Case 12

Andrey Afonin, Roland Hildebrand

February 13, 2019

Basic structure

The minimal zero supports are given by $\{1,2\},\{1,3,4\},\{1,3,5\},\{1,4,6\},\{3,5,6\},\{4,5,6\}$. This is equivalent to $\{1,2\},\{2,3,4\},\{3,4,5\},\{4,5,6\},\{5,6,2\},\{6,2,3\}$. There exists the symmetry $(123456) \mapsto (126543)$. The submatrix $A_{\{2,3,4,5,6\}}$ has then a cyclic minimal zero support set and is a T-matrix. We may write a copositive matrix with this minimal zero support set as

$$A = \begin{pmatrix} 1 & -1 & b_1 & b_2 & b_3 & b_4 \\ -1 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & \cos(\phi_4+\phi_5) & -\cos(\phi_5) \\ b_1 & -\cos(\phi_1) & 1 & -\cos(\phi_2) & \cos(\phi_2+\phi_3) & \cos(\phi_5+\phi_1) \\ b_2 & \cos(\phi_1+\phi_2) & -\cos(\phi_2) & 1 & -\cos(\phi_3) & \cos(\phi_3+\phi_4) \\ b_3 & \cos(\phi_4+\phi_5) & \cos(\phi_2+\phi_3) & -\cos(\phi_3) & 1 & -\cos(\phi_4) \\ b_4 & -\cos(\phi_5) & \cos(\phi_5+\phi_1) & \cos(\phi_3+\phi_4) & -\cos(\phi_4) & 1 \end{pmatrix},$$

where $\phi_j \in (0,\pi)$, $j = 1,\dots,5$; $\sum_{i=1}^5 \phi_i \leq \pi$.

The minimal zeros of A are given by

$$(u_1,u_2,u_3,u_4,u_5,u_6) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_2) \\ \sin(\phi_1+\phi_2) \\ \sin(\phi_1) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_3) \\ \sin(\phi_2+\phi_3) \\ \sin(\phi_2) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(\phi_4) \\ \sin(\phi_3+\phi_4) \\ \sin(\phi_3) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_4) \\ 0 \\ 0 \\ \sin(\phi_5) \\ \sin(\phi_4+\phi_5) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_5+\phi_1) \\ \sin(\phi_5) \\ 0 \\ 0 \\ \sin(\phi_1) \end{pmatrix}.$$

First order conditions

Case P: First we consider the special case $\sum_{i=1}^5 \phi_i = \pi$. Then $A_{\{2,3,4,5,6\}}$ is a PSD matrix. We have the condition $Au_1 \geq 0$, which implies

$$A \geq \begin{pmatrix} 1 & -1 & \cos\phi_1 & -\cos(\phi_1+\phi_2) & -\cos(\phi_4+\phi_5) & \cos(\phi_5) \\ -1 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & \cos(\phi_4+\phi_5) & -\cos(\phi_5) \\ \cos\phi_1 & -\cos(\phi_1) & 1 & -\cos(\phi_2) & \cos(\phi_2+\phi_3) & \cos(\phi_5+\phi_1) \\ -\cos(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) & -\cos(\phi_2) & 1 & -\cos(\phi_3) & \cos(\phi_3+\phi_4) \\ -\cos(\phi_4+\phi_5) & \cos(\phi_4+\phi_5) & \cos(\phi_2+\phi_3) & -\cos(\phi_3) & 1 & -\cos(\phi_4) \\ \cos(\phi_5) & -\cos(\phi_5) & \cos(\phi_5+\phi_1) & \cos(\phi_3+\phi_4) & -\cos(\phi_4) & 1 \end{pmatrix}$$

element-wise. But the right-hand side is also a PSD matrix and $A \in \mathcal{S}_+^6 + \mathcal{N}^6$ and it is not exceptional.

We hence assume $\sum_{i=1}^5 \phi_i < \pi$.

The conditions $(Au_i)_j \geq 0$ on the b_i amount to the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \sin(\phi_1+\phi_2) & \sin(\phi_1) & 0 & 0 \\ \sin(\phi_3) & \sin(\phi_2+\phi_3) & \sin(\phi_2) & 0 \\ 0 & \sin(\phi_4) & \sin(\phi_3+\phi_4) & \sin(\phi_3) \\ 0 & 0 & \sin(\phi_5) & \sin(\phi_4+\phi_5) \\ \sin(\phi_5) & 0 & 0 & \sin(\phi_1) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \geq \begin{pmatrix} \cos(\phi_1) \\ -\cos(\phi_1+\phi_2) \\ -\cos(\phi_4+\phi_5) \\ \cos(\phi_5) \\ \sin(\phi_2) \\ 0 \\ 0 \\ \sin(\phi_4) \\ \sin(\phi_5+\phi_1) \end{pmatrix}. \quad (1)$$

The last inequality is a positive combination of inequalities 1 and 4. Likewise the 5th and 8th inequalities are positive combinations of two of the first four. The 6th and 7th are strictly stronger than the corresponding combinations of the first four, because the right-hand side of the combinations becomes $-\sin\phi_2(\cos(\phi_1+\phi_2+\phi_3)+\cos(\phi_4+\phi_5)) < 0$ and $-\sin\phi_4(\cos(\phi_1+\phi_2)+\cos(\phi_3+\phi_4+\phi_5)) < 0$, respectively. Hence if the first three inequalities hold with equality, then the 6th is violated. Likewise, if the 2nd, 3rd and 4th hold with equality, then the 7th is violated.

In order for the vector of the b_i to be in an extremal point there must be 4 linearly independent inequalities which turn into equalities. We may consider only inequalities 1,2,3,4,6,7 because the other three are consequences of these six.

Copositivity of the submatrix $A_{\{1,4,5\}}$ gives

$$\begin{pmatrix} \sin\phi_3 \\ \sin(\phi_1+\phi_2+\phi_3) \\ \sin(\phi_1+\phi_2) \end{pmatrix}^T \begin{pmatrix} 1 & b_2 & b_3 \\ b_2 & 1 & -\cos\phi_3 \\ b_3 & -\cos\phi_3 & 1 \end{pmatrix} \begin{pmatrix} \sin\phi_3 \\ \sin(\phi_1+\phi_2+\phi_3) \\ \sin(\phi_1+\phi_2) \end{pmatrix} = 2\sin\phi_3(b_2\sin(\phi_1+\phi_2+\phi_3)+b_3\sin(\phi_1+\phi_2)+\sin$$
$$\geq 0.$$

Hence $b_2\sin(\phi_1+\phi_2+\phi_3)+b_3\sin(\phi_1+\phi_2)+\sin\phi_3 \geq 0$.

Suppose the inequalities 6 and 7 in (1) are equalities. Then system (1) reduces to the inequalities

$$\begin{pmatrix} -\sin(\phi_2+\phi_3) & -\sin\phi_2 \\ 1 & 0 \\ 0 & 1 \\ -\sin\phi_4 & -\sin(\phi_3+\phi_4) \\ -\sin(\phi_1+\phi_2+\phi_3) & -\sin(\phi_1+\phi_2) \\ -\sin(\phi_4+\phi_5) & -\sin(\phi_3+\phi_4+\phi_5) \end{pmatrix} \begin{pmatrix} b_2 \\ b_3 \end{pmatrix} \geq \begin{pmatrix} \cos\phi_1\sin\phi_3 \\ -\cos(\phi_1+\phi_2) \\ -\cos(\phi_4+\phi_5) \\ \cos\phi_5\sin\phi_3 \\ \sin\phi_3 \\ \sin\phi_3 \end{pmatrix}.$$

The fifth inequality gives $b_2\sin(\phi_1+\phi_2+\phi_3)+b_3\sin(\phi_1+\phi_2)+\sin\phi_3 \leq 0$, and this expression then has to vanish. It follows that $A_{\{1,4,5\}}$ has a zero, and we get additional minimal zeros. Hence at least one of the inequalities 6 and 7 in (1) must be strict.

Among inequalities 2,3,4 in (1) there can only two hold with equality, while among inequalities 6,7 only 1. Hence in order to have 4 equalities the first one needs to be among them. Likewise the 4th inequality must be an equality. We obtain $b_1 = \cos\phi_1$, $b_4 = \cos\phi_5$. The remaining variables obey

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \sin(\phi_2+\phi_3) & \sin\phi_2 \\ \sin\phi_4 & \sin(\phi_3+\phi_4) \end{pmatrix} \begin{pmatrix} b_2 \\ b_3 \end{pmatrix} \geq \begin{pmatrix} -\cos(\phi_1+\phi_2) \\ -\cos(\phi_4+\phi_5) \\ -\cos\phi_1\sin\phi_3 \\ -\cos\phi_5\sin\phi_3 \end{pmatrix},$$

and at least two of these inequalities are equalities. Among them are either the first or the second and either the third or the fourth, which gives 4 possibilities for the values of b_i. Under the symmetry they group into two orbits.

Case 1: 1st and 3rd are equations.
$b_2 = -cos(\phi_1+\phi_2), b_3 = cos(\phi_1+\phi_2+\phi_3)$
2nd and 4th inequalities are fulfilled. In this case we get an additional zero with support $\{1,4,5\}$.
Case 2: 1st and 4th are equations.
$b_2 = -cos(\phi_1+\phi_2)$,
$b_3 = sin(\phi_4)cos(\phi_1+\phi_2) - cos(\phi_5)sin(\phi_3)/sin(\phi_3+\phi_4)$ 2nd inequality is fulfilled. Also $b_3 \leq cos(\phi_1+\phi_2+\phi_3)$
Consider set (1,4,5):
if $b_3 = cos(\phi_1+\phi_2+\phi_3)$ we get additional zero.
if $b_3 < cos(\phi_1+\phi_2+\phi_3)$, $b_3 = cos(a) => a > \phi_1+\phi_2+\phi_3$, but from copositivity of this combination $\pi - a + \phi_1+\phi_2+\phi_3 \geq \pi => \phi_1+\phi_2+\phi_3 \geq a$. It follows, that we don't need consider Case 2.

Result

There aren't any copositive matrices with such set of zeros.