tag{2.2}$$

Then we set, $\forall \mathbf{u}$ in $(\mathrm{L}^2(\mathbb{R}^3))^3$,

$$\begin{gathered} \phi(\mathbf{u}) \text{ the unique solution of (2.2),} \\ \text{and } \mathbf{A}(\mathbf{u}) = -\mathbf{grad}\ \phi(\mathbf{u}). \end{gathered}$$

By (2.1) and (2.2) we can write $\mathbf{H}$ as

$$\begin{aligned} \mathbf{H} &= -\mathbf{grad}\ (G * \mathrm{div}\ (\mathbf{u})) \\ &= -\mathbf{grad}\ \left(G * \sum_{i=1}^{3} \frac{\partial \mathbf{u}_i}{\partial x_i} \right) \\ &= -\mathbf{grad}\ \left(\sum_{i=1}^{3} G * \frac{\partial \mathbf{u}_i}{\partial x_i} \right) \\ &= -\mathbf{grad}\ \left(\sum_{i=1}^{3} \frac{\partial}{\partial x_i} (G * \mathbf{u}) \right) \\ &= -\mathbf{grad}\ (\mathrm{div}\ (\mathbf{u} * G)), \end{aligned}$$

where G is the fundamental solution for the Laplace equation in $\mathbb{R}^3$:

$$\forall x, y \in \mathbb{R}^3, \quad G(x,y) = \frac{-1}{4\pi|x-y|}.$$

Throughout this paper we shall use the notation

$$\mathbf{A}(\mathbf{u}) = -\mathbf{H} = \mathbf{grad}\ \left(\mathrm{div}\ \left(\int_\Omega \mathbf{u}(y) . \frac{1}{4\pi|x-y|}\ dy \right) \right). \tag{2.3}$$

The operator $\mathbf{A}$ is a linear operator from $(\mathrm{L}^2(\mathbb{R}^3))^3$ into $(\mathrm{L}^2(\mathbb{R}^3))^3$. It is positive symmetric, and its norm is bounded by 1 (see [8]). Furthermore, it is singular; its kernel is given by the following lemma.

LEMMA 2.1. *The operator* $\mathbf{A}$ *satisfies the following properties:*

(i) *For any* $\mathbf{u}$ *in* $(\mathrm{L}^2(\mathbb{R}^3))^3$, $(\mathbf{A}(\mathbf{u}), \mathbf{u})_{0,\mathbb{R}^3} = 0 \Longleftrightarrow \mathbf{A}(\mathbf{u}) = 0$.

(ii) $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{u} \in (\mathrm{L}^2(\mathbb{R}^3))^3,\ \mathbf{div}\ \mathbf{u} = 0 \text{ in } \mathbb{R}^3\}$.

Proof. (i) For any $\mathbf{u}$ in $(\mathrm{L}^2(\mathbb{R}^3))^3$, we have the following relations:

$$\begin{aligned}(\mathbf{A}(\mathbf{u}), \mathbf{u})_{0,\mathbb{R}^3} = 0 &\Longleftrightarrow (\mathbf{grad}\ \phi(\mathbf{u}), \mathbf{u})_{0,\mathbb{R}^3} = 0\\ &\Longleftrightarrow -(\phi(\mathbf{u}), \mathrm{div}\ \mathbf{u})_{0,\mathbb{R}^3} = 0\\ &\Longleftrightarrow (\phi(\mathbf{u}), \triangle\phi(\mathbf{u}))_{0,\mathbb{R}^3} = 0\\ &\Longleftrightarrow (\mathbf{grad}\ \phi(\mathbf{u}), \mathbf{grad}\ \phi(\mathbf{u}))_{0,\mathbb{R}^3} = 0\\ &\Longleftrightarrow ||\mathbf{A}(\mathbf{u})||^2_{0,\mathbb{R}^3} = 0.\end{aligned}$$

(ii) For any $\mathbf{u}$ in $(\mathrm{L}^2(\mathbb{R}^3))^3$ such that div $\mathbf{u} = 0$, the uniqueness of solutions of (2.2) proves that $\phi(\mathbf{u}) = 0$, and thus $\mathbf{A}(\mathbf{u}) = 0$.

For all $\mathbf{u}$ in $\mathrm{Ker}(\mathbf{A})$, since div $\mathbf{A}(\mathbf{u}) =$ div $\mathbf{u}$, we have div $\mathbf{u} = 0$. ∎

3. The finite volume discretization.

3.1. Space discretization. The domain Ω is broken down into n cubes Ω_i of length h. A function in $(\mathrm{L}^2(\Omega))^3$ will be approximated by piecewise constant functions (constant on each cube Ω_i). $\mathbb{R}^3$ is equipped with the Euclidian product "." and norm "$|\ |$". We introduce $(\mathbb{R}^3)^n$ made of functions $u = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$, each $\mathbf{u}_i$ belonging to $\mathbb{R}^3$. The space $(\mathbb{R}^3)^n$ is furnished with the canonical Euclidian structure written as follows:

$$\begin{aligned}\forall\ (u, v) \in\ (\mathbb{R}^3)^n :&\\ (u, v)_h &= \sum_{i=1}^{n} |\Omega_i|\ \mathbf{u}_i.\mathbf{v}_i,\\ ||u||^2_h &= \sum_{i=1}^{n} |\Omega_i|\ |\mathbf{u}_i|^2.\end{aligned}$$

In order to define the discrete problem, we introduce the following operators: R_h maps $(\mathbb{R}^3)^n$ into $(\mathrm{L}^2(\Omega))^3$ and is defined by

$$\forall v \in (\mathbb{R}^3)^n,\ \ \mathrm{R}_h(v) = \sum_{i=1}^{n} \chi_i\ \mathrm{v}_i;$$

P_h maps $(\mathrm{L}^2(\Omega))^3$ into $(\mathbb{R}^3)^n$ and is defined by

$$\forall \mathbf{u} \in (\mathrm{L}^2(\Omega))^3,\ \ \mathrm{P}_h(\mathbf{u})_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} \mathbf{u}(x)\ dx,$$

where χ_i is defined for x in $\mathbb{R}^3$ by $\chi_i(x) = 1$ if x belongs to Ω_i, $\chi_i(x) = 0$ otherwise. We shall use three main properties of these operators (see [4], [7]).

PROPOSITION 3.1. *Operators R_h and P_h satisfy the following properties:*
(i) *there exists C in $\mathbb{R}^+$, such that $\forall \mathbf{u}$ in $(\mathrm{H}^1(\Omega))^3$*

$$||\mathbf{u} - R_h(P_h(\mathbf{u}))||_{0,\Omega} \leq C\ h\ |\mathbf{u}|_{1,\Omega};$$

(ii) $\forall v \in (\mathbb{R}^3)^n$, $||R_h(v)||_{0,\Omega} = ||v||_h$;
(iii) $\forall \mathbf{u} \in (\mathrm{L}^2(\Omega))^3$, $||P_h(\mathbf{u})||_h \leq ||\mathbf{u}||_{0,\Omega}$.

This allows us to approximate the operator $\mathbf{A}$ by the following finite volume operator:

$$\mathbf{A}_h = \mathrm{P}_h \circ \mathbf{A} \circ \mathrm{R}_h. \tag{3.1}$$

$\mathbf{A}_h$ is an operator from $(\mathbb{R}^3)^n$ into $(\mathbb{R}^3)^n$. We introduce the notation

$$(\mathbf{A}_h(u))_i = \sum_{j=1}^{n} \mathrm{K}_i^j(\mathbf{u}_j), \tag{3.2}$$

where $\forall u \in (\mathbb{R}^3)^n$, $u = (\mathbf{u}_i)_{i\in\{1,\dots,n\}}$ and

$$\forall \mathbf{u} \in \mathbb{R}^3, \quad \mathrm{K}_i^j(\mathbf{u}) = \frac{1}{4\pi|\Omega_i|}\int_{\Omega_i}\left[\mathbf{grad}\ \mathrm{div}\ .\int_{\Omega_j}\mathbf{u}(y)\frac{-1}{|y-x|}\,dy\right]\,dx. \tag{3.3}$$

Each K_i^j is a 3-by-3 real matrix. These matrices characterize the interaction between two cells Ω_i and Ω_j.

3.2. Properties of the approximate operator $\mathbf{A}_h$. We start with the elementary properties of $\mathbf{A}_h$.

THEOREM 3.2. *For all real $h > 0$, the discrete operator $\mathbf{A}_h$ is symmetric and a positive contraction in $\mathcal{L}((\mathbb{R}^3)^n)$; furthermore, there exists C in $\mathbb{R}_*^+$ such that $\forall \mathbf{u}$ in $(\mathrm{H}^1([0,T]\times\Omega))^3$*

$$||R_h \circ \mathbf{A}_h \circ P_h(\mathbf{u}) - \mathbf{A}(\mathbf{u})||_{0,\Omega} \leq C\ h\ |\mathbf{u}|_{1,\Omega}.$$

The proof is straightforward and will be omitted (see [13]).

We saw in Lemma 2.1 that $\mathbf{A}$ is singular. On the contrary, the discretized operator $\mathbf{A}_h$ is regular. To prove that result we will use an intermediate lemma.

LEMMA 3.3. *For all u in $(\mathbb{R}^3)^n$, one can write*

$$\mathbf{div}\ R_h(u) = 0 \iff \forall i \in \{1,\dots,n\},\ \ \mathbf{u}_i = 0.$$

Proof. We first write that $\mathrm{div}\ \mathrm{R}_h(\mathbf{u})$ vanishes if and only if the normal component of $\mathrm{R}_h(\mathbf{u})$ is continuous on the interfaces $\bar{\Omega}_i \cap \bar{\Omega}_j$. Thus, starting from one edge, since $\mathrm{R}_h(\mathbf{u})$ vanishes outside of Ω, $\mathrm{R}_h(\mathbf{u})$ vanishes everywhere. ∎

With this result, we can prove the following claim.

THEOREM 3.4. *For every $h > 0$, the discrete operator $\mathbf{A}_h$ is regular; i.e.,* $\mathbf{Ker}\mathbf{A}_h = \{0\}$.

Proof. Let u in $(\mathbb{R}^3)^n$ be such that $\mathbf{A}_h(u) = 0$. Then, for every i in $\{1,\dots,n\}$ we have the following sequence of relations:

$$\int_{\Omega_i}(\mathbf{A}(\mathrm{R}_h(u)))\ dx = 0 \Rightarrow \sum_{i=1}^{n}\left(\int_{\Omega_i}\mathbf{A}(\mathrm{R}_h(u))\,dx\right)\ \mathbf{u}_i = 0$$
$$\Rightarrow \int_{\Omega}\mathbf{A}(\mathrm{R}_h(u))\mathrm{R}_h(u)\,dx = 0.$$

This implies by Lemma 2.1 that $\mathbf{A}(\mathrm{R}_h(u)) = 0$. Then $\mathbf{A}_h(u)$ vanishes if and only if $\mathbf{A}(\mathrm{R}_h(u))$ vanishes, that is, if and only if $\mathrm{R}_h(\mathbf{u})$ is in $\mathbf{KerA} \cap \{\mathbf{v} \in (\mathrm{L}^2(\mathbb{R}^3))^3 | v = 0 \text{ a.e. in } \mathbb{R}^3 \backslash \Omega\}$. Thus, thanks to Lemma 3.3, we conclude that $\mathbf{A}_h(u)$ vanishes if and only if $u = 0$. ∎

We shall now prove an estimate on the smallest eigenvalue of $\mathbf{A}_h$.

THEOREM 3.5. *The smallest eigenvalue $\lambda_{h,min}$ of $\mathbf{A}_h$ is such that*

$$\lambda_{h,min} \geq \frac{1}{4\sqrt{34}} \frac{h^{5/2}}{d(\Omega)}, \tag{3.4}$$

where $d(\Omega)$ is the diameter of Ω, i.e., $d(\Omega) = \sup_{\mathbf{x},\mathbf{y} \in \Omega}(|\mathbf{x} - \mathbf{y}|)$.

Proof. The main idea is to use the variational formulation to estimate the Rayleigh quotient.

1. *Estimate of $\lambda_{h,min}$ through Rayleigh quotient.* To estimate the lowest eigenvalue of $\mathbf{A}_h$ we use the characterization of $\lambda_{h,min}$ by the Rayleigh quotient

$$\min_{u \in (\mathbb{R}^3)^n} \frac{(\mathbf{A}_h(u), u)_h}{\|u\|_h^2} = \lambda_{h,min},$$

or, by definition of R_h,

$$\begin{aligned} \lambda_{h,min} &= \min_{u \in (\mathbb{R}^3)^n} \frac{(\mathrm{R}_h(\mathbf{A}_h(u)), \mathrm{R}_h(u))_{0,\Omega}}{\|u\|_h^2} \\ &= \min_{u \in (\mathbb{R}^3)^n} \frac{\|\mathbf{grad}\ \phi(\mathrm{R}_h(u))\|_{0,\Omega}^2}{\|u\|_h^2}. \end{aligned}$$

2. *Definition of a convenient subset of trial functions.* We set a variational formulation for (2.2):

$$\begin{gathered} \text{find } \phi \in \mathrm{W}^1(\mathbb{R}^3), \forall \psi \in \mathrm{W}^1(\mathbb{R}^3),\ \mathbf{u} \in \mathcal{I}m(\mathrm{R}_h) \text{ we have} \\ \int_{\mathbb{R}^3} \mathbf{grad}\ \psi . \mathbf{grad}\ \phi\ d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{u} . \mathbf{grad}\ \psi\ d\mathbf{x}. \end{gathered} \tag{3.5}$$

To define trial functions, we have to set some notation. The mesh is cubic, and we denote by $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ the three main directions. For two adjacent cells in direction $\mathbf{X}$ and for the face between, we shall denote by $\Omega_j^{i,\mathrm{X}}$ the first cell, $\Omega_{j+1}^{i,\mathrm{X}}$ the following, and $\Sigma_j^{i,\mathrm{X}}$ the face between.

Then, for two adjacent cells $\Omega_j^{i,\mathrm{X}}$ and $\Omega_{j+1}^{i,\mathrm{X}}$, we define $\psi_j^{i,x}$ such that

$$\begin{gathered} \psi_j^{i,x} \in \mathrm{W}^1(\mathbb{R}^3), \\ \psi_j^{i,x}{}_{|_{\partial(\bar{\Omega}_j^{i,\mathrm{X}})}} = 0, \\ \int_{\Sigma_j^{i,\mathrm{X}}} \psi_j^{i,x}(x,y,z)\ dy\ dz = (\mathbf{u}_{j+1}^{i,\mathrm{X}} - \mathbf{u}_j^{i,\mathrm{X}}) . \mathbf{X}. \end{gathered}$$

Construction of a well chosen space of $\psi_j^{i,x}$ is extensively described in [13].

3. *Estimates.* We apply (3.5) for trial functions defined above. For $n_j^{i,\mathrm{X}}$ the normal to face $\Sigma_j^{i,\mathrm{X}}$ in direction $\mathbf{X}$ and u an element of $(\mathbb{R}^3)^n$ such that $\mathrm{R}_h(u) = \mathbf{u}$,

we find using the Green formula

$$\begin{aligned}\int_{\mathbb{R}^3} \mathbf{grad}\ \phi(\mathbf{u}).\mathbf{grad}\ \psi_j^{i,x}\ d\mathbf{x} &= \int_{\mathbb{R}^3} \mathbf{u}.\mathbf{grad}\ \psi_j^{i,x}\ d\mathbf{x}\\ &= \int_{\Sigma_j^{i,\mathrm{X}}} [\mathrm{R}_h(u).n_j^{i,\mathrm{X}}]_{|\Sigma_j^{i,\mathrm{X}}} \psi_j^{i,x}\ d\mathbf{x}\\ &= ((\mathbf{u}_{j+1}^{i,\mathrm{X}} - \mathbf{u}_j^{i,\mathrm{X}}).\mathbf{X}).\int_{\Sigma_j^{i,\mathrm{X}}} \psi_j^{i,x}\ d\mathbf{x},\end{aligned}$$

that by construction of $\psi_j^{i,x}$ we have

$$((\mathbf{u}_{j+1}^{i,\mathrm{X}} - \mathbf{u}_j^{i,\mathrm{X}}).\mathbf{X}).\int_{\Sigma_j^{i,\mathrm{X}}} \psi_j^{i,x}\ d\mathbf{x} = ((\mathbf{u}_{j+1}^{i,\mathrm{X}} - \mathbf{u}_j^{i,\mathrm{X}}).\mathbf{X})^2.$$

At this point of the proof, using the Cauchy–Schwarz inequality and expression of $\psi_j^{i,x}$, we find

$$\|\operatorname{grad}\phi\|^2_{(\mathrm{L}^2(\tilde{\Omega}_j^{i,\mathrm{X}}))^3} \geq \frac{9\ h^3}{272}((u_{j+1}^{i,\mathrm{X}} - u_j^{i,\mathrm{X}}).\mathbf{X})^2. \tag{3.6}$$

This result is also valid for directions $\mathbf{Y}$ and $\mathbf{Z}$. Now we add a layer of cells on the border of Ω in which we consider that u vanishes. Thanks to that "null layer," we can obtain by summation of (3.6) a global estimate:

$$\begin{aligned}&6\ \|\operatorname{grad}\phi\|^2_{(\mathrm{L}^2(\mathbb{R}^3))^3}\\ &\geq \frac{9\ h^3}{272}\sum_{i,j,k;l,n,m}\left(((u_{l+1}^{i,\mathrm{X}} - u_l^{i,\mathrm{X}}).\mathbf{X})^2 + ((u_{n+1}^{j,\mathrm{Y}} - u_n^{j,\mathrm{Y}}).\mathbf{Y})^2 + ((u_{m+1}^{k,\mathrm{Z}} - u_m^{k,\mathrm{Z}}).\mathbf{Z})^2\right).\end{aligned} \tag{3.7}$$

On the other hand, by a succession of discrete Cauchy–Schwarz inequalities, we can prove that

$$\begin{aligned}&\|u\|_h^2 = \sum_{i=1}^{n} |u_i|^2\\ &\leq \left(\frac{\mathrm{d}(\Omega)}{h}\right)^2 \sum_{i,j,k;l,n,m}\left(((u_{l+1}^{i,\mathrm{X}} - u_l^{i,\mathrm{X}}).\mathbf{X})^2 + ((u_{n+1}^{j,\mathrm{Y}} - u_n^{j,\mathrm{Y}}).\mathbf{Y})^2 + ((u_{m+1}^{k,\mathrm{Z}} - u_m^{k,\mathrm{Z}}).\mathbf{Z})^2\right).\end{aligned} \tag{3.8}$$

Thus, by (3.7) and (3.8) we get

$$\|\operatorname{grad}\phi\|_{0,\Omega} \geq \frac{1}{4\sqrt{34}}\ \frac{h^{5/2}}{\mathrm{d}(\Omega)}\|u\|_h,$$

which gives (3.4) and ends the proof of the theorem. □

3.3. Construction of the semianalytical operator $\mathbf{A}_{h,N}$. The two successive integrals in $\mathbf{A}_h$ practically forbid the direct use of A_h in real computations. Instead we introduce the semianalytical operator $\mathbf{A}_{h,N}$: it is obtained by analytical integration of $\mathbf{A} \circ \mathrm{R}_h$, followed by a discrete projection $\mathrm{P}_{h,N}$ computed with an N-point Gauss quadrature formula (a classical method in integral equation computation; see [16]). Thus we define the approximate discretized operator by

$$\mathbf{A}_{h,N} = \mathrm{P}_{h,N} \circ \mathbf{A} \circ \mathrm{R}_h. \tag{3.9}$$

3.3.1. Description of the numerical integration. We first introduce the integration of each submatrix K_i^j of $\mathbf{A}_h$:

$$\mathrm{K}_i^j(\mathbf{u}) = \frac{1}{|\Omega_i|}\left(\int_{\Omega_i} \mathrm{k}_j(\mathbf{x})\mathbf{u}\, d\mathbf{x}\right),$$

where $\mathbf{u}$ is an element of $\mathbb{R}^3$ and k_j is a 3-by-3 matrix defined by

$$\mathrm{k}_j(\mathbf{x})\ \mathbf{u} = \frac{1}{4\pi}\mathbf{grad}\ \mathrm{div}\ \int_{\Omega_j} \mathbf{u}\frac{-1}{|\mathbf{y}-\mathbf{x}|}\, d\mathbf{y}.$$

First of all, we remark that for $i = j$, $\mathrm{K}_i^j(\mathbf{u}) = \frac{1}{3}\mathbf{Id}_3$. Indeed, we have

$$\begin{aligned}\mathrm{k}_{i,xx}(\mathbf{x}) &= \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\int_{\Omega_i}\frac{-1}{|\mathbf{y}-\mathbf{x}|}\, d\mathbf{y},\\ \mathrm{k}_{i,xy}(\mathbf{x}) &= \frac{1}{4\pi}\frac{\partial^2}{\partial x\partial y}\int_{\Omega_i}\frac{-1}{|\mathbf{y}-\mathbf{x}|}\, d\mathbf{y};\end{aligned}$$

then, by symmetry, we obtain that the integral of the extra diagonal terms of $\mathrm{k}_j(\mathbf{x})$ over Ω_i vanish and

$$\int_{\Omega_i}\mathbf{k}_{i,xx}\, d\mathbf{x} = \int_{\Omega_i}\mathbf{k}_{i,yy}\, d\mathbf{x} = \int_{\Omega_i}\mathbf{k}_{i,zz}\, d\mathbf{x},$$

but

$$\int_{\Omega_i}(\mathbf{k}_{i,xx}+\mathbf{k}_{i,yy}+\mathbf{k}_{i,zz})\, dx = \int_{\Omega_i}\int_{\Omega_i}\triangle\left(\frac{-1}{4\pi|\mathbf{x}-\mathbf{y}|}\mathbf{xy}\right) = |\Omega_i|.$$

Then we conclude that $\mathrm{K}_i^i(\mathbf{u}) = \frac{1}{3}\mathbf{Id}_3$, and we set $\tilde{\mathrm{K}_i^i} = \frac{1}{3}\mathbf{Id}_3$.

When $i \neq j$, we have to perform a numerical integration on all $\mathrm{k}_j(\mathbf{x})$ (which are obtained by analytical integration on Ω_j). As pointed out in [15], items of matrix $\mathrm{k}_j(\mathbf{x})$ are linear combinations of functions of the following type: $\forall i, j \in \{1,\dots,n\}$ and $i \neq j$ and $r, s, t \in \{0, 1\}$ we set

$$\begin{aligned}g_{i,j}^{rst}(x,y,z) &= \tan^{-1}\left(\frac{(y-(y_i-y_j)-sh).(z-(z_i-z_j)-th)}{(x-(x_i-x_j)-rh)\mathrm{r}_{r,s,t}}\right)\\ f_{i,j}^{rst}(x,y,z) &= \mathrm{sh}^{-1}\left(\frac{(z-(z_i-z_j)-th)}{\sqrt{(x-(x_i-x_j)-rh)+(y-(y_i-y_j)-sh)}}\right),\end{aligned}$$

where $\mathrm{r}_{r,s,t} = \sqrt{((x_i-x_j)-rh)^2+((y_i-y_j)-sh)^2+((z_i-z_j)-th)^2}$ and h is the mesh step.

For each (i,j), $g_{i,j}^{rst}$ is an element of $\mathrm{C}^\infty(]0,h[^3)$, and for each (i,j) such that Ω_i and Ω_j are nonadjacent, $f_{i,j}^{rst}$ is also an element of $\mathrm{C}^\infty(]0,h[^3)$.

But, when (i,j) is such that Ω_i and Ω_j are adjacent cells, $f_{i,j}^{rst}$ is no longer an element of $\mathrm{C}^\infty(]0,h[^3)$; it is an element of $\mathrm{H}^1(]0,h[^3)$. Thus, we split $\mathrm{k}_j(\mathbf{x})$ into two parts, a singular one denoted $\mathrm{k}_j^s(\mathbf{x})$, element $\mathrm{H}^1(]0,h[^3)$, and a regular one denoted $\mathrm{k}_j^r(\mathbf{x})$, element of $\mathrm{C}^\infty(]0,h[^3)$. This splitting is such that the singular part $\mathrm{k}_j^s(\mathbf{x})$ can be integrated analytically.

We recall the Gauss quadrature formula and error estimates. For any function f sufficiently regular, we set

$$\int_{[0,1]^3} f(x)\ dx \approx \mathcal{Q}_{N,i}(f)$$
$$= h^3 \sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\sum_{j_3=1}^{N}\left(\prod_{k=1,2,3}\alpha_{j_k}\right) f(h\ \zeta_{j_1}-x_i, h\ \zeta_{j_2}-z_i, h\ \zeta_{j_3}-z_i),$$

where $(\alpha_j,\zeta_j)_{j=1,\dots,N}$ are weights and points for the one dimensional Gauss quadrature formula. We set an error formula $\forall\sigma\geq 3$ (see [2], [13]):

$$\mathrm{E}_{N,i}(f(x)) = \int_{\Omega_i} f(x)\ dx - \mathcal{Q}_{N,i}(f).$$

For $f(x)$ in $\mathrm{C}^\infty(\Omega_i)$ we have the following error estimate:

$$|\mathrm{E}_{N,i}(f(x))| \leq \frac{C}{N^\sigma}\|f\|_{\mathrm{H}^\sigma(\Omega_i)}. \tag{3.10}$$

Thus, we define $\tilde{\mathrm{K}}_i^j$ for $i\neq j$: If Ω_j and Ω_i are nonadjacent, we set

$$\tilde{\mathrm{K}}_i^j = \frac{1}{|\Omega_i|}\mathcal{Q}_{N,i}(\mathbf{k}_j);$$

else

$$\tilde{\mathrm{K}}_i^j = \frac{1}{|\Omega_i|}\left[\mathcal{Q}_{N,i}(\mathbf{k}_j^r) - \int_{\Omega_i}\mathbf{k}_j^s(x)\ dx\right].$$

We can therefore apply formula (3.10) to estimate the quadrature error $\mathrm{E}_{N,i,j}$ between K_i^j and $\tilde{\mathrm{K}}_i^j$: for $i=j$ we have

$$\mathrm{E}_{N,i,j} = 0;$$

if $i\neq j$ and Ω_j, Ω_j are nonadjacent cells,

$$\mathrm{E}_{N,i,j} \leq \frac{C}{N^\sigma|\Omega_i|}||\mathbf{k}_j||_{\mathrm{H}^\sigma(\Omega_i)}; \tag{3.11}$$

else, if Ω_j and Ω_j are adjacent cells,

$$\mathrm{E}_{N,i,j} \leq \frac{C}{N^\sigma|\Omega_i|}||\mathbf{k}_j^r||_{\mathrm{H}^\sigma(\Omega_i)}. \tag{3.12}$$

3.3.2. Estimate of the lowest eigenvalue. Now, thanks to the error estimate of the Gauss quadrature, we can establish a lower bound for the lowest eigenvalue of $\mathbf{A}_h$.

THEOREM 3.6. *Let $\sigma\geq 3$, for k_j belonging to $H^\sigma(\Omega_i)$. A sufficient condition for the positiveness of $\mathbf{A}_{h,N}$ is the existence of a real positive constant α_σ such that*

$$\alpha_\sigma\ N^\sigma\ \geq \frac{1}{h^{5/2}},$$

where N is the number of Gauss points in each space direction.

Proof. We denote by E_h the error; i.e., $\mathbf{A}_{h,N} = \mathbf{A}_h + \mathrm{E}_h$. Eigenvalues of these three operators are numbered increasingly and denoted as

$(\lambda_i)_{i=1,\dots,3n}$ the spectrum of $\mathbf{A}_h$,

$(\tilde{\lambda}_i)_{i=1,\dots,3n}$ the spectrum of $\mathbf{A}_{h,N}$,

$(\epsilon_i)_{i=1,\dots,3n}$ the spectrum of E_h.

Classical algebra results allow us to write (see [12])

$$\sup_{i\in\{1,\dots,3n\}} |\lambda_i - \tilde{\lambda}_i| \leq \sup_{i\in\{1,\dots,3n\}} |\epsilon_i|. \tag{3.13}$$

Then, we are led to find an upper bound for the eigenvalues of E_h. Since the integration on the diagonal terms (local 3-by-3 matrices) is exact, the diagonal terms in E_h (local 3-by-3 matrices) vanish. Then, the Gershgörin circles theorem gives

$$\sup_{i\in\{1,\dots,3n\}} |\epsilon_i| \leq \sup_{i\in\{1,\dots,n\}} \left(\sum_{j\in\{1,\dots,3n\}} |E_{h,ij}| \right).$$

As a consequence, if we consider the 3-by-3 submatrices K_i^j, using error estimate formulae (3.11), (3.12), we have for any $\sigma \geq 3$ the existence of a real constant α_σ such that

$$\sum_{j=1, j\neq i}^{N} \sum_{l=1}^{3} \left| \left(\tilde{\mathrm{K}}_i^j\right)_{1l} - \left(\mathrm{K}_i^j\right)_{1l} \right| \leq \alpha_\sigma \, \frac{C}{N^\sigma},$$

with $\alpha_\sigma = \sup_{i\in\{1,\dots,n\}} (\frac{1}{|\Omega_j|} \|k_j\|_{\mathrm{H}^\sigma(\Omega_j)})$ and we can write

$$\sup_{i\in\{1,\dots,3n\}} |\epsilon_i| \leq \alpha_\sigma \, \frac{C}{N^\sigma}. \tag{3.14}$$

We now build a sufficient condition for the positiveness of $\mathbf{A}_{h,N}$: the coefficients of $\mathbf{k}_j$ belong to $\mathrm{H}^\sigma(\Omega)$ ($\sigma \geq 3$), so by Theorem 3.5 and by (3.14), we have

$$\frac{1}{4\sqrt{34}} \, \frac{h^{5/2}}{\mathrm{d}(\Omega)} \geq \alpha_\sigma \, \frac{C}{N^\sigma},$$

and we can conclude that

$$N^\sigma \geq \alpha_\sigma C \, 4\sqrt{34}\mathrm{d}(\Omega) \, \frac{1}{h^{5/2}}. \qquad \square$$

3.3.3. Symmetrization of the approximate operator. In order to keep the operator symmetric, we set $\mathrm{A}_{h,N}^S = \frac{1}{2}(\mathrm{A}_{h,N} + \mathrm{A}_{h,N}^t)$, i.e.,

$$\forall u \in (\mathbb{R}^3)^n, \;\; (\mathrm{A}_{h,N}^S(u))_i = \sum_{j=1}^{n} \frac{1}{2} \left(\mathrm{K}_i^j + \tilde{\mathrm{K}}_j^i \right) (\mathbf{u}_j).$$

All the results presented here for $\mathbf{A}_{h,N}$ extend to $\mathrm{A}_{h,N}^S$. In what follows we will use $\mathrm{A}_{h,N}^S$.

3.3.4. Convergence theorem for the Gauss approximated operator. We are now able to give the convergence rate of the Gauss approximated operator as follows.

THEOREM 3.7. *For all u in $\mathrm{H}^1(\Omega)$ and N in $\mathbb{N}^*$ such that the condition given in Theorem* 3.6 *is verified, there exists C in $\mathbb{R}_*^+$ such that $\forall h$ in $\mathbb{R}_*^+$ we have*

$$\|R_h \circ A_{h,N}^S \circ P_h(u) - A(u)\|_{0,\Omega} \leq Ch|u|_{1,\Omega},$$

and the operator $A_{h,N}^S$ is symmetric, definite, and positive.

Proof. The positiveness, symmetry, and regularity of $\mathrm{A}_{h,N}^S$ are direct consequences of the hypothesis of Theorem 3.6 and the previous paragraph. The error estimate is obtained by the estimation

$$\mathrm{A}_{h,N}^S = \frac{1}{2}(\mathrm{A}_{h,N} + \mathrm{A}_{h,N}^t) = \mathrm{A}_h + \frac{1}{2}(E_{h,N} + E_{h,N}^t),$$

and then we have

$$\begin{aligned}
&\|R_h \circ \mathrm{A}_{h,N}^S \circ P_h(u) - \mathrm{A}(u)\|_{0,\Omega} \\
&\quad = \|R_h \circ \mathrm{A}_h \circ P_h(u) - \mathrm{A}(u) + \frac{1}{2}\mathrm{R}_h(E_{h,N} + E_{h,N}^t)\mathrm{P}_h(u)\|_{0,\Omega} \\
&\quad \leq \|R_h \circ \mathrm{A}_h \circ P_h(u) - \mathrm{A}(u)\|_{0,\Omega} + \frac{1}{2}\|\mathrm{R}_h(E_{h,N} + E_{h,N}^t)\mathrm{P}_h(u)\|_{0,\Omega} \\
&\quad \leq Ch|u|_{1,\Omega} + \frac{1}{2}C_1|u|_{1,\Omega} \sup_{(i,j)\in\{1,\ldots,n\}^2} |E_{h,N,ij}| \\
&\quad \leq (Ch + C_1\alpha_\sigma h^{5/2})|u|_{1,\Omega}. \qquad \square
\end{aligned}$$

To illustrate the convergence of the approximation, we compute the error between the exact and approximated solutions of the problem, a uniform field in a cube of length one (see Figure 1).

Number of cells	h	Error
64	1/4	0.0546256921
512	1/8	0.0406119569
1736	1/16	0.0264626018
32768	1/32	0.0160913191
262144	1/64	0.0093714246

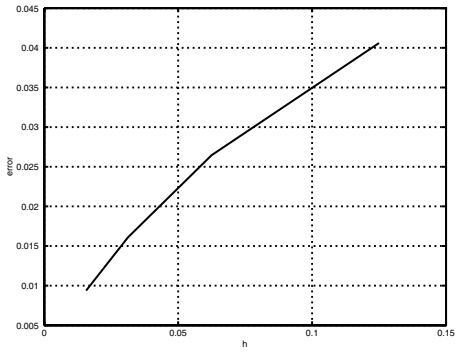

FIG. 1. *Error between exact and approximated solutions of a uniform field in a cube.*

4. Block-Toeplitz matrices: Application to the computation of the magnetostatic field. The operator $\mathbf{A}^S_{h,N}$ is represented by a full matrix. Thus, the use of this operator becomes impossible for the huge meshes used for simulations such as those of micromagnetic systems. To overcome that problem, we use a feature of this matrix: it is a block-Toeplitz matrix. We will start with a general presentation of block-Toeplitz matrices using tensored products. We will then present an application of block-Toeplitz matrix products to compute the magnetostatic field. This fast computation is not built on a truncation of the operator $\mathrm{A}^S_{h,N}$: it is an exact method. Effectively, the embedding of Toeplitz matrices in circulant matrices as presented here preserves exactly the matrix-vector product.

4.1. The block-Toeplitz vector-matrix multiplication. We recall briefly the definition of a block-Toeplitz matrix and the main ideas of the block-Toeplitz vector-matrix multiplication. An extensive study of this problem can be found in [9], [17].

DEFINITION 4.1. *T_n is a one-level Toeplitz matrix of order n if and only if*

$$T_n = (t_{i-j})_{i,j\in\{1,\dots,n\}} = \begin{pmatrix} t_0 & t_{-1} & \dots & \dots & t_{1-n} \\ t_1 & t_0 & \ddots & & t_{2-n} \\ \vdots & \ddots & \ddots & t_{-1} & \vdots \\ t_{n-2} & & t_1 & t_0 & t_{-1} \\ t_{n-1} & \dots & \dots & t_1 & t_0 \end{pmatrix},$$

with $(t_i)_{i\in\{1-n,\dots,n-1\}} \in \mathbb{R}^{2n-1}$.

The vector $\{t_{1-n}, t_{2-n}, \dots, t_{-1}, t_0, t_1, \dots, t_{n-2}, t_{n-1}\}$ is called the generator of T_n. $T_{n_1,\dots,n_p}$ is called a p-level block-Toeplitz matrix of order $\Pi_{i=1}^p n_i$ if and only if, following the notation above, the items $(t_i)_{i\in\{1-n,\dots,n-1\}}$ are $p-1$ block-Toeplitz matrices of order $\Pi_{i=2}^p n_i$.

We recall also the following definition of circulant matrices.

DEFINITION 4.2. *C_n is a one-level circulant matrix of order n if and only if C_n is a one-level Toeplitz matrix such that, $\forall i$ in $\{1,\dots,n-1\}$, $(C_n)_{i,n} = (C_n)_{i+1,1}$ and $(C_n)_{n,n} = (C_n)_{1,1}$. The multilevel circulant matrices are multilevel block-Toeplitz matrices built using the procedure.*

Then one can demonstrate that you could easily embed Toeplitz matrices in at least 2^p greater circulant matrices, where p is the number of the level considered. Such an embedding therefore permits us to compute the matrix-vector product quickly thanks to fast Fourier transformations (FFTs), using the fact that the multiplication between a circulant matrix and a vector is a discrete convolution.

Therefore, by applying the FFT algorithm to compute the products of a Fourier transform of a vector, we have the following results.

THEOREM 4.3. *The matrix-vector product algorithm using FFTs for p-level block-Toeplitz matrices of order $N_p = \prod_{k=1}^p n_k$ needs*

- *$O(3\ p\ 2^p\ N_p\ +\ 3\ 2^p\ N_p \log(N_p))$ operations,*
- *storage of $O(2^p\ N_p)$ real numbers.*

We keep in mind that a direct computation of the product would have needed $O(N_p^2)$ operations and the storage of $O(N_p^2)$ real numbers.

Proof. The algorithm requires three p-levels of the FFTs, two for the embedding (matrix and vector) and one for extraction of the result. A p-level transform $\mathrm{F}_{2^{m_k}}$ needs $2^{m_k}\log(2^{m_k})$ operations. Then, $\mathrm{F}_{\otimes p}$ on a grid $\prod_{k=1}^p\{1,\dots,2^{m_k}\}$ needs a

number of operations equal to

$$\left(\prod_{i\neq k, i=1}^{p} 2^{m_i}\right) 2^{m_k}\log 2^{m_k} = \left(\prod_{i=1}^{p} 2^{m_i}\right)\log 2^{m_k},$$

where, for any x in $\mathbb{R}_*^+$, $\log x$ is the base-2 logarithm.

Using the "power two" FFT, we set $M_p = \prod_{i=1}^{p} 2^{m_i}$. Thus, to apply $\mathrm{F}_{\otimes p}$ requires $M_p \sum_{k=1}^{p} \log 2^{m_k} = M_p \sum_{k=1}^{p} m_k$ operations. Then, using $\mathrm{Fl}(x)$ as a notation for the floor function, we set $m_k = \mathrm{Fl}(\log(n_k))+1$, and we have $\forall k \in \{1,\dots,p\}$, 2, $n_k \geq 2^{m_k}$. This allows us to bound the number of operations needed for a p-level FFT by

$$M_p \sum_{k=1}^{p} m_k \leq 2^p N_p \log\left(\prod_{k=1}^{p} 2\ n_k\right) = 2^p N_p \log(2^p N_p).$$

We conclude that the algorithm needs $O(3\ p\ 2^p\ N_p\ +\ 32^p\ \log(N_p))$ operations.

We only need to store the generator vectors of the monolevel Toeplitz submatrices of the p-level block-Toeplitz matrix. The storage of each monolevel structure needs 2^{m_k} reals, so we can estimate the global storage by

$$\prod_{k=1}^{p} 2^{m_k} \leq \prod_{k=1}^{p} 2\ n_k \leq 2^p\ N_p. \qquad \square$$

4.2. Application to magnetostatic computations for micromagnetic simulations. Let us come back to problem (1.1). First of all, we have the following.

THEOREM 4.4. *The discretized operator* $\mathbf{A}_h$ *is a* 3*-level block-Toeplitz matrix.*

Proof. As we saw previously,

$$\mathrm{A}_{h,I,J}\mathbf{u} = \frac{1}{4\pi\ h^3}\int_{\Omega_{\mathrm{ind}_3(I)}} \mathrm{grad}_x \,\mathrm{div}_{\ x}. \int_{\Omega_{\mathrm{ind}_3(J)}} \mathbf{u}\frac{1}{|y-x|}\ dy\ dx \quad \forall \mathbf{u}\in\mathbb{R}^3.$$

We apply to this formula the following change of variables:

$$x = x_{\mathrm{ind}_3(I)} + \hat{x}, \qquad \hat{x}\in[0,h]^3,$$
$$y = x_{\mathrm{ind}_3(I)} + \hat{y}, \quad \hat{y}\in\prod_{k=1}^{3}[(i_k-j_k)h,(i_k-j_k+1)h] = \Omega_{|IJ|},$$

so that

$$\mathrm{A}_{h,I,J}\mathbf{u} = \frac{1}{4\pi\ h^3}\int_{\Omega_{|IJ|}} \mathrm{grad}_x \,\mathrm{div}_{\ x}. \int_{\Omega_{|IJ|}} \mathbf{u}\frac{1}{|\hat{y}-\hat{x}|}\ d\hat{y}\ d\hat{x} \quad \forall \mathbf{u}\in\mathbb{R}^3.$$

Then, $\forall I$ and J in $\prod_{k=1}^{3}\{1,\dots,n_k\}$, $\mathbf{A}_{h,I,J}$ depends only on $(I-J)$. We conclude, by using Definition 4.1, that $\mathbf{A}_h$ is a 3-level block-Toeplitz matrix. $\square$

A comparison of the computational time for magnetic bricks of various sizes is presented in Tables 1 and 2. The time unit used in the tables is 10^{-2}s.

5. Some numerical results. In this section, efficiency of the method is tested by comparing numerical and theoretical results. The results, by Joseph and Schlömann (see [11]), are valid for a rectangular magnetic prism whose basis length a is negligible with respect to its height b (see Figure 2); the magnetization field is considered to be

TABLE 1
Computational time. The computations are made with the optimized LAPACK library for full matrices and the fast solving method discussed in the text. The FFT used for the fast solving method is a plain Fortran code.

Number of cells in each direction	Total number of cells	LAPACK	Block-Toeplitz algorithm
4×4	16	0.01	0.17
$4 \times 4 \times 2$	32	0.04	0.32
$4 \times 4 \times 4$	64	0.17	0.62
$4 \times 4 \times 8$	128	0.78	1.31
$4 \times 8 \times 8$	256	3.93	2.55
$8 \times 8 \times 8$	512	16.49	5.35
$8 \times 8 \times 16$	1024	70.77	11.03

TABLE 2
The assembly time. Assembly has to be made only when the geometry is changed.

Number of cells in each direction	Total number of cells	Assembly time for a full matrix	Assembly time for a block-Toeplitz matrix
4×4	16	52	24
$4 \times 4 \times 2$	32	199	57
$4 \times 4 \times 4$	64	779	133
$4 \times 4 \times 8$	128	3107	290
$4 \times 8 \times 8$	256	12364	611
$8 \times 8 \times 8$	512	49153	1325
$8 \times 8 \times 16$	1024	199938	2725

TABLE 3

$p = \frac{a}{b}$	Number of cells on basis	Number of cells on length	Total number of cells
0.5	16×16	32	8192
0.25	16×16	64	16384
0.125	8×8	64	4096
0.06255	8×8	128	8192

uniformly parallel to the height. The authors give the magnetic field along the great axis between two points of the domain: the center of the prism and the center of one of the bases. Figure 3 gives the magnetostatic field (projected on the prism height) along the computation line for various ratios $p = \frac{a}{b}$, as given in Table 3.

The results are quite satisfactory. We see that the theoretical results tend to the numerical results when the length ratio tends to zero.

6. Conclusion. The method developed in this article for computing the magnetostatic field is performant. It is useful for dynamic computations like micromagnetic simulations, which need to compute the magnetostatic field at each time step. For these simulations (see [13], [14]), the embedding 3-level block circulant matrix is computed before the first time step. Then, the only computation at each time step is the matrix-vector block circulant product and the extraction.

There exist other methods to solve the Poisson equation, one of the most competitive being the fast multipole method [6]. However, it turns out that this method is not adapted to our problem. First, the nonexact preservation of the negativeness of the magnetostatic operator, as explained in the introduction, is essential to obtain consistent equilibrium states for ferromagnetic problems. Second, the use of a regular

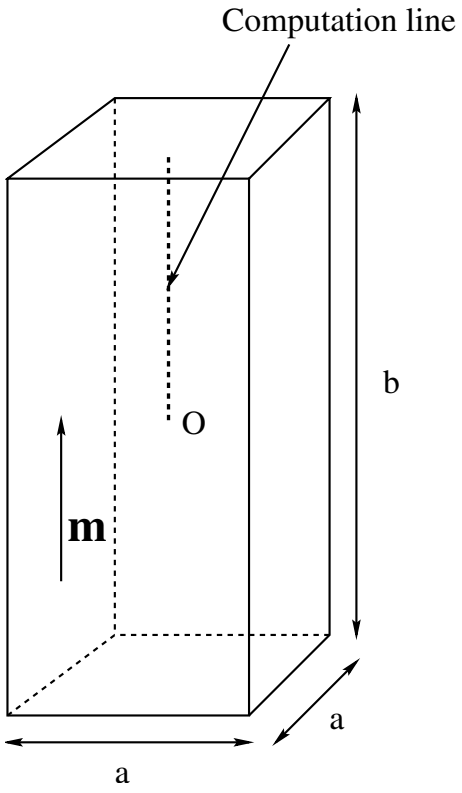

FIG. 2. *The magnetic domain* Ω.

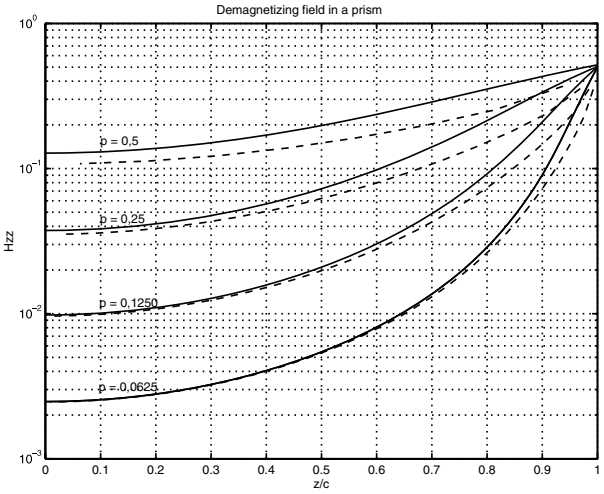

FIG. 3. *Comparison between theoretical and numerical results.*

grid is an advantage in the context of dynamical simulation; indeed, the structures we want to catch are very fine, and nonregular grids may adversely influence the results [14]. In sum, using regular grids, our method is clearly easier to implement than the fast multipole method for the same complexity.

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