Solutions to exercises

Cones

1. At \((0,0,0)\) any hyperplane defined by a point in the dual cone is supporting. We may take the hyperplane \(\{(x_0, x_1, x_2) \mid x_0 = 0\}\).

At any non-zero point of the boundary \(\partial L_3\) the hyperplane defined by the gradient of the function \(f(x) = x_0^2 - x_1^2 - x_2^2\) will be the unique supporting hyperplane.

At \((1,1,0)\) this yields the plane \(\{(x_0, x_1, x_2) \mid x_0 - x_1 = 0\}\), at \((5,3,4)\) the plane \(\{(x_0, x_1, x_2) \mid 5x_0 - 3x_1 - 4x_2 = 0\}\).

We look for a supporting hyperplane to \(S^3\) at a point \(X^* \in \partial S^3\). Any supporting hyperplane to \(S^3\) is of the form \(\{X \in S(3) \mid \langle A, X \rangle = 0\}\) for some \(A \in S^3_+\), because \(S^3_+\) is self-dual. It is supporting at \(X^*\) if \(\langle A, X^* \rangle = 0\).

At \((1, 0, 0)\) we may hence choose \(A\) of the form
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & a_{22} & a_{23} \\
0 & a_{23} & a_{33}
\end{pmatrix}
\]
with the lower right \(2 \times 2\) sub-block of \(A\) non-zero and positive semi-definite.

At \((1, 1, 0)\) we may choose \(A\) of the form
\[
\begin{pmatrix}
a_{11} & -a_{11} & a_{13} \\
-a_{11} & a_{11} & -a_{13} \\
a_{13} & -a_{13} & a_{33}
\end{pmatrix}
\]
with the lower right \(2 \times 2\) sub-block of \(A\) non-zero and positive semi-definite.

At \((1, 1, 0)\) we may choose \(A\) of the form
\[
\begin{pmatrix}
a_{11} & -a_{11} & 0 \\
-a_{11} & a_{11} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
with \(a_{11} > 0\).

2. The cone \(K\) can be represented as the intersection of the cones
\[
K_1 = \{(x, y, z) \mid x \geq 0\}, \quad K_2 = \{(x, y, z) \mid z \geq \sqrt{x^2 + y^2}\}.
\]
The cone \(K_2\) is isomorphic to a Lorentz cone, and \(K_2^* = K_2\). The cone \(K_1\) is a half-space, and its dual is given by \(K_1^* = \{(x, y, z) \mid x \geq 0, \ y = z = 0\}\).

Let us show that \(K^* = K_1^* + K_2^*\). We have
\[
(K_1^* + K_2^*)^* = \{u \mid \langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle \geq 0 \ \forall \ v_i \in K_i^*\} = \{u \mid \langle u, v_1 \rangle, \ \langle u, v_2 \rangle \geq 0 \ \forall \ v_i \in K_i^*\} = K_1 \cap K_2.
\]
The central equality relation holds because the summands \(\langle u, v_1 \rangle\), \(\langle u, v_2 \rangle\) may be multiplied individually by arbitrarily large positive numbers. Since the cones are closed, the claim follows.

Hence
\[
K^* = \{(x, y, z) \mid z \geq \sqrt{x^2 + y^2} \} + \{(x, y, z) \mid x \geq 0, \ y = z = 0\} = \{(x, y, z) \mid z \geq |y|, \ x \geq -\sqrt{z^2 - y^2}\}.
\]