Case 42

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Linear relations

The minimal zero supports are given by (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 6), (2, 4, 6), (3, 5, 6), (4, 5, 6)

$$A = \begin{pmatrix} 1 & -\cos(\phi_2) & -\cos(\phi_1) & \cos(\phi_2+\phi_3) & \cos(\phi_2+\phi_4) & \cos(\phi_1+\phi_5) \\ -\cos(\phi_2) & 1 & \cos(\phi_1+\phi_2) & -\cos(\phi_3) & -\cos(\phi_4) & \cos(\phi_3+\phi_6) \\ -\cos(\phi_1) & \cos(\phi_1+\phi_2) & 1 & b_1 & \cos(\phi_5+\phi_7) & -\cos(\phi_5) \\ \cos(\phi_2+\phi_3) & -\cos(\phi_3) & b_1 & 1 & \cos(\phi_6+\phi_7) & -\cos(\phi_6) \\ \cos(\phi_2+\phi_4) & -\cos(\phi_4) & \cos(\phi_5+\phi_7) & \cos(\phi_6+\phi_7) & 1 & -\cos(\phi_7) \\ \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_6) & -\cos(\phi_5) & -\cos(\phi_6) & -\cos(\phi_7) & 1 \end{pmatrix}$$

The zeros are given by the columns:

$$(u_1,u_2,u_3,u_4,u_5,u_6,u_7) = \begin{pmatrix} \sin(\phi_1+\phi_2) \\ \sin(\phi_1) \\ \sin(\phi_2) \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_3) \\ \sin(\phi_2+\phi_3) \\ 0 \\ \sin(\phi_2) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_4) \\ \sin(\phi_2+\phi_4) \\ 0 \\ 0 \\ \sin(\phi_2) \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_5) \\ 0 \\ \sin(\phi_1+\phi_5) \\ 0 \\ 0 \\ \sin(\phi_1) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_6) \\ 0 \\ \sin(\phi_3+\phi_6) \\ 0 \\ \sin(\phi_3) \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ \sin(\phi_7) \\ 0 \\ \sin(\phi_5) \\ \sin(\phi_5+\phi_7) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(\phi_7) \\ \sin(\phi_6) \\ \sin(\phi_6+\phi_7) \end{pmatrix}$$

1. $$Au_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_1\sin(\phi_2) + \sin(\phi_2)\cos(\phi_1+\phi_2+\phi_3) \\ \sin(\phi_2)\cos(\phi_1+\phi_2+\phi_4) + \sin(\phi_2)\cos(\phi_5+\phi_7) \\ \sin(\phi_1)\cos(\phi_1+\phi_2+\phi_5) + \sin(\phi_1)\cos(\phi_3+\phi_6) \end{pmatrix} \Rightarrow \begin{cases} b_1 \geq -\cos(\phi_1+\phi_2+\phi_3) \\ \cos(\phi_1+\phi_2+\phi_4) \geq -\cos(\phi_5+\phi_7) \\ \cos(\phi_1+\phi_2+\phi_5) \geq -\cos(\phi_3+\phi_6) \end{cases}$$

2. $$Au_2 = \begin{pmatrix} 0 \\ 0 \\ b_1\sin(\phi_2) + \sin(\phi_2)\cos(\phi_1+\phi_2+\phi_3) \\ 0 \\ -\sin(\phi_2)\cos(\phi_3-\phi_4) + \sin(\phi_2)\cos(\phi_6+\phi_7) \\ \sin(\phi_3)\cos(\phi_1+\phi_5) + \sin(\phi_3)\cos(\phi_2+\phi_3+\phi_6) \end{pmatrix} \Rightarrow \begin{cases} b_1 \geq -\cos(\phi_1+\phi_2+\phi_3) \\ \cos(\phi_6+\phi_7) \geq \cos(\phi_3-\phi_4) \\ \cos(\phi_1+\phi_5) \geq -\cos(\phi_2+\phi_3+\phi_6) \end{cases}$$

3. $$Au_3 = \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_2)\cos(\phi_1+\phi_2+\phi_4) + \sin(\phi_2)\cos(\phi_5+\phi_7) \\ -\sin(\phi_2)\cos(\phi_3-\phi_4) + \sin(\phi_2)\cos(\phi_6+\phi_7) \\ 0 \\ \sin(\phi_4)\cos(\phi_1+\phi_5) + \sin(\phi_2+\phi_4)\cos(\phi_3+\phi_6) - \sin(\phi_2)\cos(\phi_7) \end{pmatrix} \Rightarrow \begin{cases} \cos(\phi_1+\phi_2+\phi_4) \geq -\cos(\phi_5+\phi_7) \\ \cos(\phi_6+\phi_7) \geq \cos(\phi_3-\phi_4) \\ \sin(\phi_4)\cos(\phi_1+\phi_5) + \sin(\phi_2+\phi_4)\cos(\phi_3 + \end{cases}$$

4. $$Au_4 = \begin{pmatrix} 0 \\ \sin(\phi_1)\cos(\phi_1+\phi_2+\phi_5) + \sin(\phi_1)\cos(\phi_3+\phi_6) \\ 0 \\ b_1\sin(\phi_1+\phi_5) + \sin(\phi_5)\cos(\phi_2+\phi_3) - \sin(\phi_1)\cos(\phi_6) \\ \sin(\phi_5)\cos(\phi_2+\phi_4) + \sin(\phi_5)\cos(\phi_1+\phi_5+\phi_7) \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \cos(\phi_1+\phi_2+\phi_5) \geq -\cos(\phi_3+\phi_6) \\ b_1\sin(\phi_1+\phi_5) + \sin(\phi_5)\cos(\phi_2+\phi_3) - \sin(\phi_1)\cos(\phi_6 \\ \cos(\phi_2+\phi_4) \geq -\cos(\phi_1+\phi_5+\phi_7) \end{cases}$$

5. $$Au_5 = \begin{pmatrix} \sin(\phi_3)\cos(\phi_1+\phi_5)+\sin(\phi_3)\cos(\phi_2+\phi_3+\phi_6) \\ 0 \\ -\sin(\phi_3)\cos(\phi_5)+\sin(\phi_6)\cos(\phi_1+\phi_2)+b_1\sin(\phi_3+\phi_6) \\ 0 \\ -\sin(\phi_6)\cos(\phi_4)+\sin(\phi_6)\cos(\phi_3+\phi_6+\phi_7) \\ 0 \end{pmatrix} \Rightarrow$$
$$\begin{cases} \cos(\phi_1+\phi_5) \geq -\cos(\phi_2+\phi_3+\phi_6) \\ -\sin(\phi_3)\cos(\phi_5)+\sin(\phi_6)\cos(\phi_1+\phi_2)+b_1\sin(\phi_3+\phi_6) \geq 0 \\ \cos(\phi_3+\phi_6+\phi_7) \geq \cos(\phi_4) \end{cases}$$
6. $$Au_6 = \begin{pmatrix} \sin(\phi_5)\cos(\phi_2+\phi_4)+\sin(\phi_5)\cos(\phi_1+\phi_5+\phi_7) \\ -\sin(\phi_5)\cos(\phi_4)+\sin(\phi_7)\cos(\phi_1+\phi_2)+\sin(\phi_5+\phi_7)\cos(\phi_3+\phi_7) \\ 0 \\ b_1\sin(\phi_7)-\sin(\phi_7)\cos(\phi_5-\phi_6) \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$
$$\begin{cases} \cos(\phi_2+\phi_4) \geq -\cos(\phi_1+\phi_5+\phi_7) \\ -\sin(\phi_5)\cos(\phi_4)+\sin(\phi_7)\cos(\phi_1+\phi_2)+\sin(\phi_5+\phi_7)\cos(\phi_3+\phi_7) \geq 0 \\ b_1 \geq \cos(\phi_5-\phi_6) \end{cases}$$
7. $$Au_7 = \begin{pmatrix} \sin(\phi_6)\cos(\phi_2+\phi_4)+\sin(\phi_7)\cos(\phi_2+\phi_3)+\sin(\phi_6+\phi_7)\cos(\phi_1+\phi_5) \\ -\sin(\phi_6)\cos(\phi_4)+\sin(\phi_6)\cos(\phi_3+\phi_6+\phi_7) \\ b_1\sin(\phi_7)-\sin(\phi_7)\cos(\phi_5-\phi_6) \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$
$$\begin{cases} \sin(\phi_6)\cos(\phi_2+\phi_4)+\sin(\phi_7)\cos(\phi_2+\phi_3)+\sin(\phi_6+\phi_7)\cos(\phi_1+\phi_5) \geq 0 \\ \cos(\phi_3+\phi_6+\phi_7) \geq \cos(\phi_4) \\ b_1 \geq \cos(\phi_5-\phi_6) \end{cases}$$

Consider equation 4.5 and let $\phi_1+\phi_5+\phi_7 \geq \pi \Rightarrow \phi_1+\phi_5+\phi_7-\phi_2-\phi_4 \geq \pi$ - doesn't hold, because of inequality from set (2, 3, 5) in Copositivity section $\Rightarrow$
$\phi_1+\phi_5+\phi_7+\phi_2+\phi_4 \leq \pi$

From inequalities on ϕ_i we get:
$$\begin{cases} \phi_3+\phi_6+\phi_7 \leq \phi_4 \\ \phi_1+\phi_2+\phi_3+\phi_5+\phi_6 \leq \pi \\ \phi_1+\phi_5+\phi_7+\phi_2+\phi_4 \leq \pi \end{cases} \Rightarrow$$

Copositivity:

Sets $(1,3,4), (2,3,4), (1,3,5), (2,3,5), (1,4,5), (2,4,5), (3,4,5), (1,3,4,5), (2,3,4,5), (1,2,6), (2,3,6), (1,4,6), (3,4,6), (1,5,6), (2,5,6)$ have to be considered.

1. $I = (2,5,6), \phi_4+\phi_7 \geq \phi_3+\phi_6$ performs strictly
2. $I = (1,3,4,5), (1,3,5), (1,3,4) : u = e_1+e_3 : \phi_2+\phi_3+|\phi_5-\phi_6| \leq \pi, \phi_2+\phi_4+\phi_5+\phi_7 \leq \pi$ performs strictly
3. $I = (2,3,4,5), (2,3,4), (2,4,5) : u = e_2+e_4 : \phi_1+\phi_2+|\phi_5-\phi_6| \leq \pi, \phi_6+\phi_7 \leq \phi_4$ performs strictly
4. $I = (1,5,6) : \phi_7+\pi-\phi_2-\phi_4+\pi-\phi_1-\phi_5 \geq \pi \Rightarrow \pi \geq \phi_1+\phi_5-\phi_7+\phi_2+\phi_4$ performs strictly
5. $I = (3,4,6) : \pi-|\phi_5-\phi_6|+\phi_5+\phi_6 \geq \pi$ performs strictly
6. $I = (1,4,6) : \pi-\phi_2-\phi_3+\pi-\phi_1-\phi_5+\phi_6 \geq \pi$ performs strictly
7. $I = (2,3,6) : \pi-\phi_1-\phi_2+\pi-\phi_3-\phi_6+\phi_5 \geq \pi$ performs strictly
8. $I = (1,2,6) : \pi-\phi_1-\phi_5+\pi-\phi_3-\phi_6+\phi_2 \geq \pi$ performs strictly

9. $I=(2,3,5): \pi-\phi_1-\phi_2+\pi-\phi_5-\phi_7+\phi_4 \geq \pi$ performs strictly
10. $I=(1,4,5): \pi-\phi_2-\phi_3+\pi-\phi_2-\phi_4-\phi_6-\phi_7 \geq 0$ performs strictly
11. $I=(3,4,5): \pi-\phi_6-\phi_7+\pi-\phi_5-\phi_7-|\phi_6-\phi_7| \geq 0$ performs strictly

Extremality:

$X = FPF^T$

$$X = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & * \\ b_{31} & b_{32} & b_{33} & * & * & b_{36} \\ b_{41} & b_{42} & * & b_{44} & * & * \\ b_{51} & b_{52} & * & * & b_{55} & * \\ b_{61} & * & b_{63} & * & * & b_{66} \end{pmatrix}, F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix},$$

$F * (u_1 u_2 u_3 u_4) = 0$

$$F = \begin{pmatrix} \sin(\phi_2) & 0 \\ 0 & \sin(\phi_2) \\ -\sin(\phi_1+\phi_2) & -\sin(\phi_1) \\ -\sin(\phi_3) & -\sin(\phi_2+\phi_3) \\ -\sin(\phi_4) & -\sin(\phi_2+\phi_4) \\ \sin(\phi_1+\phi_2+\phi_5) & \sin(\phi_1+\phi_5) \end{pmatrix}$$

$Y = GQG^T$

$$Y = \begin{pmatrix} b_{22} & * & b_{24} & * & b_{26} \\ * & b_{33} & * & b_{35} & b_{36} \\ b_{42} & * & b_{44} & b_{45} & b_{46} \\ * & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{pmatrix}, G = \begin{pmatrix} g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{pmatrix},$$

$G * (u_5 u_6 u_7) = 0$

$$G = \begin{pmatrix} -\sin(\phi_3+\phi_6) & -\sin(\phi_3) \\ \sin(\phi_5) & \sin(\phi_5-\phi_6) \\ \sin(\phi_6) & 0 \\ -\sin(\phi_7) & -\sin(\phi_6+\phi_7) \\ 0 & \sin(\phi_6) \end{pmatrix}$$

Necessary equalities, that are constraints on elements of matrices P and Q:

$$\begin{cases} x_{22} = y_{22} \\ x_{24} = y_{24} \\ x_{33} = y_{33} \\ x_{36} = y_{36} \\ x_{44} = y_{44} \\ x_{55} = y_{55} \\ x_{66} = y_{66} \end{cases} \Rightarrow$$

The coefficient matrix of the system is given by $\begin{pmatrix} 0 & 0 & -\sin^2(\phi_2) \\ 0 & \sin(\phi_3)\sin(\phi_2) & \sin(\phi_2+\phi_3)\sin(\phi_2) \\ -\sin^2(\phi_1+\phi_2) & -2\sin(\phi_1+\phi_2)\sin(\phi_1) & -\sin^2(\phi_1) \\ \sin(\phi_1+\phi_2)\sin(\phi_1+\phi_2+\phi_5) & \sin(\phi_1+\phi_2)\sin(\phi_1+\phi_5)+\sin(\phi_1)\sin(\phi_1+\phi_2+\phi_5) & \sin(\phi_1)\sin(\phi_1+\phi_5) \\ -\sin^2(\phi_3) & -2\sin(\phi_2+\phi_3)\sin(\phi_3) & -\sin^2(\phi_2+\phi_3) \\ -\sin^2(\phi_4) & -2\sin(\phi_2+\phi_4)\sin(\phi_4) & -\sin^2(\phi_2+\phi_4) \\ \sin^2(\phi_1+\phi_2+\phi_5) & 2\sin(\phi_1+\phi_2+\phi_5)\sin(\phi_1+\phi_5) & \sin^2(\phi_1+\phi_5) \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \\ -\sin^2(\phi_1+\phi_2) \\ \sin(\phi_1+\phi_2)\sin(\phi_1+\phi_2+\phi_5) \\ -\sin^2(\phi_3) \\ -\sin^2(\phi_4) \\ \sin^2(\phi_1+\phi_2+\phi_5) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_3)\sin(\phi_2) \\ -2\sin(\phi_1+\phi_2)\sin(\phi_1) \\ \sin(\phi_1+\phi_2)\sin(\phi_1+\phi_5)+\sin(\phi_1)\sin(\phi_1+\phi_2+\phi_5) \\ -2\sin(\phi_2+\phi_3)\sin(\phi_3) \\ -2\sin(\phi_2+\phi_4)\sin(\phi_4) \\ 2\sin(\phi_1+\phi_2+\phi_5)\sin(\phi_1+\phi_5) \end{pmatrix}$$

$$\begin{pmatrix} -\sin^2(\phi_2) & \sin^2(\phi_3+\phi_6) & 2\sin(\phi_3+\phi_6)\sin(\phi_3) & \sin^2(\phi_3) \\ \sin(\phi_2+\phi_3)\sin(\phi_2) & -\sin(\phi_3+\phi_6)\sin(\phi_6) & -\sin(\phi_6)\sin(\phi_3) & 0 \\ -\sin^2(\phi_1) & \sin^2(\phi_5) & 2\sin(\phi_5-\phi_6)\sin(\phi_5) & \sin^2(\phi_5-\phi_6) \\ \sin(\phi_1)\sin(\phi_1+\phi_5) & 0 & \sin(\phi_5)\sin(\phi_6) & \sin(\phi_5-\phi_6)\sin(\phi_6) \\ -\sin^2(\phi_2+\phi_3) & \sin^2(\phi_6) & 0 & 0 \\ -\sin^2(\phi_2+\phi_4) & \sin^2(\phi_7) & 2\sin(\phi_6+\phi_7)\sin(\phi_7) & \sin^2(\phi_6+\phi_7) \\ \sin^2(\phi_1+\phi_5) & 0 & 0 & -\sin^2(\phi_6) \end{pmatrix}$$

Consider determinant of submatrix with rows (1,2,5,6,7) and columns (1,2,4,5,6) :
$det = -2\sin^3(\phi_3)\sin^3(\phi_6)\sin(\phi_2)\sin(\phi_1+\phi_2+\phi_5+\phi_3+\phi_6)\sin(\phi_4-\phi_7-\phi_3-\phi_6)\sin(\phi_1+\phi_2+\phi_5+\phi_4+\phi_7)=$¿

rank of this matrix less than 5 if $\left[\begin{array}{c} \phi_1+\phi_2+\phi_5+\phi_3+\phi_6=\pi \\ \phi_4=\phi_7+\phi_3+\phi_6 \\ \phi_1+\phi_2+\phi_5+\phi_4+\phi_7=\pi \end{array}\right.$

The first equation cannot hold because the other two hold with inequality and $\phi_7 > 0$. In the other two cases of equalities, the zeros are contained in a proper subspace because every 6×6 minor of the matrix U of the zeros vanishes.

Result

The extremal copositive matrices with such minimal zero support set are given by

$$A = \begin{pmatrix} 1 & -\cos(\phi_2) & -\cos(\phi_1) & \cos(\phi_2+\phi_3) & \cos(\phi_2+\phi_4) & \cos(\phi_1+\phi_5) \\ -\cos(\phi_2) & 1 & \cos(\phi_1+\phi_2) & -\cos(\phi_3) & -\cos(\phi_4) & \cos(\phi_3+\phi_6) \\ -\cos(\phi_1) & \cos(\phi_1+\phi_2) & 1 & b_1 & \cos(\phi_5+\phi_7) & -\cos(\phi_5) \\ \cos(\phi_2+\phi_3) & -\cos(\phi_3) & b_1 & 1 & \cos(\phi_6+\phi_7) & -\cos(\phi_6) \\ \cos(\phi_2+\phi_4) & -\cos(\phi_4) & \cos(\phi_5+\phi_7) & \cos(\phi_6+\phi_7) & 1 & -\cos(\phi_7) \\ \cos(\phi_1+\phi_5) & \cos(\phi_3+\phi_6) & -\cos(\phi_5) & -\cos(\phi_6) & -\cos(\phi_7) & 1 \end{pmatrix}$$

under the conditions $\phi_i \in (0,\pi)$, $\begin{cases} \phi_3+\phi_6+\phi_7 < \phi_4 \\ \phi_1+\phi_5+\phi_7+\phi_2+\phi_4 < \pi \end{cases}$

References

[1] Peter J.C. Dickinson. *A new certificate for copositivity. Optimization Online.*

[2] Peter J.C. Dickinson and Roland Hildebrand. *Considering copositivity locally.* J.Math. Anal.Appl., 437(2):11841195, 2016

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