Exactness of SOS relaxations in copositive programming

MSIAM2 Modelling seminar project

Supervisor: Roland Hildebrand

A real symmetric matrix A of size $n \times n$ is called *copositive* if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n_+$. The set of all such matrices forms the *copositive cone* $\mathcal{COP}^n$. This cone is invariant under the action of the multiplicative group $\mathbb{R}^n_{++}$, acting as $A \mapsto DAD$ with $D = \mathrm{diag}(d)$, $d \in \mathbb{R}^n_{++}$. Every matrix $A \in \mathcal{COP}^n$ with positive diagonal can hence be "scaled" to a copositive matrix with unit diagonal by this action.

A conic program over the cone $\mathcal{COP}^n$ is called a *copositive program*. Copositive programs are formally convex, but model a wide variety of non-convex hard optimization problems. Among these are combinatorial problems such as the bandwidth problem [12], graph partitioning [13], computing the stability number [6], clique number [16], and chromatic number [10] of graphs, and the quadratic assignment problem [14]. Copositive formulations have been derived for quadratic programming problems [15, 4, 2] and mixed-integer programs [5]. More applications of copositive programming can be found in the surveys [9, 3].

Verifying copositivity of a given matrix is a co-NP-complete problem [11]. This is not surprising given the extraordinary descriptive power of copositive programs. Therefore much research has been focussed on finding tractable approximations of the copositive cone, in particular, semi-definite approximations.

The commonest approximation of the cone $\mathcal{C}^n$ is that by the sum of the cone $\mathcal{S}^n_+$ of positive semi-definite matrices and the cone $\mathcal{N}^n$ of element-wise nonnegative symmetric matrices. It is a classical result by Diananda [7, Theorem 2] that for $n \leq 4$ the relation $\mathcal{C}^n = \mathcal{S}^n_+ + \mathcal{N}^n$ holds. In general, only the inclusion $\mathcal{S}^n_+ + \mathcal{N}^n \subset \mathcal{COP}^n$ holds. A. Horn showed that for $n \geq 5$ this inclusion is indeed strict [7, p.25]. The approximation by $\mathcal{K}^0_n = \mathcal{S}^n_+ + \mathcal{N}^n$ is the simplest one in a whole family of semi-definite approximations, using the so-called *Parrilo cones* $\mathcal{K}^r_n$, $r \in \mathbb{N}$. With increasing r the relaxation becomes increasingly tighter, but for $n \geq 5$ it fails to be exact for every r.

However, for $n = 5$ a weaker result holds. Namely, every $A \in \mathcal{COP}^5$ with *unit diagonal* is in $\mathcal{K}^1_5$ [8]. Therefore there exists a simple algorithm to check the inclusion $A \in \mathcal{COP}^5$. First, the matrix has to be scaled to a matrix with unit diagonal by the action of the group $\mathbb{R}^5_{++}$, and then the inclusion of the scaled matrix in $\mathcal{K}^1_5$ has to be checked by solving a semi-definite program.

The Parrilo cone $\mathcal{K}^r_n$ is a sums of squares (SOS) approximation. A matrix A belongs to this cone if the homogeneous polynomial

$$\left(\sum_{k=1}^n x_k^2\right)^r \cdot \sum_{k,l=1}^n A_{kl} x_k^2 x_l^2$$

of degree $4 + 2r$ can be represented as a SOS of polynomials of degree $2 + r$.

The goal of the project is to check the exactness of the Parrilo relaxations on unit diagonal copositive matrices in the $n = 6$ case. The convexity structure of the cone $\mathcal{COP}^6$ being explicitly known [1], the task is reduced to checking exactness for 5 different types of special copositive matrices parameterized by a finite number of parameters. It can be split into a computational and a theoretical part.

The computational part consists of randomly generating instances of the special matrices and checking the existence of a SOS decomposition of the corresponding polynomial. This involves implementing a semi-definite program which is equivalent to a SOS relaxation of a concrete polynomial optimization problem.

The theoretical part is invoked in case the relaxation turns out to be exact computationally. Here the existence of the decomposition has to be proven by constructing the factor polynomials as functions of the original parameters. In the course of the investigation the students have the opportunity to become acquainted with the structure of SOS decompositions, in particular Newton polytopes.

SOS decompositions and the corresponding semi-definite relaxations of polynomial optimization problems are part of the "Efficient Methods in Optimization" course.

References

[1] Andrei Afonin, Roland Hildebrand, and Peter J.C. Dickinson. The extreme rays of the 6×6 copositive cone. Optimization Online.

[2] Immanuel Bomze and Etienne de Klerk. Solving standard quadratic optimization problems via linear, semidefinite and copositive programming. *J. Global Optim.*, 24(2):163–185, 2002.

[3] Immanuel M. Bomze. Copositive optimization — recent developments and applications. *European J. Oper. Res.*, 216(3):509–520, 2012.

[4] Immanuel M. Bomze, Mirjam Dür, Etienne de Klerk, Cornelis Roos, Arie J. Quist, and Tamás Terlaky. On copositive programming and standard quadratic optimization problems. *J. Global Optim.*, 18:301–320, 2000.

[5] Samuel Burer. On the copositive representation of binary and continuous nonconvex quadratic programs. *Math. Program., Ser. A*, 120:479–495, 2009.

[6] Etienne de Klerk and Dmitri V. Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM J. Optim.*, 12(4):875–892, 2002.

[7] P. H. Diananda. On nonnegative forms in real variables some or all of which are nonnegative. *Proc. Cambridge Philos. Soc.*, 58:17–25, 1962.

[8] Peter J.C. Dickinson, Mirjam Dür, Luuk Gijben, and Roland Hildebrand. Scaling relationship between the copositive cone and Parrilo's first level approximation. *Optim. Lett.*, 7:1669–1679, 2013.

[9] Mirjam Dür. Copositive programming - a survey. In Moritz Diehl, François Glineur, Elias Jarlebring, and Wim Michiels, editors, *Recent Advances in Optimization and its Applications in Engineering*, pages 3–20. Springer, Berlin, Heidelberg, 2010.

[10] Nebojša Gvozdenović and Monique Laurent. The operator ψ for the chromatic number of a graph. *SIAM J. Optim.*, 19(2):572–591, 2008.

[11] Katta G. Murty and Santosh N. Kabadi. Some NP-complete problems in quadratic and nonlinear programming. *Math. Program.*, 39:117–129, 1987.

[12] Janez Povh. *Application of semidefinite and copositive programming in combinatorial optimization.* PhD thesis, University of Ljubljana, 2006.

[13] Janez Povh and Franz Rendl. A copositive programming approach to graph partitioning. *SIAM J. Optim.*, 18(1):223–241, 2007.

[14] Janez Povh and Franz Rendl. Copositive and semidefinite relaxations of the quadratic assignment problem. *Discrete Optimization*, 6(3):231–241, 2009.

[15] Arie J. Quist, Etienne de Klerk, Cornelis Roos, and Tamás Terlaky. Copositive relaxation for general quadratic programming. *Optim. Methods Softw.*, 9(1–3):185–208, 1998.

[16] Julius Žilinskas and Mirjam Dür. Depth-first simplicial partition for copositivity detection, with an application to MaxClique. *Optim. Methods Softw.*, 26(3):499–510, 2011.