Uniform controllability of semidiscrete approximations of parabolic control systems

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Abstract

Controlling an approximation model of a controllable infinite dimensional linear control system does not necessarily yield a good approximation of the control needed for the continuous model. In the present paper, under the main assumptions that the discretized semigroup is uniformly analytic, and that the control operator is mildly unbounded, we prove that the semidiscrete approximation models are uniformly controllable. Moreover, we provide a computationally efficient way to compute the approximation controls. An example of application is implemented for the one- and two-dimensional heat equation with Neumann boundary control.

Key words: Controllability, partial differential equation, discretization, observability inequality, Hilbert uniqueness method.
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1 Introduction

Consider an infinite dimensional linear control system

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0, \tag{1}$$

where the state $y(t)$ belongs to a Hilbert space X, the control $u(t)$ belongs to a Hilbert space U, $A : D(A) \to X$ is an operator, and B is a control operator (in general, unbounded) on U. Discretizing this partial differential equation,

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using for instance a finite difference, or a finite element scheme, leads to a family of finite dimensional linear control systems

$$\dot{y}_h(t) = A_h y_h(t) + B_h u_h(t), \quad y_h(0) = y_{0h}, \tag{2}$$

where $y_h(t) \in X_h$ and $u_h(t) \in U_h$, for $0 < h < h_0$.

Let $y_1 \in X$; if the control system (1) is controllable in time T, then there exists a solution $y(\cdot)$ of (1), associated with a control u, such that $y(T) = y_1$. The following question arises naturally: is it possible to find controls u_h, for $0 < h < h_0$, converging to the control u as the mesh size h of the discretization process tends to zero, and such that the associated trajectories $y_h(\cdot)$, solutions of (2), converge to $y(\cdot)$? Moreover, does there exist an efficient algorithmic way to determine the controls u_h?

For controllable linear control systems of the type (1), we have available many methods in order to realize the controllability. A well known method, adapted to numerical implementations, is the Hilbert Uniqueness Method (HUM), introduced in [13], which consists in minimizing a cost function, namely, the L^2 norm of the control. In this paper, we investigate the above question in the case where controllability of (1) is achieved using the HUM method. Our objective is to establish conditions ensuring a *uniform controllability property* of the family of discretized control systems (2), and to establish a computationally feasible approximation method for realizing controllability.

The question of uniform controllability and/or observability of the family of approximation control systems (2) has been investigated by E. Zuazua and collaborators in a series of articles [4,8,12,15,17,25,30–33], for different discretization processes, on different examples. When the observability constant of the finite dimensional approximation systems does not depend on h, one says that the property of *uniform observability* holds. For classical finite difference schemes, a uniform boundary observability property holds for one-dimensional heat equations [15], beam equations [12], Schrödinger equations [33], but does not hold for 1-D wave equations [8]. It is actually well known that discretization processes generate high frequency spurious solutions that do not exist in the continuous problem, and that may lead, in the exact controllability problem, to the divergence of the control approximations.

From the practical point of view, it is well known that, in the parabolic case, this defect of the high frequencies is compensated by the dissipative properties: these higher modes are damped out, and then good convergence properties of the controls can be obtained (see [7]). However, up to our knowledge, except on quite particular cases, no general theoretical result has been established to prove this fact, and it is precisely the aim of this article. Note that the situation is completely different for conservative systems, for which no such mechanism is available to deal with the spurious high frequencies.

The discretization framework used in this paper is in the same spirit as the one of [1,2,6,9,11,14,19]. In these references, approximation results are provided for the linear quadratic regulator (LQR) problem in the parabolic case, that show, in particular, the convergence of the controls of the semidiscrete models to the control of the continuous model. However, in the LQR problem, the final point is not fixed, and the exact controllability problem is a very different matter.

In the present paper, we prove a uniform controllability property of the discretized models (2), in the case where the operator A generates an analytic semigroup. Of course, due to regularization properties, the control system (1) is not exactly controllable in general. Hence, we focus on exact null controllability. Our main result, Theorem 3.1, states that, for an exactly null controllable parabolic system (1), under standard assumptions on the discretization process (that are satisfied for most of classical schemes), if the discretized semigroup is *uniformly analytic* (see [11]), and if the *degree of unboundedness* of the control operator B with respect to A (see [20]) is lower than $1/2$, then the approximating control systems are uniformly controllable, in the following sense: we are able to construct a bounded sequence of approximating controls u_h, that converge to the HUM control of the continuous model. A uniform observability and admissibility type inequality is proved. We stress that we do not prove a uniform exact null controllability property for the approximating systems (2), however the property proved is stronger than uniform approximate controllability. A minimization procedure to compute the approximation controls is provided. Note that the condition on the degree of unboundedness of B is satisfied for distributed controls (that is, if B is bounded), and, if B is unbounded, it is for instance satisfied for the heat equation with Neumann boundary control, but not with Dirichlet boundary control.

The outline of the article is as follows. In Section 2, we briefly review some well known facts on controllability for finite and infinite dimensional linear control systems. The main result is stated in Section 3, and proved in Section 4. An example of application, and numerical simulations, are provided in Section 5, for the one- and two-dimensional heat equation with Neumann boundary control. In Section 6, we formulate some further comments, and open problems. Section 7 is an appendix devoted to the proof of a lemma.

2 A short review on controllability

2.1 *Controllability of finite dimensional linear control systems*

Let $T > 0$ fixed. Consider the linear control system (S) given by $\dot{x}(t) = Ax(t) + Bu(t)$, where $x(t) \in \mathbb{R}^n$, A is a $(n \times n)$-matrix, B is a $(n \times m)$-matrix, with real coefficients, and $u(\cdot) \in L^2(0, T; \mathbb{R}^m)$.

Let $x_0 \in \mathbb{R}^n$. The system (S) is *controllable from x_0 in time T* if, for every $x_1 \in \mathbb{R}^n$, there exists $u(\cdot) \in L^2(0, T; \mathbb{R}^m)$ so that the corresponding solution $x(\cdot)$, with $x(0) = x_0$, satisfies $x(T) = x_1$.

It is well known that the system (S) is controllable in time T if and only if the matrix $\int_0^T \mathrm{e}^{(T-t)A} BB^* \mathrm{e}^{(T-t)A^*} dt$, called *Gramian* of the system, is nonsingular (here, M^* denotes the transpose of the matrix M). In finite dimension, this is equivalent to the existence of $\alpha > 0$ so that $\int_0^T \|B^* \mathrm{e}^{(T-t)A^*} \psi\|^2 dt \geq \alpha \|\psi\|^2$, for every $\psi \in \mathbb{R}^n$ (observability inequality).

It is also well known that, if such a linear system is controllable from x_0 in time $T > 0$, then it is controllable in any time T', and from any initial state $x'_0 \in \mathbb{R}^n$. Indeed, another necessary and sufficient condition for controllability is the Kalman condition $\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n$, which is independent on x_0 and T.

2.2 *Controllability of linear partial differential equations in Hilbert spaces*

In this section, we review some known facts on controllability of infinite dimensional linear control systems in Hilbert spaces (see [28,29,27]).

Throughout the paper, the notation $L(E, F)$ stands for the set of linear continuous mappings from E to F, where E and F are Hilbert spaces.

Let X be a Hilbert space. Denote by $\langle\ ,\ \rangle_X$ the inner product on X, and by $\|\ \|_X$ the associated norm. Let $S(t)$ denote a strongly continuous semigroup on X, of generator $(A, D(A))$. Let X_{-1} denote the completion of X for the norm $\|x\|_{-1} = \|(\beta I - A)^{-1} x\|$, where $\beta \in \rho(A)$ is fixed. Note that X_{-1} does not depend on the specific value of $\beta \in \rho(A)$. The space X_{-1} is isomorphic to $(D(A^*))'$, the dual space of $D(A^*)$ with respect to the pivot space X, and $X \subset X_{-1}$, with a continuous and dense embedding. The semigroup $S(t)$ extends to a semigroup on X_{-1}, still denoted $S(t)$, whose generator is an extension of the operator A, still denoted A. With these notations, A is a linear operator from X to X_{-1}.

Let U be a Hilbert space. Denote by $\langle\ ,\ \rangle_U$ the inner product on U, and by $\|\ \|_U$ the associated norm.

A linear continuous operator $B : U \to X_{-1}$ is *admissible* for the semigroup $S(t)$ if every solution of

$$\dot{y}(t) = Ay(t) + Bu(t), \tag{3}$$

with $y(0) = y_0 \in X$ and $u(\cdot) \in L^2(0, +\infty; U)$, satisfies $y(t) \in X$, for every $t \geq 0$. In this case, $y(\cdot) \in H^1(0, +\infty; X)$, the differential equation (3) on X_{-1} holds almost everywhere on $[0, +\infty)$, and $y(t) = S(t)y_0 + \int_0^t S(t-s)Bu(s)ds$, for every $t \geq 0$.

For $T > 0$, define $L_T : L^2(0, T; U) \to X_{-1}$ by

$$L_T u = \int_0^T S(t-s)Bu(s)ds. \tag{4}$$

A control operator $B \in \mathcal{L}(U, X_{-1})$ is admissible if and only if $\operatorname{Im} L_T \subset X$, for some (and hence for every) $T > 0$. Note that the adjoint L_T^* of L_T satisfies

$$L_T^* y_0(t) = B^* S(T-t)^* y_0 \quad \text{a.e. on } [0, T], \tag{5}$$

for every $y_0 \in D(A^*)$.

Let $B \in \mathcal{L}(U, X_{-1})$ denote an admissible control operator.

For $y_0 \in X$, and $T > 0$, the system (3) is *exactly controllable* from y_0 in time T if, for every $y_1 \in X$, there exists $u(\cdot) \in L^2(0, T; U)$ so that the corresponding solution of (3), with $y(0) = y_0$, satisfies $y(T) = y_1$.

It is clear that the system (3) is exactly controllable from y_0 in time T if and only if L_T is onto, that is $\operatorname{Im} L_T = X$. In particular, if the system (3) is exactly controllable from y_0 in time T, then it is exactly controllable from any point $y_0' \in X$ in time T. One says that the system (3) is exactly controllable in time T. It is well known that the system (1) is exactly controllable in time T if and only if there exists $\alpha > 0$ so that $\int_0^T \|B^* S^*(t)\psi\|_U^2 dt \geq \alpha \|\psi\|_X^2$, for every $\psi \in D(A^*)$ (observability inequality).

For $T > 0$, the system (3) is said to be *exactly null controllable* in time T if, for every $y_0 \in X$, there exists $u(\cdot) \in L^2(0, T; U)$ so that the corresponding solution of (3), with $y(0) = y_0$, satisfies $y(T) = 0$. The system (1) is exactly null controllable in time T if and only if there exists $\alpha > 0$ so that

$$\int_0^T \|B^* S^*(t)\psi\|_U^2 dt \geq \alpha \|S(T)^*\psi\|_X^2, \tag{6}$$

for every $\psi \in D(A^*)$.

Remark 2.1 *Assume that B is admissible and that the control system (3) is exactly null controllable in time T. Let $y_0 \in X$. For every $\psi \in D(A^*)$, set*

$$J(\psi) = \frac{1}{2}\int_0^T \|B^* S(t)^*\psi\|_U^2 dt + \langle S(T)^*\psi, y_0\rangle_X. \tag{7}$$

The functional J is strictly convex, and, from the observability inequality (6), is coercive. Define the control u by

$$u(t) = B^* S(T-t)^*\psi, \tag{8}$$

for every $t \in [0,T]$, and let $y(\cdot)$ be the solution of (3), such that $y(0) = y_0$, associated with the control u. Then, one has $y(T) = 0$, and moreover, u is the control of minimal L^2 norm, among all controls whose associated trajectory satisfies $y(T) = 0$.

This remark proves that observability implies controllability, and gives a constructive way to build the control of minimal L^2 norm (see [32]). This is more or less the contents of the Hilbert Uniqueness Method (see [13]). Hence, in what follows, we refer to the control (8) as the HUM control.

3 The main result

Let X and U be Hilbert spaces, and let $A : D(A) \to X$ be a linear operator, generating a strongly continuous semigroup $S(t)$ on X. Let $B \in L(U, D(A^*)')$ be a control operator. We make the following assumptions.

($\mathbf{H_1}$) *The semigroup $S(t)$ is analytic.*

Therefore (see [18]), there exist positive real numbers C_1 and ω such that

$$\|S(t)y\|_X \leq C_1 e^{\omega t}\|y\|_X, \text{ and } \|AS(t)y\|_X \leq C_1 \frac{e^{\omega t}}{t}\|y\|_X, \tag{9}$$

for all $t > 0$ and $y \in D(A)$, and such that, if we set $\hat{A} = A - \omega I$, then the fractional powers $(-\hat{A})^\theta$ of $\hat{A}$ are well defined, for $\theta \in [0, 1]$, and there holds

$$\|(-\hat{A})^\theta S(t) y\|_X \leq C_1 \frac{\mathrm{e}^{\omega t}}{t^\theta} \|y\|_X, \tag{10}$$

for all $t > 0$ and $y \in D(A)$.

Of course, inequalities (9) hold as well if one replaces A by A^*, $S(t)$ by $S(t)^*$, for $y \in D(A^*)$.

Moreover, if $\rho(A)$ denotes the *resolvent set* of A, then there exists $\delta \in (0, \pi/2)$ such that

$$\rho(A) \supset \Delta_\delta = \{\omega + \rho \mathrm{e}^{i\theta} \mid \rho > 0,\ |\theta| \leq \frac{\pi}{2} + \delta\}. \tag{11}$$

For $\lambda \in \rho(A)$, denote by $R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent of A. It follows from the previous estimates that there exists $C_2 > 0$ such that

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{C_2}{|\lambda - \omega|},\ \|AR(\lambda, A)\|_{L(X)} \leq C_2, \tag{12}$$

for every $\lambda \in \Delta_\delta$, and

$$\|R(\lambda, \hat{A})\|_{L(X)} \leq \frac{C_2}{|\lambda|},\ \|\hat{A}R(\lambda, \hat{A})\|_{L(X)} \leq C_2, \tag{13}$$

for every $\lambda \in \{\Delta_\delta + \omega\}$. Similarly, inequalities (12) and (12) hold as well with A^* and $\hat{A}^*$.

($\mathbf{H_2}$) *The degree of unboundedness of B is lower than $1/2$, i.e., there exists $\gamma \in [0, 1/2)$ such that $B \in L(U, D((-\hat{A}^*)^\gamma)')$.*

In these conditions, the domain of B^* is $D(B^*) = D((-\hat{A}^*)^\gamma)$, and there exists $C_3 > 0$ such that

$$\|B^*\psi\|_U \leq C_3 \|(-\hat{A}^*)^\gamma \psi\|_X, \tag{14}$$

for every $\psi \in D((-\hat{A}^*)^\gamma)$.

Note that this assumption implies that the control operator B is *admissible*.

We next introduce adapted approximation assumptions, inspired by [11] (see also [1,2,6,9,14,19]). Consider two families $(X_h)_{0<h<h_0}$ and $(U_h)_{0<h<h_0}$ of finite dimensional spaces, where h is the discretization parameter.

($\mathbf{H_3}$) *For every $h \in (0, h_0)$, there exist linear mappings $P_h : D((-\hat{A}^*)^{1/2})' \to X_h$ and $\widetilde{P}_h : X_h \to D((-\hat{A}^*)^{1/2})$ (resp., there exist linear mappings $Q_h : U \to U_h$ and $\widetilde{Q}_h : U_h \to U$), satisfying the following requirements:*

($\mathbf{H_{3.1}}$) *For every $h \in (0, h_0)$, there holds*

$$P_h \widetilde{P}_h = id_{X_h}, \text{ and } Q_h \widetilde{Q}_h = id_{U_h}. \tag{15}$$

($\mathbf{H_{3.2}}$) *There exist $s > 0$ and $C_4 > 0$ such that there holds, for every $h \in (0, h_0)$,*

$$\|(I - \widetilde{P}_h P_h)\psi\|_X \leq C_4 h^s \|A^*\psi\|_X, \tag{16}$$

$$\|(-\hat{A}^*)^\gamma (I - \widetilde{P}_h P_h)\psi\|_X \leq C_4 h^{s(1-\gamma)} \|A^*\psi\|_X, \quad \forall \psi \in D(A^*), \tag{17}$$

$$\|(I - \widetilde{Q}_h Q_h)u\|_U \to 0 \text{ as } h \to 0, \quad \forall u \in U, \tag{18}$$

$$\|(I - \widetilde{Q}_h Q_h)B^*\psi\|_U \leq C_4 h^{s(1-\gamma)} \|A^*\psi\|_X, \quad \forall \psi \in D(A^*). \tag{19}$$

Note that (17) makes sense since, by assumption, $\gamma < 1/2$, and thus, $\mathrm{Im}\, \widetilde{P}_h \subset D((-\hat{A}^*)^{1/2}) \subset D((-\hat{A}^*)^\gamma)$.

For every $h \in (0, h_0)$, the vector space X_h (resp. U_h) is endowed with the norm $\| \ \|_{X_h}$ (resp., $\| \ \|_{U_h}$) defined by

$$\|y_h\|_{X_h} = \|\widetilde{P}_h y_h\|_X, \tag{20}$$

for $y_h \in X_h$ (resp., $\|u_h\|_{U_h} = \|\widetilde{Q}_h u_h\|_U$, for $u_h \in U_h$). In these conditions, it is clear that

$$\|\widetilde{P}_h\|_{L(X_h, X)} = \|\widetilde{Q}_h\|_{L(U_h, U)} = 1, \tag{21}$$

for every $h \in (0, h_0)$. Moreover, it follows from (16), (17), (18), and from the Banach-Steinhaus Theorem, that there exists $C_5 > 0$ such that

$$\|P_h\|_{L(X, X_h)} \leq C_5, \text{ and } \|Q_h\|_{L(U, U_h)} \leq C_5, \tag{22}$$

and

$$\|(-\hat{A}^*)^\gamma (I - \widetilde{P}_h P_h)\psi\|_X \leq C_5 \|(-\hat{A}^*)^\gamma \psi\|_X, \tag{23}$$

for all $h \in (0, h_0)$ and $\psi \in D((-\hat{A}^*)^\gamma)$.

($\mathbf{H_{3.3}}$) *For every $h \in (0, h_0)$, there holds*

$$P_h = \widetilde{P}_h^*, \text{ and } Q_h = \widetilde{Q}_h^*, \tag{24}$$

where the adjoint operators are considered with respect to the pivot spaces X, U, X_h, and U_h.

Note that this assumption indeed holds for most of classical schemes (Galerkin or spectral approximations, centered finite differences, ...).

($\mathbf{H_{3.4}}$) *There exists $C_6 > 0$ such that*

$$\|B^* \widetilde{P}_h \psi_h\|_U \le C_6 h^{-\gamma s} \|\psi_h\|_{X_h}, \tag{25}$$

for all $h \in (0, h_0)$ and $\psi_h \in X_h$.

For every $h \in (0, h_0)$, we define the approximation operators $A_h^* : X_h \to X_h$ of A^*, and $B_h^* : X_h \to U_h$ of B^*, by

$$A_h^* = P_h A^* \widetilde{P}_h, \text{ and } B_h^* = Q_h B^* \widetilde{P}_h. \tag{26}$$

Due to $(H_{3.3})$, it is clear that $B_h = P_h B \widetilde{Q}_h$, for every $h \in (0, h_0)$. On the other part, we set $A_h = (A_h^*)^*$ (with respect to the pivot space X_h). Note that, if A is selfadjoint, then $A_h = P_h A \widetilde{P}_h$.

($\mathbf{H_4}$) *The following properties hold:*

($\mathbf{H_{4.1}}$) *The family of operators $\mathrm{e}^{tA_h^*}$ is uniformly analytic, in the sense that there exists $C_7 > 0$ such that*

$$\|\mathrm{e}^{tA_h}\|_{L(X_h)} \le C_7 \mathrm{e}^{\omega t}, \text{ and } \|A_h \mathrm{e}^{tA_h}\|_{L(X_h)} \le C_7 \frac{\mathrm{e}^{\omega t}}{t}, \quad \forall t > 0. \tag{27}$$

Under $(H_{4.1})$, there exists $C_8 > 0$ such that

$$\|R(\lambda, A_h)\|_{L(X_h)} \le \frac{C_8}{|\lambda - \omega|}, \tag{28}$$

for every $\lambda \in \Delta_\delta$. Note that (27) and (28) hold as well if one replaces A_h with A_h^*.

($\mathbf{H_{4.2}}$) *There exists $C_9 > 0$ such that, for every $f \in X$ and every $h \in (0, h_0)$, the respective solutions of $\hat{A}^* \psi = f$ and $\hat{A}_h^* \psi_h = P_h f$ satisfy*

$$\|P_h \psi - \psi_h\|_{X_h} \le C_9 h^s \|f\|_X. \tag{29}$$

In other words, there holds $\|P_h \hat{A}^{*-1} - \hat{A}_h^{*-1} P_h\|_{L(X, X_h)} \le C_9 h^s$. This is a (strong) rate of convergence assumption.

Remark 3.1 *Assumptions (H_3) and $(H_{4.2})$ hold for most of the classical numerical approximation schemes, such as Galerkin methods, spectral methods, centered finite difference schemes, ... As noted in [11], the assumption $(H_{4.1})$ of uniform analyticity is not standard, and has to be checked in each specific case. However, it can be shown to hold, under Assumption (H_1), provided the bilinear form associated with A_h is uniformly coercive (see [3] for the selfadjoint case, and [10, Lemma 4.2] for the general nonselfadjoint case).*

The main result of the paper is the following.

Theorem 3.1 *Under the previous assumptions, the control system $\dot{y} = Ay + Bu$ is exactly null controllable in time $T > 0$, if and only if the family of discretized control systems $\dot{y}_h = A_h y_h + B_h u_h$ is uniformly controllable in the following sense. There exist $\beta > 0$, $h_1 > 0$, and positive real numbers c, c', such that the uniform observability and admissibility inequality*

$$c\|\mathrm{e}^{TA_h^*}\psi_h\|_{X_h}^2 \leq \int_0^T \|B_h^*\mathrm{e}^{tA_h^*}\psi_h\|_{U_h}^2 dt + h^\beta\|\psi_h\|_{X_h}^2 \leq c'\|\psi_h\|_{X_h}^2 \tag{30}$$

holds, for every $h \in (0, h_1)$ and every $\psi_h \in X_h$.

In these conditions, for every $y_0 \in X$, and every $h \in (0, h_1)$, there exists a unique $\psi_h \in X_h$ minimizing the functional

$$J_h(\psi_h) = \frac{1}{2}\int_0^T \|B_h^*\mathrm{e}^{tA_h^*}\psi_h\|_{U_h}^2 dt + \frac{1}{2}h^\beta\|\psi_h\|_{X_h}^2 + \langle \mathrm{e}^{TA_h^*}\psi_h, P_h y_0\rangle_{X_h}, \tag{31}$$

and the sequence of controls $(\widetilde{Q}_h u_h)_{0<h<h_1}$, where u_h is defined by

$$u_h(t) = B_h^*\mathrm{e}^{(T-t)A_h^*}\psi_h, \tag{32}$$

for every $t \in [0, T]$, converges weakly (up to a subsequence), in the space $L^2(0, T; U)$, to a control u such that the solution of

$$\dot{y} = Ay + Bu, \ y(0) = y_0, \tag{33}$$

satisfies $y(T) = 0$. For every $h \in (0, h_1)$, let $y_h(\cdot)$ denote the solution of

$$\dot{y}_h = A_h y_h + B_h u_h, \ y_h(0) = P_h y_0. \tag{34}$$

Then,

- *$y_h(T) = -h^\beta\psi_h$;*
- *the sequence $(\widetilde{P}_h y_h(\cdot))_{0<h<h_1}$ converges weakly (up to a subsequence), in the space $L^2(0, T; X)$, to $y(\cdot)$ on $[0, T]$;*
- *for every $t \in (0, T]$, the sequence $(\widetilde{P}_h y_h(t))_{0<h<h_1}$ converges weakly (up to a subsequence), in the space X, to $y(t)$.*

Furthermore, there holds

$$\int_0^T \|u(t)\|_U^2 dt \leq \frac{1}{c}\|y_0\|_X^2, \tag{35}$$

and there exists $M > 0$ such that, for every $h \in (0, h_1)$,

$$\int_0^T \|u_h(t)\|_{U_h}^2 dt \leq M^2 \|y_0\|_X^2, \quad h^\beta \|\psi_h\|_{X_h}^2 \leq M^2 \|y_0\|_X^2,$$
$$\text{and } \ \|y_h(T)\|_{X_h} \leq M h^{\beta/2} \|y_0\|_X. \tag{36}$$

Remark 3.2 *The left-hand side of (30) is a uniform observability type inequality for the control systems $\dot{y}_h = A_h y_h + B_h u_h$. The right-hand side of that inequality implies that the control operators B_h are uniformly admissible.*

Remark 3.3 *As noticed in the introduction, we stress that this is not a result of uniform exact null controllability for the approximating systems (34). Such a result is indeed wrong in general: for some semi-discretized two-dimensional heat equations, finite difference approximations are not uniformly exactly null controllable (see [33]). However, the result proved here is stronger than the uniform approximate controllability property. Indeed, the sequence of controls u_h defined by (32) converges to a control u realizing the exact null controllability for the continuous model. In contrast, the uniform approximate controllability property means that the sequence of controls u_h converges to a control u steering the system (33) to a prescribed neighborhood of zero only.*

Note that the notion of uniform controllability provided by this result is relevant from the numerical point of view. Indeed, numerically, it is impossible to realize exactly zero.

Remark 3.4 *A similar result holds if the control system $\dot{y} = Ay + Bu$ is exactly controllable in time $T > 0$. However, due to Assumption (H_1), the semigroup $S(t)$ enjoys in general regularity properties. Therefore, the solution $y(\cdot)$ of the control system may belong to a subspace of X, whatever the control u is. For instance, in the case of the heat equation with a Dirichlet or Neumann boundary control, the solution is a smooth function of the state variable x, as soon as $t > 0$, for every control and initial condition $y_0 \in L^2$. Hence, exact controllability does not hold in this case in the space L^2.*

The theorem states that the controls u_h defined by (32) tend to a control u realizing the exact null controllability for (33). On may wonder under which assumptions the control u is the HUM control such that $y(T) = 0$ (see Remark 2.1). The following result provides an answer.

Proposition 3.2 *With the notations of Theorem 3.1, if the sequence of real numbers $\|\psi_h\|_{X_h}$, $0 < h < h_1$, is moreover bounded, then the control u is the unique HUM control such that $y(T) = 0$.*

A sufficient condition on $y_0 \in X$, ensuring the boundedness of the sequence

$(\|\psi_h\|_{X_h})_{0<h<h_1}$, *is the following: there exists* $\eta > 0$ *such that the control system* $\dot{y} = Ay + Bu$ *is exactly null controllable in time* t*, for every* $t \in [T-\eta, T+\eta]$*, and the trajectory* $t \mapsto S(t)y_0$ *in* X*, for* $t \in [T-\eta, T+\eta]$*, is not contained in a hyperplane of* X*.*

An example where this situation indeed occurs is the following. Additionally to the previous assumptions, assume that the operator A admits a Hilbertian basis of eigenvectors e_k, associated with eigenvalues λ_k, for $k \in \mathbb{N}$, satisfying

$$\sum_{k=1}^{+\infty} \frac{-1}{\lambda_k} < +\infty. \tag{37}$$

Let $y_0 = \sum_{k\in\mathbb{N}} y_{0k} e_k$ a point of X such that $y_{0k} \neq 0$, for every $k \in \mathbb{N}$. Then, the assumption of Proposition 3.2 is satisfied. Indeed, if the trajectory $t \mapsto S(t)y_0$ in X, for $t \in [T-\eta, T+\eta]$, were contained in a hyperplane of X, there would exist $\Phi = \sum_{k\in\mathbb{N}} \Phi_k e_k \in X \setminus \{0\}$ so that $\sum_{k\in\mathbb{N}} \mathrm{e}^{\lambda_k t} y_{0k} \Phi_k = 0$, for every $t \in [T-\eta, T+\eta]$. It is well known that the condition (37) implies that the functions $\mathrm{e}^{\lambda_k t}$, $k \in \mathbb{N}$, are independent in L^2. Hence, $y_{0k}\Phi_k = 0$, for every $k \in \mathbb{N}$. This yields a contradiction.

4 Proof of the main results

4.1 Proof of Theorem 3.1

The proof is based on the following approximation lemma, whose proof readily follows that of [11, Lemma 4.3.1 p. 446]. To be self-contained, a proof of this lemma is provided in Appendix.

Lemma 4.1 *There exists* $C_{10} > 0$ *such that, for all* $t \in (0, T]$ *and* $h \in (0, h_0)$*, there holds*

$$\|(\mathrm{e}^{tA_h^*} P_h - P_h S(t)^*)\psi\|_{X_h} \leq C_{10} \frac{h^s}{t} \|\psi\|_X, \ \forall \psi \in D(A^*), \tag{38}$$

$$\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*} \psi_h\|_U \leq \frac{C_{10}}{t^\gamma} \|\psi_h\|_{X_h}, \ \forall \psi_h \in X_h, \tag{39}$$

and moreover, for every $\theta \in [0, 1]$*,*

$$\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*} \psi_h - B^* S(t)^* \widetilde{P}_h \psi_h\|_U \leq \ C_{10} \frac{h^{s(1-\gamma)\theta}}{t^{\theta+(1-\theta)\gamma}} \|\psi_h\|_{X_h}, \ \forall \psi_h \in X_h. \tag{40}$$

Remark 4.1 *It follows from the proof of this lemma that the estimate (38) can be improved into*

$$\|(\mathrm{e}^{tA_h^*}P_h - P_h S(t)^*)\psi\|_{X_h} \leq C_{10}\frac{h^{s\theta}}{t^{\theta}}\|\psi\|_X,\ \forall \psi \in D(A^*). \tag{41}$$

Let us prove that, if the system $\dot{y} = Ay + Bu$ is exactly null controllable, then the uniform inequality (30) holds. Since the degree of unboundedness γ of the control operator B is lower than $1/2$, there exists $\theta \in (0,1)$ such that $0 < \theta + (1-\theta)\gamma < 1/2$.

For all $h \in (0, h_0)$ and $\psi_h \in X_h$, we have

$$\begin{aligned}\int_0^T \|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*}\psi_h\|_U^2 dt = &\int_0^T \|B^*S(t)^*\widetilde{P}_h\psi_h\|_U^2 dt \\ &+ \int_0^T \left(\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*}\psi_h\|_U^2 - \|B^*S(t)^*\widetilde{P}_h\psi_h\|_U^2\right) dt. \end{aligned} \tag{42}$$

Since the control system $\dot{y} = Ay + Bu$ is exactly null controllable in time T, there exists a positive real number $c > 0$ such that

$$\int_0^T \|B^*S(t)^*\widetilde{P}_h\psi_h\|_U^2 dt \geq c\|S(T)^*\widetilde{P}_h\psi_h\|_X^2. \tag{43}$$

Using (16), (21), (9), and the estimate (38),

$$\begin{aligned}&\|S(T)^*\widetilde{P}_h\psi_h - \widetilde{P}_h\mathrm{e}^{TA_h^*}\psi_h\|_X \\ &\leq \|(I - \widetilde{P}_h P_h)S(T)^*\widetilde{P}_h\psi_h\|_X + \|\widetilde{P}_h(P_h S(T)^*\widetilde{P}_h\psi_h - \mathrm{e}^{TA_h^*}\psi_h)\|_X \\ &\leq C_4 h^s\|A^*S(T)^*\widetilde{P}_h\psi_h\|_X + \|P_h S(T)^*\widetilde{P}_h\psi_h - \mathrm{e}^{TA_h^*}\psi_h\|_{X_h} \\ &\leq C_{11}h^s\|\psi_h\|_{X_h},\end{aligned}$$

where $C_{11} > 0$ is independent on h. Hence, using (9) and (27), we get

$$\begin{aligned}&\left|\|S(T)^*\widetilde{P}_h\psi_h\|_X^2 - \|\mathrm{e}^{TA_h^*}\psi_h\|_{X_h}^2\right| \\ &\leq \|S(T)^*\widetilde{P}_h\psi_h - \widetilde{P}_h\mathrm{e}^{TA_h^*}\psi_h\|_X \left(\|S(T)^*\widetilde{P}_h\psi_h\|_X + \mathrm{e}^{TA_h^*}\psi_h\|_{X_h}\right) \\ &\leq C_{12}h^s\|\psi_h\|_{X_h}^2,\end{aligned}$$

where $C_{12} > 0$ is independent on h. Therefore,

$$\|S(T)^*\widetilde{P}_h\psi_h\|_X^2 \geq \|\mathrm{e}^{TA_h^*}\psi_h\|_{X_h}^2 - C_{12}h^s\|\psi_h\|_{X_h}^2. \tag{44}$$

Concerning the second term of the right-hand side of (42), one has, using (40), (39), (14) and (10),

$$\begin{aligned}
&\left| \|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*}\psi_h\|_U^2 - \|B^* S(t)^* \widetilde{P}_h \psi_h\|_U^2 \right| \\
&\leq \|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*}\psi_h - B^* S(t)^* \widetilde{P}_h \psi_h\|_U \left(\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*}\psi_h\|_U + \|B^* S(t)^* \widetilde{P}_h \psi_h\|_U \right) \\
&\leq C_{13} h^{s(1-\gamma)\theta} \frac{1}{t^{\theta+(1-\theta)\gamma+\gamma}} \|\psi_h\|_{X_h}^2,
\end{aligned}$$

where $C_{13} > 0$ is independent on h. Since $\theta + (1-\theta)\gamma + \gamma < 1$, we get, by integration,

$$\left| \int_0^T \left(\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*}\psi_h\|_U^2 - \|B^* S(t)^* \widetilde{P}_h \psi_h\|_U^2 \right) dt \right| \leq C_{14} h^{s(1-\gamma)\theta} \|\psi_h\|_{X_h}^2. \quad (45)$$

If we choose a real number β such that $0 < \beta < s(1-\gamma)\theta$, then the left-hand side of the inequality (30) (*i.e.*, the uniform observability inequality) follows from (42), (44), and (45), for $h \in (0, h_1)$, where $h_1 > 0$ is small enough. The right-hand side of (30), that is, the uniform admissibility inequality, is proved similarly.

For $h \in (0, h_1)$, the functional J_h is strictly convex, and, from (30), is coercive. Hence, it admits a unique minimum at $\psi_h \in X_h$ so that

$$0 = \nabla J_h(\psi_h) = M_h(T)\psi_h + h^\beta \psi_h + \mathrm{e}^{TA_h} P_h y_0,$$

where $M_h(T) = \int_0^T \mathrm{e}^{tA_h} B_h B_h^* \mathrm{e}^{tA_h^*} dt$ is the Gramian of the semidiscrete system. Then, the solution y_h of (34) satisfies

$$y_h(T) = \mathrm{e}^{TA_h} y_h(0) + \int_0^T \mathrm{e}^{TA_h} B_h u_h(t) dt = \mathrm{e}^{TA_h} P_h y_0 + M_h(T)\psi_h = -h^\beta \psi_h.$$

Note that, since $J_h(0) = 0$, there must hold, at the minimum, $J_h(\psi_h) \leq 0$. Hence, using the observability inequality (30) and the Cauchy-Schwarz inequality, one gets

$$c\|\mathrm{e}^{TA_h^*}\psi_h\|_{X_h}^2 \leq \int_0^T \|B_h^* \mathrm{e}^{tA_h^*}\psi_h\|_{U_h}^2 dt + h^\beta \|\psi_h\|_{X_h}^2 \leq \|\mathrm{e}^{TA_h^*}\psi_h\|_{X_h} \|P_h y_0\|_{X_h},$$

and thus,

$$\|\mathrm{e}^{TA_h^*}\psi_h\|_{X_h} \leq \frac{1}{c} \|P_h y_0\|_{X_h}. \quad (46)$$

As a consequence,

$$\int_0^T \|B_h^* \mathrm{e}^{tA_h^*}\psi_h\|_{U_h}^2 dt \leq \frac{1}{c}\|P_h y_0\|_{X_h}^2, \quad \text{and} \quad h^\beta \|\psi_h\|_{X_h}^2 \leq \frac{1}{c}\|P_h y_0\|_{X_h}^2,$$

and the estimates (36) follow.

In particular, the sequence of controls $(\widetilde{Q}_h u_h)_{0<h<h_1}$ is bounded in $L^2(0,T;U)$, and thus, up to a subsequence, converges to a control u. Let $y(\cdot)$ denote the solution of (33), associated with this control u. According to (4), one has

$$y(t) = S(t)y_0 + \int_0^t S(t-s)Bu(s)ds = S(t)y_0 + L_t u,$$

for every $t \in [0,T]$, and, with a similar notation,

$$y_h(t) = \mathrm{e}^{tA_h} P_h y_0 + \int_0^t \mathrm{e}^{(t-s)A_h} B_h u_h(s)ds = \mathrm{e}^{tA_h} P_h y_0 + L_{th} u_h,$$

for every $h \in (0,h_1)$. First of all, using a dual version of (41), and using (16), (9), one has

$$\begin{aligned}
&\|\widetilde{P}_h \mathrm{e}^{tA_h} P_h y_0 - S(t)y_0\|_X \\
&\leq \|\widetilde{P}_h(\mathrm{e}^{tA_h} P_h y_0 - P_h S(t)y_0)\|_X + \|(\widetilde{P}_h P_h - I)S(t)y_0\|_X \\
&\leq C_{10}\frac{h^{s\theta}}{t^\theta}\|S(t)y_0\|_X + C_4 h^s \|S(t)y_0\|_X \\
&\leq \mathrm{Cste}\frac{h^{s\theta}}{t^\theta}\|y_0\|_X.
\end{aligned} \tag{47}$$

On the other part, for every $h \in (0,h_1)$, one gets, from a dual version of (40),

$$\begin{aligned}
\|\widetilde{P}_h L_{th} u_h - L_t \widetilde{Q}_h u_h\|_X &\leq \int_0^t \|(\widetilde{P}_h \mathrm{e}^{(t-s)A_h} B_h - S(t-s)B\widetilde{Q}_h)u_h(s)ds\|_X \\
&\leq \mathrm{Cste}\ h^{s(1-\gamma)\theta}\|u_h\|_{L^2(0,T;U_h)}^2 \\
&\leq \mathrm{Cste}\ h^{s(1-\gamma)\theta}\|y_0\|_X^2.
\end{aligned} \tag{48}$$

Finally, let us prove that $L_t \widetilde{Q}_h u_h$ converges weakly to $L_t u$ in X, for every $t \in [0,T]$. Using (5), one has

$$\langle \psi, L_t \widetilde{Q}_h u_h \rangle_X = \langle L_t^* \psi, \widetilde{Q}_h u_h \rangle_{L^2(0,t;U)} = \int_0^t \langle B^* S(t-s)^* \psi, \widetilde{Q}_h u_h(s) \rangle_U ds,$$

for every $\psi \in X$, and since $\widetilde{Q}_h u_h$ converges weakly to u in U, the scalar product $\langle \psi, L_t \widetilde{Q}_h u_h \rangle_X$ tends, as h tends to zero, to

$$\int_0^t \langle B^* S(t-s)^* \psi, u(s) \rangle_U ds = \langle L_t^* \psi, u \rangle_{L^2(0,t;U)} = \langle \psi, L_t u \rangle_X,$$

and thus, $L_t \widetilde{Q}_h u_h$ converges weakly to $L_t u$ in X, for every $t \in [0, T]$. Now, writing

$$\begin{aligned} \widetilde{P}_h y_h(t) - y(t) = (\widetilde{P}_h \mathrm{e}^{tA_h} P_h y_0 - S(t) y_0) &+ (\widetilde{P}_h L_{th} u_h - L_t \widetilde{Q}_h u_h) \\ &+ L_t(\widetilde{Q}_h u_h - u), \end{aligned}$$

it follows from (47), (48), from the weak convergence of $\widetilde{Q}_h u_h$ to u, and from the continuity of L_t, that, on the one part, the sequence $(\widetilde{P}_h y_h(\cdot))_{0<h<h_1}$ converges weakly in $L^2(0, T; X)$ (up to subsequence) to $y(\cdot)$ on $[0, T]$, and, on the other part, for every $t \in (0, T]$, the sequence $(\widetilde{P}_h y_h(t))_{0<h<h_1}$ converges weakly in X (up to subsequence) to $y(t)$.

In particular, since $y_h(T) = -h^\beta \psi_h \to 0$ as $h \to 0$, it follows that $y(T) = 0$.

To prove the converse statement of Theorem 3.1, it suffices to notice that, if the uniform inequality (30) holds, then, the controls $\widetilde{Q}_h u_h$, where u_h is defined by (32), where ψ_h minimizes J_h, are bounded in L^2, and thus, up to a subsequence, converge to a control u. As previously, one proves that the trajectory $y(\cdot)$, solution of (33), associated with the control u, satisfies $y(T) = 0$.

Theorem 3.1 is proved.

4.2 Proof of Proposition 3.2

If the sequence $(\|\widetilde{P}_h \psi_h\|_X)_{0<h<h_1}$ is bounded, then, up to a subsequence, it converges weakly to an element $\psi \in X$. It follows from the estimate (40) that $u(t) = B^* S(T-t)^* \psi$, for every $t \in [0, T]$. The control u is such that $y(T) = 0$, hence the vector ψ must be solution of $\nabla J(\psi) = 0$, where J is defined by (7) (see Remark 2.1). Since J is strictly convex, ψ is the minimum of J, that is, u is the HUM control such that $y(T) = 0$.

We next prove, by contradiction, that the sufficient condition provided in the statement of the proposition implies that the sequence $(\|\psi_h\|_{X_h})_{0<h<h_1}$ is bounded. If it is not bounded, then, up to subsequence, $\psi_h / \|\psi_h\|_{X_h}$ converges weakly to $\Phi \in X$, as h tends to 0. For every $t \in [T-\eta, T+\eta]$, the control system is exactly null controllable in time t, and thus, from (46), the sequence

$\langle e^{tA_h^*}\psi_h, P_h y_0\rangle_{X_h}$ is bounded, uniformly for $h \in (0, h_1)$. Thus, passing to the limit, one gets $\langle \Phi, S(t)y_0\rangle_X = 0$. This contradicts the fact that the trajectory $t \mapsto S(t)y_0$, $t \in [T-\eta, T+\eta]$, is not contained in a hyperplane of X.

5 Numerical simulations for the heat equation with Neumann boundary control

In this section, we give an example of a situation where Theorem 3.1 applies, provide some numerical simulations, and comment on their practical implementation.

Let $d \geq 1$ be an integer, c a real number, Ω an open bounded connected subset of $\mathbb{R}^d$, and $\Gamma = \partial\Omega$. Let Γ_1 and Γ_2 be subsets of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Consider the Neumann boundary control system

$$\begin{aligned} \frac{\partial y}{\partial t} &= \triangle y + cy && \text{in } (0,T)\times\Omega, \\ y(0,\cdot) &= y_0(\cdot) && \text{in } \Omega, \\ \frac{\partial y}{\partial n} &= u && \text{on } [0,T]\times\Gamma_1, \\ \frac{\partial y}{\partial n} &= 0 && \text{on } [0,T]\times\Gamma_2, \end{aligned} \tag{49}$$

where $y_0 \in L^2(\Omega)$ and $u \in L^2(0,T;L^2(\Gamma_1))$.

Set $X = L^2(\Omega)$ and $U = L^2(\Gamma_1)$. It is well known (see [16,21–24]) that the control system (49) is exactly null controllable in X, with controls $u \in L^2(0,T;U)$. It can be written in the form (33), where the selfadjoint operator $A : D(A) \to X$ is defined by $Ay = \triangle y + cy$, on $D(A) = \{y \in H^2(\Omega) \mid \frac{\partial y}{\partial n} = 0 \text{ on } \Gamma_2\}$, and $B = -AN \in L(U, D(A^*)')$, where N is the Neumann mapping, defined by

$$Nu = y \Leftrightarrow \begin{cases} Ay = 0 & \text{in } \Omega, \\ \frac{\partial y}{\partial n} = u & \text{on } \Gamma_1, \\ \frac{\partial y}{\partial n} = 0 & \text{on } \Gamma_2. \end{cases}$$

Note that $N : H^s(\Gamma_1) \longrightarrow H^{s+3/2}(\Gamma_1)$ is continuous, for every $s \in \mathbb{R}$. Here, we assume that $-c$ is not an eigenvalue of the Laplacian operator $\triangle$ on $D(A)$. The adjoint $B^* \in L(D(A^*), U)$ of B is given by $B^*\psi = \psi_{\Gamma_1}$, for every $\psi \in D(A^*)$. Morever, the degree of unboundedness of B is $\gamma = 1/4 + \varepsilon$, for every $\varepsilon > 0$. Hence, assumptions (H_1) and (H_2) are satisfied. Note that $D((-A)^{1/2}) = H^1(\Omega)$.

We next introduce a semi-discretized model of the system (49), using finite elements of order one. Consider a family of simplex meshes $(\mathcal{T}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$, with $\mathcal{T}_n = (K_k)_{k\in\{1,\dots,N_n\}}$, where, for every $k \in \{1,...,N_n\}$, K_k is an open d-simplex such that, for every $l \in \{1,...,N_n\}\backslash\{k\}$, there holds $K_k \cap K_l = \emptyset$ and $\bigcup_{k=1}^{N_n} \overline{K}_k = \overline{\Omega}$. Let ∂I_n be the set of indexes such that, for every $k \in \partial I_n$, there holds $\overline{K}_k \cap \Gamma_1 \neq \emptyset$. Let S_n be the number of distinct vertices $(p_k)_{k\in\{1,\dots,S_n\}}$ of $\mathcal{T}_n$, and let ∂S_n be the set of indexes of the distinct vertices of $(K_k \cap \Gamma_1)_{k\in\partial I_n}$. Let $h_n = \max\limits_{k\in\{1,\dots,N_n\}} \mathrm{diam}(K_k)$, where $\mathrm{diam}(K_k)$ denotes the diameter of K_k. Assume that $(h_n)_{n\in\mathbb{N}}$ is decreasing. For the sake of clarity, the index n is replaced by h in the following notations. Set

$$
\begin{aligned}
X_h &= \{y \in C^0(\Omega) \mid \forall k \in \{1,...,N_h\},\ y_{|K_k} \text{ is linear}\},\\
U_h &= \{u \in C^0(\Gamma_1) \mid \forall k \in \partial I_h,\ y_{|K_k\cap\Gamma_1} \text{ is linear}\}.
\end{aligned}
$$

The spaces X_h and U_h are respectively generated by $\Phi_h = (\varphi_k)_{k\in\{1,\dots,S_h\}}$ and $\Upsilon_h = (\upsilon_k)_{k\in\partial S_h}$, with

$$
\begin{aligned}
\forall k \in \{1,...,S_h\},\ \varphi_k(p_k) = 1,\ \forall l \in \{1,...,S_h\}\backslash\{k\},\ \varphi_k(p_l) = 0,\\
\forall k \in \partial S_h,\ \upsilon_k(p_k) = 1, \qquad \forall l \in \partial S_h\backslash\{k\},\ \upsilon_k(p_l) = 0.
\end{aligned}
$$

Note that $X_h \subset D((-A)^{1/2}) = H^1(\Omega)$ and $U_h \subset U$. Define $\widetilde{P}_h$ (resp., $\widetilde{Q}_h$), as the canonical injection from X_h into $D((-A)^{1/2})$ (resp., from U_h into U). For all $x_h, y_h \in X_h$ and $u_h, v_h \in U_h$, set, according to (20),

$$\langle x_h, y_h\rangle_{X_h} = \langle \widetilde{P}_h x_h, \widetilde{P}_h y_h\rangle_X, \text{ and } \langle u_h, v_h\rangle_{U_h} = \langle \widetilde{Q}_h u_h, \widetilde{Q}_h v_h\rangle_U.$$

For every $y \in D((-A)^{1/2})' = H^1(\Omega)'$ (with respect to the pivot space $X = L^2(\Omega)$), set

$$P_h y = (M_h^{-1}\ \langle y, \widetilde{P}_h\Phi_h\rangle_{H^1(\Omega)',H^1(\Omega)}).\Phi_h,$$

and, for every $u \in U$, set

$$Q_h u = (M_{\partial,h}^{-1}\ \langle u, \widetilde{Q}_h\Upsilon_h\rangle_U).\Upsilon_h,$$

where $M_h = \langle \Phi_h, \Phi_h^T\rangle_{X_h}$, and $M_{\partial,h} = \langle \Upsilon_h, \Upsilon_h^T\rangle_{U_h}$, are mass matrices.

Assumptions $(H_{3.1})$ and $(H_{3.3})$ are obviously satisfied; the assumption $(H_{3.2})$, with $s = 2$, follows from the classical finite element theory (see [5]), and the assumption $(H_{3.4})$ follows from a standard approximation property (see [26]).

The variational version of (49) is the following. For $u \in C^1([0,T], U)$, one has to determine y in $C^1([0,T],X)$ such that

$$\langle y_t, w\rangle_X = -\langle \nabla u, \nabla w\rangle_X + c\langle y, w\rangle_X - \langle u, w_{|\Gamma_1}\rangle_U,$$

for every $w \in X$. We next derive a similar formulation in the approximating spaces X_h and U_h, and, more precisely, in their respective representations in $\mathbb{R}^{S_h}$ and $\mathbb{R}^{\#\partial S_h}$. For $V \in C^1([0,T],\mathbb{R}^{\#\partial S_h})$, one has to determine $Y \in C^1([0,T],\mathbb{R}^{S_h})$ such that

$$\langle \dot{Y}.\Phi_h, W.\Phi_h\rangle_{X_h} = -\langle \nabla Y.\Phi_h, \nabla W.\Phi_h\rangle_{X_h} + c\langle Y.\Phi_h, W.\Phi_h\rangle_{X_h} - \langle U.\Upsilon_h, W.\Phi_{h|\Gamma_1}\rangle_{U_h},$$

for every $W \in \mathbb{R}^{S_h}$. Hence, the finite element semi-discretization model of (49) writes

$$M_h\dot{Y}(t) = A_hY(t) + B_hV(t), \quad Y(0) = Y_0, \tag{50}$$

where $Y_0 \in \mathbb{R}^{S_h}$, $V(t) \in \mathbb{R}^{\#\partial S_h}$, $Y(t) \in \mathbb{R}^{S_h}$, and

$$A_h = -\langle \nabla\Phi_h, \nabla\Phi_h^T\rangle_{X_h} + cM_h, \;\; B_h = -\langle \Upsilon_h, \Phi_{h|\Gamma_1}^T\rangle_{U_h}.$$

Remark 5.1 *For implementation issues, this approximation model is not written in the abstract spaces X_h and U_h, but rather in $\mathbb{R}^{S_h}$ and $\mathbb{R}^{\#\partial S_h}$. This does not alter the uniformity of the observability and admissibility inequality (30). Indeed, the mappings*

$$\begin{aligned} \iota_h : X_h &\longrightarrow \mathbb{R}^{S_h} \\ y_h &\longmapsto M_h^{-1}\langle y_h, \Phi_h\rangle_X, \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \iota_{\partial,h} : U_h &\longrightarrow \mathbb{R}^{\#\partial S_h} \\ u_h &\longmapsto M_{\partial,h}^{-1}\langle u_h, \Upsilon_h\rangle_U, \end{aligned}$$

are isomorphisms, such that there exist $m > 0$ and $M > 0$ so that

$$m\|y_h\|_{X_h} \leq \|\iota_h(y_h)\|_{\mathbb{R}^{S_h}} \leq M\|y_h\|_{X_h}, \; m\|u_h\|_{U_h} \leq \|\iota_{\partial,h}(u_h)\|_{\mathbb{R}^{\#\partial S_h}} \leq M\|u_h\|_{U_h},$$

for every $y_h \in X_h$, and every $u_h \in U_h$. This follows from a standard property of the spectrum of mass matrices (see [5]).

The uniform analyticity assumption $(H_{4.1})$ follows from [3]. The assumption $(H_{4.2})$ is satisfied with $s = 2$ (see [5,11,26]). Hence, Theorem 3.1 applies to our situation, with $\beta = 0.45$ for instance.

5.2 Numerical simulations

The minimization procedure described in Theorem 3.1 has been implemented for $d = 1$ and $d = 2$, using a standard gradient type method, that has the advantage not to require the computation of the gradient of the functional.

Indeed, this computation is expensive, since the gradient is related to the Gramian matrix. In the following numerical simulations, provided using *Matlab*, we choose $c = 1$. Then, the operator A has a positive eigenvalue, and the uncontrolled system (49) (i.e., $u \equiv 0$) is unstable.

5.2.1 The one-dimensional heat equation

Set $\Omega = (0, 1)$, $\Gamma_1 = \{1\}$, $\Gamma_2 = \{0\}$, $c = 1$, and $T = 1$. With the previous notations, consider the subdivision of Ω $p_k = (k-1)h$, for $k \in \{1, ..., S_h\}$ and $h = 1/(S_h - 1)$. Numerical simulations are lead with a time discretization step equal to 0.001, with the following data.

name	S_h	h	y_0
1D_10	11	10^{-1}	$y_0(x) = x$
1D_100	101	10^{-2}	$y_0(x) = x$
1D_1000	1001	10^{-3}	$y_0(x) = x$
sin1D_10	11	10^{-1}	$y_0(x) = \sin^2(\pi x)$
sin1D_100	101	10^{-2}	$y_0(x) = \sin^2(\pi x)$
sin1D_1000	1001	10^{-3}	$y_0(x) = \sin^2(\pi x)$

The numerical results are the following.

name	$\|\psi_h\|_X$	h^β	$\|h^\beta \psi_h + y_h(T)\|_X$	$\|y_h(T)\|_{X_h}$	$\|y_h^u(T)\|_{X_h}$
1D_10	0.41380476	0.3720411	0.0030567	0.1567023	1.3577812
1D_100	0.53185422	0.1264632	0.0036999	0.0700055	1.3577812
1D_1000	0.70536645	0.0446885	0.0039907	0.0339719	1.3577812
sin1D_10	0.41383134	0.3720411	0.0030884	0.1567131	1.3577812
sin1D_100	0.53222526	0.1264632	0.0040701	0.0700721	1.3577812
sin1D_1000	0.70565021	0.0446885	0.0039909	0.0339841	1.3577812

The notation $y_h^u(T)$ stands for the extremity at time T of the solution of (49), for $u_h \equiv 0$.

The convergence of the method is slow. For an exact time integration, the final state $y_h(T)$ is equal to $-h^\beta \psi_h$. This is however in accordance with the estimates (36) of Theorem 3.1. Indeed, it follows from these estimates that $y_h(T)$ converges very slowly to zero (here, $\beta/2 = 0.225$). These results illustrate the difficulty in using the HUM method to compute controls. In our case, to divide $\|y_h(T)\|_{X_h}$ by 10, one has to divide h by 30000. On the other part, three days of computations are required, on a bi-processor (Xeon 2.40 GHz, 512 Go RAM), to compute controls for $h = 10^{-3}$. Nevertheless, the convergence of the gradient procedure is fast and does not seem to depend on the size of the space discretization (6 to 8 iterations, in our numerical simulations). Hence, optimizing the computation of J_h may improve the performance of the algorithm and decrease computation times.

5.2.2 *The two-dimensional heat equation*

Let Ω be the unit disk of $\mathbb{R}^2$, let $c = 1$, and $T = 1$. Numerical simulations are lead with a time discretization step equal to 0.001, with the following data.

name	S_h	y_0	Γ_1	Γ_2
`disk_1`	55	$y_0(x,y) = x + y$	Γ	$\emptyset$
	104	$y_0(x,y) = x + y$	Γ	$\emptyset$
`disk_2`	55	$y_0(x,y) = x + y$	$\{(x,y) \in \Gamma \mid x \geq 0 \text{ and } y \geq 0\}$	$\Gamma \backslash \Gamma_1$
	104	$y_0(x,y) = x + y$	$\{(x,y) \in \Gamma \mid x \geq 0 \text{ and } y \geq 0\}$	$\Gamma \backslash \Gamma_1$

The numerical results are the following.

name	S_h	$\|\psi_h\|_X$	h^β	$\|h^\beta \psi_h + y_h(T)\|_X$	$\|y_h(T)\|_{X_h}$	$\|y_h^u(T)\|_{X_h}$
`disk_1`	55	0.1587292	0.6142423	0.0174738	0.0803068	0.1035126
	104	0.2309178	0.4511246	0.0195391	0.0849494	0.1085287
`disk_2`	55	0.1649749	0.6142423	0.0116253	0.0962398	0.1035126
	104	0.1751393	0.4511246	0.0106206	0.1025904	0.1086115

The bench `disk_1` gives slightly better results than the bench `disk_2`, due to the fact that the control is located on one part of the boundary only. The simulations provided here involve 55 and 104 cells (see Figure 1). As previously, the times of computations are very long, and, to divide $\|y_h(T)\|_{X_h}$ by 10, one has to divide the cell diameters by 30000, that is, one has to multiply the number of cells by 9.10^8; this is clearly not feasible on a standard machine, within reasonable time, and indicates the limits of the method.

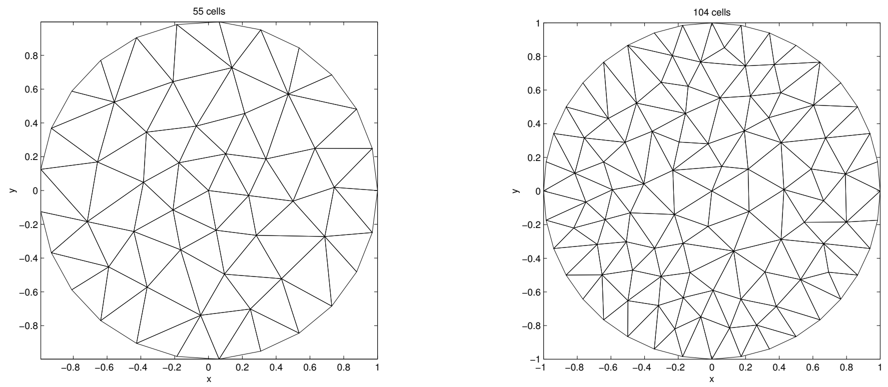

Fig. 1. Meshes used for `disk` benches.

6 Conclusion and open problems

We have shown that, under standard assumptions on the discretization process, for an exactly null controllable linear control system, if the semigroup of the approximating system is uniformly analytic, and if the degree of unboundedness of the control operator is lower than 1/2, then the semidiscrete approximation models are uniformly controllable, in a certain sense. This property is weaker than the uniform exact null controllability (that does not hold in general), but it is however stronger than uniform approximate controllability. A uniform observability and admissibility type inequality was proved, and moreover, a minimization procedure was provided to build the approximation controls. This was implemented in the case of the one- and two-dimensional heat equation with Neumann boundary control.

No rates of convergence were given for the convergence of controls of the semidiscrete models, and this is an open problem.

The condition on the degree of unboundedness γ of the control operator B is very stringent, and an interesting open problem is to investigate whether the results of this article still hold whenever $\gamma \geq 1/2$. Note that, if $\gamma < 1/2$, then B is automatically admissible; this does not hold necessarily whenever $\gamma \geq 1/2$, and may cause some technical difficulties. However, there are many important and relevant problems for which $\gamma \geq 1/2$, that are not covered by the results of this paper, such as, for instance, the heat equation with Dirichlet boundary control. Note that, in this case, the finite difference semidiscrete models are uniformly exactly null controllable in the one dimensional case (see [15]).

Another open and challenging question, much more difficult, is to remove the assumption of uniform analyticity of the discretized semigroup. The properties of analyticity of the semigroup have been used in an essential way in the proof of the results, and it is not clear how to adapt the results in the case of hyperbolic equations. For instance, in the case of the one dimensional wave equation, a result of uniform controllability was proved when using a mixed finite element discretization process (see [4]); the extension to higher dimensions is not clear (see [33]). However, a general result, stating uniform stabilization properties, was derived in [19] for general hyperbolic systems, and it would be interesting to try to adapt the techniques of proof used in this paper to the problem of exact controllability.

Finally, the question of uniform controllability of semidiscrete approximations of controlled partial differential equations is completely open in semilinear (more generally, nonlinear) case. It seems reasonable to investigate, in a first step, whether similar results hold in the case of globally Lipschitzian nonlinearities. Indeed, using fixed point arguments combined with the HUM method (see for instance [31]), it should be possible to reduce the study of the controllability to the linear case.

7 Appendix: proof of Lemma 4.1

We first prove (38). Using (H_1), there holds, for every $t>0$ (see [18]), $S(t)^* = \frac{1}{2i\pi}\int_{\partial\Delta_\delta} \mathrm{e}^{\lambda t} R(\lambda, A^*) d\lambda$, and, similarly, $\mathrm{e}^{tA_h^*} = \frac{1}{2i\pi}\int_{\partial\Delta_\delta} \mathrm{e}^{\lambda t} R(\lambda, A_h^*) d\lambda$. It follows that

$$\mathrm{e}^{tA_h^*} P_h - P_h S(t)^* = \frac{1}{2i\pi}\int_{\partial\Delta_\delta} \mathrm{e}^{\lambda t}\left(R(\lambda, A^*)P_h - P_h R(\lambda, A_h^*)\right) d\lambda. \tag{51}$$

From the resolvent identity $R(\beta_1, A) = R(\beta_2, A) + (\beta_2 - \beta_1) R(\beta_1, A) R(\beta_2, A)$, one gets

$$R(\lambda, A_h^*)P_h = R(\omega, A_h^*)P_h + (\omega-\lambda) R(\lambda, A_h^*) R(\omega, A_h^*) P_h, \tag{52}$$

$$P_h R(\lambda, A^*) = P_h R(\omega, A^*) + (\omega-\lambda) P_h R(\lambda, A^*) R(\omega, A^*). \tag{53}$$

It follows from (52) and (53) that

$$\begin{aligned}&(R(\lambda, A_h^*)P_h - P_h R(\lambda, A^*))(I - (\lambda-\omega)\hat{A}^{*-1}))\\ &= (I + (\omega-\lambda) R(\lambda, A_h^*))(\hat{A}_h^{*-1})P_h - P_h \hat{A}^{*-1}).\end{aligned} \tag{54}$$

Note that $I - (\lambda-\omega)\hat{A}^{*-1} = (\hat{A}^* - (\lambda-\omega)I)\hat{A}^{*-1}$, and thus, $(I - (\lambda-\omega)\hat{A}^{*-1})^{-1} = -\hat{A}R(\lambda-\omega, \hat{A})$. Using (13), one gets

$$\|(I - (\lambda-\omega)\hat{A}^{*-1})^{-1}\|_{L(X)} \leq C_2, \tag{55}$$

for every $\lambda \in \Delta_\delta$. On the other part, from (28),

$$\|(\omega - \lambda)R(\lambda, A_h^*)\|_{L(X_h)} \leq C_8. \tag{56}$$

We deduce from (54), (55), and (56), that there exists a constant C_{15} such that

$$\|R(\lambda, A_h^*)P_h - P_h R(\lambda, A^*)\|_{L(X,X_h)} \leq C_{15}\|\hat{A}_h^{*-1}P_h - P_h\hat{A}^{-1}\|_{L(X,X_h)}, \tag{57}$$

for every $\lambda \in \Delta_\delta$. The estimate (38) follows from (51), (57), and from the estimate (29) of $(H_{4.2})$.

We next prove (39). For every $\psi \in D(A^*)$, one has

$$\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*} P_h\psi - B^*\widetilde{P}_h P_h S(t)^*\psi\|_U \leq \|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*} P_h\psi\|_U + \|B^*\widetilde{P}_h P_h S(t)^*\psi\|_U. \tag{58}$$

We estimate each term of the right-hand side of (58). From (26), $B_h^* = Q_h B^* \widetilde{P}_h$, and thus, using (21), (22), (25), and (27), one gets

$$\begin{aligned}\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*} P_h\psi\|_U &\leq C_5\|B^*\widetilde{P}_h \mathrm{e}^{tA_h^*} P_h\psi\|_U \\ &\leq C_5 C_6 h^{-\gamma s}\|\mathrm{e}^{tA_h^*} P_h\psi\|_{X_h} \\ &\leq C_5^2 C_6 C_7 h^{-\gamma s}\mathrm{e}^{\omega t}\|\psi\|_X.\end{aligned} \tag{59}$$

On the other part, from (25), (22), and (9),

$$\begin{aligned}\|B^*\widetilde{P}_h P_h S(t)^*\psi\|_U &\leq C_6 h^{-\gamma s}\|P_h S(t)^*\psi\|_{X_h} \\ &\leq C_5 C_6 h^{-\gamma s}\|S(t)^*\psi\|_X \\ &\leq C_1 C_5 C_6 h^{-\gamma s}\mathrm{e}^{\omega t}\|\psi\|_X.\end{aligned} \tag{60}$$

Hence, using (58), (59), and (60), there exists $C_{16} > 0$ such that

$$\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*} P_h\psi - B^*\widetilde{P}_h P_h S(t)^*\psi\|_U \leq C_{16} h^{-\gamma s}\|\psi\|_X, \tag{61}$$

for every $\psi \in D(A^*)$, every $t \in [0, T]$, and every $h \in (0, h_0)$. Let us get another estimate of this term. Using successively (22), (25), (14), (19), (38), (17), and (9), one gets

$$\begin{aligned}&\|\widetilde{Q}_h B_h^* \mathrm{e}^{tA_h^*} P_h\psi - B^*\widetilde{P}_h P_h S(t)^*\psi\|_U \\ &= \|\widetilde{Q}_h Q_h B^*\widetilde{P}_h \mathrm{e}^{tA_h^*} P_h\psi - B^*\widetilde{P}_h P_h S(t)^*\psi\|_U \\ &\leq \|\widetilde{Q}_h Q_h B^*\widetilde{P}_h(\mathrm{e}^{tA_h^*} P_h\psi - P_h S(t)^*\psi)\|_U + \|\widetilde{Q}_h Q_h B^*(\widetilde{P}_h P_h - I)S(t)^*\psi\|_U\end{aligned}$$

$$
\begin{aligned}
&\quad +\|(\widetilde{Q}_hQ_h - I)B^*S(t)^*\psi\|_U + \|B^*(I - \widetilde{P}_hP_h)S(t)^*\psi\|_U \\
&\leq C_5C_6h^{-\gamma s}\|\mathrm{e}^{tA_h^*}P_h\psi - P_hS(t)^*\psi\|_{X_h} + C_5C_3\|(-\hat{A})^\gamma(\widetilde{P}_hP_h - I)S(t)^*\psi\|_X \\
&\quad +C_4h^{s(1-\gamma)}\|A^*S(t)^*\psi\|_X + C_3\|(-\hat{A})^\gamma(\widetilde{P}_hP_h - I)S(t)^*\psi\|_X \\
&\leq C_5C_6C_{10}\frac{h^{s(1-\gamma)}}{t}\|\psi\|_X + (C_3(C_5+1)+1)C_4h^{s(1-\gamma)}\|A^*S(t)^*\psi\|_X \\
&\leq C_{17}\frac{h^{s(1-\gamma)}}{t}\|\psi\|_X, \qquad (62)
\end{aligned}
$$

for every $\psi \in D(A^*)$, every $t \in [0,T]$, and every $h \in (0,h_0)$, where $C_{17} > 0$. Then, raising (61) to the power $1-\gamma$, (62) to the power γ, and multiplying both resulting estimates, we obtain $\|\widetilde{Q}_hB_h^*\mathrm{e}^{tA_h^*}P_h\psi - B^*\widetilde{P}_hP_hS(t)^*\psi\|_U \leq \frac{C_{18}}{t^\gamma}\|\psi\|_X$, and hence,

$$
\|\widetilde{Q}_hB_h^*\mathrm{e}^{tA_h^*}P_h\psi\|_U \leq \frac{C_{18}}{t^\gamma}\|\psi\|_X + \|B^*\widetilde{P}_hP_hS(t)^*\psi\|_U. \qquad (63)
$$

From (14), (23), and (10), there holds

$$
\begin{aligned}
\|B^*\widetilde{P}_hP_hS(t)^*\psi\|_U &\leq \|B^*(\widetilde{P}_hP_h - I)S(t)^*\psi\|_U + \|B^*S(t)^*\psi\|_U \\
&\leq C_3\Big(\|(-\hat{A})^\gamma(\widetilde{P}_hP_h - I)S(t)^*\psi\|_X + \|(-\hat{A})^\gamma S(t)^*\psi\|_X\Big) \\
&\leq C_3(C_5+1)\|(-\hat{A})^\gamma S(t)^*\psi\|_X \\
&\leq C_3(C_5+1)C_1\frac{\mathrm{e}^{\omega t}}{t^\gamma}\|\psi\|_X,
\end{aligned}
$$

and thus, using (63), the estimate (39) follows.

Finally, we prove the estimate (40). On the one part, reasoning as above for obtaining the estimate (62), we get

$$
\|\widetilde{Q}_hB_h^*\mathrm{e}^{tA_h^*}P_h\psi - B^*S(t)^*\psi\|_U \leq C_{19}\frac{h^{s(1-\gamma)}}{t}\|\psi\|_X, \qquad (64)
$$

for every $\psi \in D(A^*)$, every $t \in [0,T]$, and every $h \in (0,h_0)$, where C_{19} is a positive constant. On the other part, from (39), (14), and (10),

$$
\begin{aligned}
\|\widetilde{Q}_hB_h^*\mathrm{e}^{tA_h^*}P_h\psi - B^*S(t)^*\psi\|_U &\leq \|\widetilde{Q}_hB_h^*\mathrm{e}^{tA_h^*}P_h\psi\|_U + \|B^*S(t)^*\psi\|_U \\
&\leq \frac{C_{10}}{t^\gamma}\|\psi_h\|_{X_h} + C_3\|(-\hat{A})^\gamma S(t)^*\psi\|_X \\
&\leq \frac{C_{20}}{t^\gamma}\|\psi_h\|_{X_h}, \qquad (65)
\end{aligned}
$$

where $C_{20} > 0$. Raising (64) to the power θ, (65) to the power $1-\theta$, and multiplying both resulting estimates, we obtain (40). Lemma 4.1 is proved.

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