Case 43

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After the permutation (125634) of the indices the minimal zero support set becomes $\{1,2,5\},\{1,2,3\}$, $\{2,3,4\},\{3,4,5\},\{1,4,5\},\{3,4,6\},\{1,2,6\},\{1,4,6\}$. By possibly exchanging the indices 5,6 we may assume that $A_{16} \geq A_{15}$. The principal submatrices $A_{\{12345\}}$ and $A_{\{12346\}}$ are T-matrices, and there exist angles $\phi_1,\dots,\phi_5,\xi_1,\dots,\xi_5 > 0$ such that $\sum_{i=1}^5 \phi_i < \pi$ and

$$A = \begin{pmatrix} 1 & -\cos\phi_4 & \cos(\phi_4+\phi_5) & \cos(\phi_2+\phi_3) & -\cos\phi_3 & -\cos\xi_1 \\ -\cos\phi_4 & 1 & -\cos\phi_5 & \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_4) & -\cos\xi_2 \\ \cos(\phi_4+\phi_5) & -\cos\phi_5 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & -\cos\xi_3 \\ \cos(\phi_2+\phi_3) & \cos(\phi_1+\phi_5) & -\cos\phi_1 & 1 & -\cos\phi_2 & -\cos\xi_4 \\ -\cos\phi_3 & \cos(\phi_3+\phi_4) & \cos(\phi_1+\phi_2) & -\cos\phi_2 & 1 & -\cos\xi_5 \\ -\cos\xi_1 & -\cos\xi_2 & -\cos\xi_3 & -\cos\xi_4 & -\cos\xi_5 & 1 \end{pmatrix}.$$

The zeros with support $\{3,4,6\},\{1,2,6\},\{1,4,6\}$ lead to the conditions

$$\phi_1+\xi_3+\xi_4 = \phi_4+\xi_1+\xi_2 = \pi-\phi_2-\phi_3+\xi_1+\xi_4 = \pi.$$

Resolving with respect to $\xi_1,\dots,\xi_4$, we obtain

$$\xi_1 = \phi_3+\xi,\ \xi_2 = \pi-\phi_3-\phi_4-\xi,\ \xi_3 = \pi-\phi_1-\phi_2+\xi,\ \xi_4 = \phi_2-\xi,$$

where $\xi \in [0,\phi_2)$ is another parameter.

Since A is irreducible with respect to E_{56}, there must exist a minimal zero u with support equal to one of the sets $\{3,4,5\},\{1,2,5\},\{1,4,5\},\{3,4,6\},\{1,2,6\},\{1,4,6\}$ such that $(Au)_5 = (Au)_6 = 0$. Each of the relations leads to the same condition $\cos\xi + \cos\xi_5 = 0$. We hence obtain $\xi_5 = \pi - \xi$. The submatrices $A_{\{1256\}}, A_{\{1456\}}, A_{\{3456\}}$ are then positive semi-definite.

Lemma 0.1. *Let A be a real symmetric $n\times n$ matrix with the following properties. The principal submatrix $A_{\{1,\dots,n-1\}}$ is positive semi-definite. There exists an index set $J \subset \{1,\dots,n\}$ such that $n \in J$ and A_J is positive semi-definite. Moreover, for every index $i \notin J$, there exists a zero $u^i \in \mathbb{R}^n_+$ such that $u^i_n = 0$, $\operatorname{supp} u^i \setminus J = \{i\}$, and such that $(u^i)^T A u^i = 0$, $(Au^i)_n \geq 0$. Then $A \in S_+(n) + \mathcal{N}_n$.*

Proof. Suppose without restriction of generality that there exists a vector $v \in \mathbb{R}^n_+$ with $v_n > 0$ and $\operatorname{supp} v \subset J$ such that $v^T A v = 0$. Otherwise we may subtract a positive number from the element A_{nn} to enforce this condition.

Let B be a partial positive semi-definite matrix whose principal submatrices $B_{\{1,\dots,n-1\}}, B_J$ coincide with those of A and whose other elements are undetermined. Then B has a positive semi-definite completion. This completion is unique and determined by the relation $Bv = 0$. Note that this relation can be resolved with respect to the undetermined elements due to the relation $v_n \neq 0$. We shall denote the positive semi-definite completion also by B.

Let now $i \notin J$. Then we have $(u^i)^T B u^i = (u^i)^T A u^i = 0$, and hence $Bu^i = 0$. We have $0 \leq (Au^i)_n = ((A-B)u^i)_n = (A_{ni} - B_{ni})u^i_i$, which in view of $u^i_i > 0$ yields $A_{ni} \geq B_{ni}$. Hence the difference $A - B$ is a nonnegative matrix, which completes the proof of the lemma. □

Consider now the submatrix A_{23456}. The principal submatrices A_{3456}, A_{234} are positive semi-definite, and there exist zeros $u^{345}, u^{346} \in \mathbb{R}^6_+$ of A with supports $\{345\},\{346\}$, respectively. Since A_{2345}, A_{2346} are copositive, we have $(Au^{345})_2 \geq 0$, $(Au^{346})_2 \geq 0$. Application of the lemma leads to $A_{23456} \in S_+(5) + \mathcal{N}_5$.

Consider now the submatrix A_{12356}. The principal submatrices A_{1256}, A_{123} are positive semi-definite, and there exist zeros $u^{125}, u^{126} \in \mathbb{R}^6_+$ of A with supports $\{125\},\{126\}$, respectively. Since A_{1235}, A_{1236} are copositive, we have $(Au^{125})_3 \geq 0$, $(Au^{126})_3 \geq 0$. Application of the lemma leads to $A_{12356} \in S_+(5) + \mathcal{N}_5$.

Consider now the submatrix A_{13456}. The principal submatrices A_{1456}, A_{345} are positive semi-definite, and there exist zeros $u^{145}, u^{146} \in \mathbb{R}^6_+$ of A with supports $\{145\},\{146\}$, respectively. Since

A_{1345}, A_{1346} are copositive, we have $(Au^{145})_3 \geq 0$, $(Au^{146})_3 \geq 0$. Application of the lemma leads to $A_{13456} \in S_+(5) + \mathcal{N}_5$.

Consider now the submatrix A_{12456}. The principal submatrices A_{1456}, A_{125} are positive semi-definite, and there exist zeros $u^{145}, u^{146} \in \mathbb{R}^6_+$ of A with supports $\{145\}, \{146\}$, respectively. Since A_{1245}, A_{1246} are copositive, we have $(Au^{145})_2 \geq 0$, $(Au^{146})_2 \geq 0$. Application of the lemma leads to $A_{12456} \in S_+(5) + \mathcal{N}_5$.

Hence all principal 5×5 submatrices of A are copositive. If now A is not copositive, then the set $\{u \in \mathbb{R}^n_+ \,|\, u^T Au < 0\}$ must be confined to the interior of $\mathbb{R}^6_+$. This is possible only if A is of signature $+++++-$ with the cones of elements $\{u \in \mathbb{R}^6 \,|\, u^T Au \leq 0\}$ confined to $\pm\mathbb{R}^6_+$. Thus all 5×5 principal submatrices of A must be positive semi-definite. But then A_{12345} is positive semi-definite and $\sum_{i=1}^5 \phi_i = \pi$, a contradiction. Hence $A \in \mathcal{C}_6$.

Finally, we shall show that A is extremal. The zeros of A are given by the columns of the matrix

$$\begin{pmatrix} \sin\phi_5 & 0 & 0 & \sin\phi_2 & \sin(\phi_3+\phi_4) & 0 & \sin(\phi_2-\xi) & \sin(\phi_3+\phi_4+\xi) \\ \sin(\phi_4+\phi_5) & \sin\phi_1 & 0 & 0 & \sin\phi_3 & 0 & 0 & \sin(\phi_3+\xi) \\ \sin\phi_4 & \sin(\phi_1+\phi_5) & \sin\phi_2 & 0 & 0 & \sin(\phi_2-\xi) & 0 & 0 \\ 0 & \sin\phi_5 & \sin(\phi_1+\phi_2) & \sin\phi_3 & 0 & \sin(\phi_1+\phi_2-\xi) & \sin(\phi_3+\xi) & 0 \\ 0 & 0 & \sin\phi_1 & \sin(\phi_2+\phi_3) & \sin\phi_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin\phi_1 & \sin(\phi_2+\phi_3) & \sin\phi_4 \end{pmatrix}.$$

For these zeros u we have $(Au)_J = 0$ for
$J = \{1,2,3\}, \{2,3,4\}, \{3,4,5,6\}, \{1,4,5,6\}, \{1,2,5,6\}, \{3,4,5,6\}, \{1,4,5,6\}, \{1,2,5,6\}$, respectively. Let C be a copositive matrix having these zeros and satisfying this linear system. The first 5 zeros imply that C_{12345} is proportional to the T-matrix A_{12345}. The zeros 1,2,6,7,8 imply that C_{12346} is proportional to the T-matrix A_{12346}. By multiplication of C with a positive constant we achieve that these submatrices of C are equal to the corresponding submatrices of A. Finally, if u is the third zero, then $(Cu)_6 = 0$ implies $C_{56} = A_{56}$. Hence $C = A$ and A is extremal.

Note that if we assume $A_{16} \leq A_{15}$, then we obtain $\xi \in (-\phi_3, 0]$, the rest of the discussion being similar. Hence the variety of extremal exceptional copositive matrices corresponding to this case is given by

$$\begin{pmatrix} 1 & -\cos\phi_4 & \cos(\phi_4+\phi_5) & \cos(\phi_2+\phi_3) & -\cos\phi_3 & -\cos(\phi_3+\xi) \\ -\cos\phi_4 & 1 & -\cos\phi_5 & \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_4) & \cos(\phi_3+\phi_4+\xi) \\ \cos(\phi_4+\phi_5) & -\cos\phi_5 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & \cos(\phi_1+\phi_2-\xi) \\ \cos(\phi_2+\phi_3) & \cos(\phi_1+\phi_5) & -\cos\phi_1 & 1 & -\cos\phi_2 & -\cos(\phi_2-\xi) \\ -\cos\phi_3 & \cos(\phi_3+\phi_4) & \cos(\phi_1+\phi_2) & -\cos\phi_2 & 1 & \cos\xi \\ -\cos(\phi_3+\xi) & \cos(\phi_3+\phi_4+\xi) & \cos(\phi_1+\phi_2-\xi) & -\cos(\phi_2-\xi) & \cos\xi & 1 \end{pmatrix}$$

with $\phi_i > 0$, $\sum_{i=1}^5 \phi_i < \pi$, $\xi \in (-\phi_3, \phi_2)$.