Case 23

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December 1, 2018

Preliminaries

The presence of a minimal zero with support of size 4 leads us to study positive semi-definite 4×4 matrices of rank 3 with positive kernel vector. Let

$$A = \begin{pmatrix} 1 & -\cos\phi_5 & -\cos\phi_7 & \star \\ -\cos\phi_5 & 1 & -\cos\phi_8 & -\cos\phi_1 \\ -\cos\phi_7 & -\cos\phi_8 & 1 & -\cos\phi_9 \\ \star & -\cos\phi_1 & -\cos\phi_9 & 1 \end{pmatrix}$$

be a partially positive definite matrix, i.e., $A_{123} \succ 0$, $A_{234} \succ 0$, where $\phi_1, \phi_5, \phi_7, \phi_8, \phi_9 \in (0, \pi)$. The positive definiteness of the principal submatrices yields the conditions

$$\max(|\pi - \phi_5 - \phi_7|, |\pi - \phi_1 - \phi_9|) < \phi_8 < \min(\pi - |\phi_5 - \phi_7|, \pi - |\phi_1 - \phi_9|). \tag{1}$$

A consequence of these relations is

$$\phi_7 < \phi_1 + \phi_5 + \phi_9, \qquad \phi_9 < \phi_1 + \phi_5 + \phi_7. \tag{2}$$

The matrix is positive semi-definite completable, and there exists a closed interval $[a_{\min}, a_{\max}]$ for possible values of the missing corner element a which makes A positive semi-definite. The extremal values of the interval yield a rank 3 matrix. If a equals an extremal value, then there exists a non-zero kernel vector u of A. Since $A_{123} \succ 0$, the element u_4 cannot vanish. Likewise, u_1 does not vanish, because $A_{234} \succ 0$.

The matrix uu^T generates the face which is orthogonal to the face of A in the cone $\mathcal{S}_+^4$. If $a = a_{\max}$, then adding a positive multiple of E_{14} to A leads out of the cone $\mathcal{S}_+^4$, while adding a small negative multiple leads into the interior of $\mathcal{S}_+^4$. Hence $\langle E_{14}, uu^T \rangle = 2u_1u_4 < 0$. Likewise, for $a = a_{\min}$ we get $\langle E_{14}, uu^T \rangle = 2u_1u_4 > 0$. Thus $a = a_{\min}$ is a necessary condition for having a positive kernel vector. We shall henceforth assume that $a = a_{\min}$. We have the explicit value

$$a = \frac{\cos\phi_1\cos\phi_5 + \cos\phi_1\cos\phi_7\cos\phi_8 + \cos\phi_5\cos\phi_8\cos\phi_9 + \cos\phi_7\cos\phi_9}{\sin^2\phi_8} -$$

$$- \frac{\sqrt{(\sin^2\phi_1\sin^2\phi_8 - (\cos\phi_9 + \cos\phi_1\cos\phi_8)^2)(\sin^2\phi_5\sin^2\phi_8 - (\cos\phi_7 + \cos\phi_5\cos\phi_8)^2)}}{\sin^2\phi_8}.$$

We also assume that the elements u_1, u_4 of the kernel vector are positive.

The submatrix A_{124} is also positive semi-definite, and hence $a \geq \cos(\phi_1 + \phi_5)$. Likewise, $A_{134} \succeq 0$ yields $a \geq \cos(\phi_7 + \phi_9)$. Note that $a = \cos(\phi_1 + \phi_5)$ and $a = \cos(\phi_7 + \phi_9)$ implies

$$u = (\sin\phi_1, \sin(\phi_1 + \phi_5), 0, \sin\phi_5)^T, \qquad u = (\sin\phi_9, 0, \sin(\phi_7 + \phi_9), \sin\phi_7)^T, \tag{3}$$

respectively.

Define $v = (\sin\phi_1, \sin(\phi_1 + \phi_5), 0, \sin\phi_5)^T$ and $w = (\sin\phi_9, 0, \sin(\phi_7 + \phi_9), \sin\phi_7)^T$. We obtain

$$Av = ((a - \cos(\phi_1 + \phi_5))\sin\phi_5, 0, \star, (a - \cos(\phi_1 + \phi_5))\sin\phi_1)^T \quad Aw = ((a - \cos(\phi_7 + \phi_9))\sin\phi_7, \star, 0, (a - \cos(\phi_7 + \phi_9))\sin\phi_9)^T.$$

We have $u^TAv = u^TAw = 0$, because u is the kernel vector of A. This yields

$$(Av)_3u_3 = -(a - \cos(\phi_1 + \phi_5))(\sin\phi_5 u_1 + \sin\phi_1 u_4), \quad (Aw)_2u_2 = -(a - \cos(\phi_7 + \phi_9))(\sin\phi_7 u_1 + \sin\phi_9 u_4). \tag{4}$$

Hence $u_3 = 0$ implies $a = \cos(\phi_1 + \phi_5)$ and $u_2 = 0$ implies $a = \cos(\phi_7 + \phi_9)$. This in turn gives the values (3) for u, respectively, which by $Au = 0$ implies $(Av)_3 = 0$ and $(Aw)_2 = 0$, respectively. Hence $u_3 = 0$ is equivalent to $(Av)_3 = 0$ and $u_2 = 0$ to $(Aw)_2 = 0$.

If $u_3 \neq 0$ and $u_2 \neq 0$, then the right-hand sides of equations (4) are negative, and we obtain

$$u_3 > 0 \;\Leftrightarrow\; (Av)_3 < 0, \qquad u_2 > 0 \;\Leftrightarrow\; (Aw)_2 < 0,$$

respectively.

Thus $u > 0$ if and only if

$$\cos\phi_7 \sin\phi_1 + \cos\phi_8 \sin(\phi_1+\phi_5) + \cos\phi_9 \sin\phi_5 > 0, \quad \cos\phi_5 \sin\phi_9 + \cos\phi_8 \sin(\phi_7+\phi_9) + \cos\phi_1 \sin\phi_7 > 0, \tag{5}$$

implying also

$$a > \cos(\phi_1 + \phi_5), \qquad a > \cos(\phi_7 + \phi_9). \tag{6}$$

Let us now derive a condition on the kernel vector. Divide A into 2×2 blocks I, II of equal size. Then the kernel vector satisfies

$$A_{I,I} u_I + A_{I,II} u_{II} = A_{I,II}^T u_I + A_{II,II} u_{II} = 0.$$

It follows that

$$u_I^T A_{I,I} u_I = -u_I^T A_{I,II} u_{II} = u_{II}^T A_{II,II} u_{II}. \tag{7}$$

Basic structure

The minimal zero support set is equivalent to $\{\{3,4,5\},\{4,5,1\},\{5,1,2\},\{1,2,3\},\{1,5,6\},\{2,3,4,6\}\}$. Let the corresponding zeros be $u^1,\dots,u^6$. It follows that A has the form

$$A = \begin{pmatrix} 1 & -\cos\phi_4 & \cos(\phi_4+\phi_5) & \cos(\phi_2+\phi_3) & -\cos\phi_3 & \cos(\phi_3+\phi_6) \\ -\cos\phi_4 & 1 & -\cos\phi_5 & a_{24} & \cos(\phi_3+\phi_4) & a_{26} \\ \cos(\phi_4+\phi_5) & -\cos\phi_5 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & a_{36} \\ \cos(\phi_2+\phi_3) & a_{24} & -\cos\phi_1 & 1 & -\cos\phi_2 & a_{46} \\ -\cos\phi_3 & \cos(\phi_3+\phi_4) & \cos(\phi_1+\phi_2) & -\cos\phi_2 & 1 & -\cos\phi_6 \\ \cos(\phi_3+\phi_6) & a_{26} & a_{36} & a_{46} & -\cos\phi_6 & 1 \end{pmatrix}.$$

We also set $a_{26} = -\cos\phi_7$, $a_{36} = -\cos\phi_8$, $a_{46} = -\cos\phi_9$. Here $\phi_1,\dots,\phi_9 \in (0,\pi)$, and $\phi_1+\phi_2, \phi_2+\phi_3, \phi_3+\phi_4, \phi_4+\phi_5, \phi_3+\phi_6 < \pi$.

The minimal zeros $u_1,\dots,u_6$ are given by the columns of the matrix

$$\begin{pmatrix} 0 & \sin\phi_2 & \sin(\phi_3+\phi_4) & \sin\phi_5 & \sin\phi_6 & 0 \\ 0 & 0 & \sin\phi_3 & \sin(\phi_4+\phi_5) & 0 & u_{62} \\ \sin\phi_2 & 0 & 0 & \sin\phi_4 & 0 & u_{63} \\ \sin(\phi_1+\phi_2) & \sin\phi_3 & 0 & 0 & 0 & u_{64} \\ \sin\phi_1 & \sin(\phi_2+\phi_3) & \sin\phi_4 & 0 & \sin(\phi_3+\phi_6) & 0 \\ 0 & 0 & 0 & 0 & \sin\phi_3 & 1 \end{pmatrix}.$$

We have the symmetry $(123456) \mapsto (543216)$.

First order conditions

The minimal zero supports cover all elements, and irreducibility with respect to $\mathcal{N}^6$ follows.

Conditions on $(Au_1)_1, (Au_2)_3, (Au_3)_3, (Au_4)_5$: We have the inequality

$$(Au_1)_1 = \cos(\phi_4+\phi_5)\sin\phi_2 + \cos(\phi_2+\phi_3)\sin(\phi_1+\phi_2) - \cos\phi_3 \sin\phi_1 = (\cos(\phi_4+\phi_5) + \cos(\phi_1+\phi_2+\phi_3))\sin\phi_2 \geq 0,$$

which yields either $\phi_1+\phi_2+\phi_3+\phi_4+\phi_5 \leq \pi$ or $\phi_1+\phi_2+\phi_3 \geq \pi + \phi_4 + \phi_5$. Further

$$(Au_2)_3 = \cos(\phi_4+\phi_5)\sin\phi_2 - \cos\phi_1 \sin\phi_3 + \cos(\phi_1+\phi_2)\sin(\phi_2+\phi_3) = (Au_1)_1,$$

$$(Au_3)_3 = \cos(\phi_4+\phi_5)\sin(\phi_3+\phi_4) - \cos\phi_5\sin\phi_3 + \cos(\phi_1+\phi_2)\sin\phi_4 = (\cos(\phi_3+\phi_4+\phi_5)+\cos(\phi_1+\phi_2))\sin\phi_4 \geq 0,$$

which yields either $\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 \leq \pi$ or $\phi_3 + \phi_4 + \phi_5 \geq \pi + \phi_1 + \phi_2$. Hence if the first case does not hold, then combination with the above inequality yields $\phi_3 \geq \pi$, a contradiction. Thus

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 \leq \pi. \tag{8}$$

Similar to the relation $(Au_2)_3 = (Au_1)_1$ we prove

$$(Au_4)_5 = -\cos\phi_3\sin\phi_5 + \cos(\phi_3+\phi_4)\sin(\phi_4+\phi_5) + \cos(\phi_1+\phi_2)\sin\phi_4 = (Au_3)_3.$$

The four quantities are hence zero if and only if in (8) we have a equality.

Conditions on $(Au_1)_2, (Au_2)_2, (Au_3)_4, (Au_4)_4$**:** The submatrix A_{2346} is positive semi-definite of rank 3 with positive kernel vector. This yields conditions (1),(5), and by virtue of (6) $a_{24} > \cos(\phi_1+\phi_5)$, $a_{24} > \cos(\phi_7+\phi_9)$.

The strict lower bound $\cos(\phi_1+\phi_5)$ on a_{24} implies that the submatrix A_{12345} is a sum of a T-matrix and a positive multiple of E_{24}, and in particular copositive. It implies also that $(Au_1)_2, (Au_2)_2, (Au_3)_4, (Au_4)_4 > 0$, because these quantities are already nonnegative if $a_{24} = \cos(\phi_1+\phi_5)$.

Conditions on $(Au_3)_6, (Au_5)_2, (Au_2)_6, (Au_5)_4$**:** The first two quantities give rise to the condition

$$a_{26} \geq -\cos(\phi_3+\phi_4+\phi_6), \tag{9}$$

while the last two yield

$$a_{46} \geq \cos(\phi_2-\phi_6). \tag{10}$$

Conditions on $(Au_1)_6, (Au_4)_6$**:** The inequalities amount to

$$\begin{pmatrix} & \sin\phi_2 & \sin(\phi_1+\phi_2) \\ \sin(\phi_4+\phi_5) & \sin\phi_4 & \end{pmatrix} \begin{pmatrix} a_{26} \\ a_{36} \\ a_{46} \end{pmatrix} \geq \begin{pmatrix} \cos\phi_6\sin\phi_1 \\ -\cos(\phi_3+\phi_6)\sin\phi_5 \end{pmatrix}. \tag{11}$$

Conditions on $(Au_6)_1, (Au_6)_5$**:** Set $u = (u_{62}, u_{63}, u_{64})^T$. Then the inequalities can be written as

$$\begin{pmatrix} -\cos\phi_4 & \cos(\phi_4+\phi_5) & \cos(\phi_2+\phi_3) \\ \cos(\phi_3+\phi_4) & \cos(\phi_1+\phi_2) & -\cos\phi_2 \end{pmatrix} u \geq \begin{pmatrix} -\cos(\phi_3+\phi_6) \\ \cos\phi_6 \end{pmatrix}.$$

Set further $P = A_{234} \succ 0$ and $a = (a_{26}, a_{36}, a_{46})^T$. Then $a = -Pu$ and conditions (11) can be rewritten as

$$\begin{pmatrix} \cos\phi_5\sin\phi_2 - a_{24}\sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2)\sin\phi_1 & -\sin\phi_1\cos\phi_2 \\ -\cos\phi_4\sin\phi_5 & \cos(\phi_4+\phi_5)\sin\phi_5 & -a_{24}\sin(\phi_4+\phi_5)+\cos\phi_1\sin\phi_4 \end{pmatrix} u \geq \begin{pmatrix} \cos\phi_6\sin\phi_1 \\ -\cos(\phi_3+\phi_6)\sin\phi_5 \end{pmatrix}.$$

Now

$$-a_{24}\sin(\phi_4+\phi_5)+\cos\phi_1\sin\phi_4 < -\cos(\phi_1+\phi_5)\sin(\phi_4+\phi_5)+\cos\phi_1\sin\phi_4 = -\cos(\phi_1+\phi_4+\phi_5)\sin\phi_5 \leq \cos(\phi_2+\phi_3)\sin\phi_5,$$

and hence $(Au_4)_6 \geq 0$ implies $(Au_6)_1 > 0$. Likewise,

$$\cos\phi_5\sin\phi_2 - a_{24}\sin(\phi_1+\phi_2) < \cos(\phi_3+\phi_4)\sin\phi_5,$$

and $(Au_1)_6 \geq 0$ implies $(Au_6)_5 > 0$.

Condition on $(Au_5)_3$**:** The corresponding inequality yields

$$a_{36} \geq \frac{-\cos(\phi_4+\phi_5)\sin\phi_6 - \cos(\phi_1+\phi_2)\sin(\phi_3+\phi_6)}{\sin\phi_3}.$$

From $A_{346} \succ 0$ we obtain $\phi_9 < \pi - |\phi_1 - \phi_8|$ and hence $\cos\phi_9 > -\cos(\phi_1-\phi_8)$. Combined with the first condition in (11) this yields

$$\cos(\phi_1+\phi_2-\phi_8)\sin\phi_1 = -\sin\phi_2\cos\phi_8+\sin(\phi_1+\phi_2)\cos(\phi_1-\phi_8) > -\sin\phi_2\cos\phi_8-\sin(\phi_1+\phi_2)\cos\phi_9 \geq \cos\phi_6\sin\phi_1.$$

Thus $|\phi_1 + \phi_2 - \phi_8| < \phi_6$, which yields $\phi_8 > \phi_1 + \phi_2 - \phi_6$.

Now suppose that $\phi_1 + \phi_2 - \phi_6 \geq 0$. Then we obtain $a_{36} = -\cos\phi_8 > -\cos(\phi_1 + \phi_2 - \phi_6)$. Then, however,

$$\frac{-\cos(\phi_4+\phi_5)\sin\phi_6 - \cos(\phi_1+\phi_2)\sin(\phi_3+\phi_6)}{\sin\phi_3} \leq \frac{\cos(\phi_1+\phi_2+\phi_3)\sin\phi_6 - \cos(\phi_1+\phi_2)\sin(\phi_3+\phi_6)}{\sin\phi_3}$$
$$= -\cos(\phi_1+\phi_2-\phi_6) < a_{36}.$$

Likewise, from $A_{236} \succ 0$ we obtain $\phi_7 < \pi - |\phi_5 - \phi_8|$ and hence $\cos\phi_7 > -\cos(\phi_5 - \phi_8)$. Combined with the second condition in (11) this yields

$$\cos(\phi_4+\phi_5-\phi_8)\sin\phi_5 = \sin(\phi_4+\phi_5)\cos(\phi_5-\phi_8) - \sin\phi_4\cos\phi_8 > -\sin(\phi_4+\phi_5)\cos\phi_7 - \sin\phi_4\cos\phi_8 \geq -\cos(\phi_3+\phi_6)\sin\phi_5.$$

Thus $|\phi_4 + \phi_5 - \phi_8| < \pi - \phi_3 - \phi_6$, which yields $\phi_3 + \phi_4 + \phi_5 + \phi_6 < \pi + \phi_8$.

Now suppose that $\phi_3 + \phi_4 + \phi_5 + \phi_6 \geq \pi$, then we get $\cos(\phi_3 + \phi_4 + \phi_5 + \phi_6) < -\cos\phi_8 = a_{36}$. It follows that

$$\frac{-\cos(\phi_4+\phi_5)\sin\phi_6 - \cos(\phi_1+\phi_2)\sin(\phi_3+\phi_6)}{\sin\phi_3} \leq \frac{-\cos(\phi_4+\phi_5)\sin\phi_6 + \cos(\phi_3+\phi_4+\phi_5)\sin(\phi_3+\phi_6)}{\sin\phi_3}$$
$$= \cos(\phi_3+\phi_4+\phi_5+\phi_6) < a_{36}.$$

Finally, assume that $\phi_1 + \phi_2 - \phi_6 < 0$ and $\phi_3 + \phi_4 + \phi_5 + \phi_6 < \pi$. Then we get

$$\frac{-\cos(\phi_4+\phi_5)\sin\phi_6 - \cos(\phi_1+\phi_2)\sin(\phi_3+\phi_6)}{\sin\phi_3} < \frac{-\cos(\pi-\phi_3-\phi_6)\sin\phi_6 - \cos\phi_6\sin(\phi_3+\phi_6)}{\sin\phi_3} = -1 < a_{36}.$$

Thus in any case

$$a_{36} > \frac{-\cos(\phi_4+\phi_5)\sin\phi_6 - \cos(\phi_1+\phi_2)\sin(\phi_3+\phi_6)}{\sin\phi_3}$$

and $(Au_5)_3 > 0$.

Further relations on ϕ_k: From the first condition in (11) and the constraint $\phi_8 < \pi - |\phi_1 - \phi_9|$ we get

$$\sin\phi_2\cos(\phi_1-\phi_9) - \sin(\phi_1+\phi_2)\cos\phi_9 = -\cos(\phi_2+\phi_9)\sin\phi_1 > \cos\phi_6\sin\phi_1.$$

Hence $|\pi - \phi_9 - \phi_2| < \phi_6$ and $-\phi_6 < \pi - \phi_9 - \phi_2$. On the other hand, (10) yields $\pi - \phi_9 \leq |\phi_6 - \phi_2|$, and hence $\phi_2 - \phi_6 < |\phi_6 - \phi_2|$, implying $\phi_6 > \phi_2$.

Likewise, from the second condition in (11) and $\phi_8 < \pi - |\phi_5 - \phi_7|$ we get

$$-\sin(\phi_4+\phi_5)\cos\phi_7 + \sin\phi_4\cos(\phi_5-\phi_7) = -\cos(\phi_4+\phi_7)\sin\phi_5 > -\cos(\phi_3+\phi_6)\sin\phi_5.$$

Hence $|\pi - \phi_7 - \phi_4| < \pi - \phi_3 - \phi_6$ and $\phi_3 + \phi_6 - \pi < \pi - \phi_7 - \phi_4$. On the other hand, by (9) we have $\pi - \phi_7 \leq |\pi - \phi_3 - \phi_6 - \phi_4|$, and hence $-\pi + \phi_3 + \phi_6 + \phi_4 < |\pi - \phi_3 - \phi_6 - \phi_4|$, implying $\pi > \phi_3 + \phi_4 + \phi_6$.

Thus conditions (9),(10) can be rewritten as

$$\phi_7 \geq \phi_3 + \phi_4 + \phi_6, \quad \phi_9 \geq \pi + \phi_2 - \phi_6, \tag{12}$$

and we obtain as a consequence

$$\phi_2 < \phi_6 < \pi - \phi_3 - \phi_4, \qquad \phi_7 + \phi_9 > \pi.$$

Note that $\phi_3 + \phi_6 < \pi$ also follows.

Again conditions on $(Au_1)_6, (Au_4)_6$: We obtain $0 < \pi - \phi_6 \leq \phi_9 - \phi_2 < \pi$ and hence $\cos\phi_6 \leq -\cos(\phi_9 - \phi_2)$. Multiplying with $\sin\phi_1$ this further gives

$$-\cos\phi_9\cos\phi_2\sin\phi_1 - \sin\phi_9\sin\phi_2\sin\phi_1 = -\cos\phi_9\sin(\phi_1+\phi_2) + \sin\phi_2\cos(\phi_1+\phi_9) \geq \cos\phi_6\sin\phi_1.$$

Now $\cos(\phi_1+\phi_9) = -\cos(|\pi-\phi_1-\phi_9|) < -\cos\phi_8$, and hence

$$-\cos\phi_9\sin(\phi_1+\phi_2) - \sin\phi_2\cos\phi_8 > \cos\phi_6\sin\phi_1.$$

Thus $(Au_1)_6 > 0$.

Likewise, $0 < \phi_3+\phi_6 \le \phi_7-\phi_4 < \pi$ and hence $-\cos(\phi_3+\phi_6) \le -\cos(\phi_4-\phi_7)$. Multiplying with $\sin\phi_5$ this further gives

$$-\cos\phi_4\cos\phi_7\sin\phi_5 - \sin\phi_4\sin\phi_7\sin\phi_5 = -\cos\phi_7\sin(\phi_4+\phi_5) + \sin\phi_4\cos(\phi_5+\phi_7) \ge -\cos(\phi_3+\phi_6)\sin\phi_5.$$

Now $\cos(\phi_5+\phi_7) = -\cos(|\pi-\phi_5-\phi_7|) < -\cos\phi_8$, and hence

$$-\cos\phi_7\sin(\phi_4+\phi_5) - \sin\phi_4\cos\phi_8 > -\cos(\phi_3+\phi_6)\sin\phi_5.$$

Thus $(Au_4)_6 > 0$.

Again conditions on $(Au_1)_1, (Au_2)_3, (Au_3)_3, (Au_4)_5$**:** Inserting $\cos(\phi_5+\phi_7) < -\cos\phi_8$ into the first condition of (5) we obtain

$$\begin{aligned} 0 &< \cos\phi_7\sin\phi_1 + \cos\phi_8\sin(\phi_1+\phi_5) + \cos\phi_9\sin\phi_5 < \cos\phi_7\sin\phi_1 - \cos(\phi_5+\phi_7)\sin(\phi_1+\phi_5) + \cos\phi_9\sin\phi_5 \\ &= (\cos\phi_9 - \cos(\phi_1+\phi_5+\phi_7))\sin\phi_5. \end{aligned}$$

This yields $|\pi-\phi_1-\phi_5-\phi_7| < \pi-\phi_9$ and in particular

$$\phi_1+\phi_5+\phi_7+\phi_9 < 2\pi. \tag{13}$$

Combining with (12) we obtain

$$\phi_1+\phi_2+\phi_3+\phi_4+\phi_5 < \pi,$$

and $(Au_1)_1, (Au_2)_3, (Au_3)_3, (Au_4)_5 > 0$.

Conditions $u_{62}, u_{63}, u_{64}, u_{66} > 0$

We now consider constraints (5). Note that $\phi_7+\phi_9 > \pi$, $\phi_1+\phi_5 < \pi$. Hence (5) can be written as

$$-\frac{\cos\phi_7\sin\phi_1 + \cos\phi_9\sin\phi_5}{\sin(\phi_1+\phi_5)} < \cos\phi_8 < -\frac{\cos\phi_5\sin\phi_9 + \cos\phi_1\sin\phi_7}{\sin(\phi_7+\phi_9)}.$$

From (13) and taking into account $\phi_7+\phi_9 > \pi$ we obtain $\cos\phi_5 > \cos(\phi_1+\phi_7+\phi_9)$. After multiplying by $\sin\phi_9$ and transforming we get

$$\cos\phi_5\sin\phi_9 + \cos\phi_1\sin\phi_7 > \cos(\phi_1+\phi_9)\sin(\phi_7+\phi_9).$$

This further yields

$$-\frac{\cos\phi_5\sin\phi_9 + \cos\phi_1\sin\phi_7}{\sin(\phi_7+\phi_9)} > \cos(\pi-\phi_1-\phi_9) > \cos\phi_8.$$

Thus $u_{62} > 0$ is satisfied automatically.

By the first relation in (2) and $\phi_1+\phi_5 < \pi < \phi_7+\phi_9$ we have $-\phi_9 < \phi_1+\phi_5-\phi_7 < \phi_9$ and hence $\cos(\phi_1+\phi_5-\phi_7) > \cos\phi_9$. Multiplying by $\sin\phi_5$ we get

$$\cos(\phi_5-\phi_7)\sin(\phi_1+\phi_5) - \cos\phi_7\sin\phi_1 = \cos(\phi_1+\phi_5-\phi_7)\sin\phi_5 > \cos\phi_9\sin\phi_5.$$

It follows that

$$\cos(\pi-|\phi_5-\phi_7|) = \cos(\pi-\phi_5+\phi_7) < -\frac{\cos\phi_9\sin\phi_5 + \cos\phi_7\sin\phi_1}{\sin(\phi_1+\phi_5)}$$

Likewise, by the second relation in (2) we get $-\phi_7 < \phi_1+\phi_5-\phi_9 < \phi_7$ and hence $\cos(\phi_1+\phi_5-\phi_9) > \cos\phi_7$. Multiplying by $\sin\phi_1$ we get

$$\cos(\phi_1-\phi_9)\sin(\phi_1+\phi_5) - \cos\phi_9\sin\phi_5 = \cos(\phi_1+\phi_5-\phi_9)\sin\phi_1 > \cos\phi_7\sin\phi_1.$$

It follows that

$$\cos(\pi-|\phi_1-\phi_9|) = \cos(\pi-\phi_1+\phi_9) < -\frac{\cos\phi_9\sin\phi_5 + \cos\phi_7\sin\phi_1}{\sin(\phi_1+\phi_5)}.$$

Thus $u_{64} > 0$ implies $\max(|\pi-\phi_5-\phi_7|, |\pi-\phi_1-\phi_9|) < \phi_8$.

Parametrization

We parameterize A by the angles $\phi_1, \dots, \phi_9$. We are left with the constraints $\phi_k \in (0, \pi)$, $\phi_7 \geq \phi_3 + \phi_4 + \phi_6$, $\phi_9 \geq \pi - \phi_6 + \phi_2$,

$$-\frac{\cos\phi_7 \sin\phi_1 + \cos\phi_9 \sin\phi_5}{\sin(\phi_1 + \phi_5)} < \cos\phi_8 < -\max(\cos(\phi_5 + \phi_7), \cos(\phi_1 + \phi_9)). \qquad (14)$$

Note that (2) and (13) follow from this relation, and these are the conditions obtained by projecting out ϕ_8. Projecting out ϕ_9, ϕ_7, ϕ_6 consecutively we obtain

$$\phi_7 - \phi_1 - \phi_5 < \phi_9 < \min(2\pi - \phi_1 - \phi_5 - \phi_7, \phi_1 + \phi_5 + \phi_7), \quad \pi - \phi_6 + \phi_2 \leq \phi_9,$$

$$\pi - \phi_1 + \phi_2 - \phi_5 - \phi_6 < \phi_7 < \min(\pi, \pi - \phi_1 - \phi_2 - \phi_5 + \phi_6), \quad \phi_3 + \phi_4 + \phi_6 \leq \phi_7,$$

$$\phi_2 < \phi_6 < \pi - \phi_3 - \phi_4,$$

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 < \pi.$$

The element a_{24} is given by

$$a_{24} = \frac{\cos\phi_1\cos\phi_5 + \cos\phi_1\cos\phi_7\cos\phi_8 + \cos\phi_5\cos\phi_8\cos\phi_9 + \cos\phi_7\cos\phi_9}{\sin^2\phi_8} - \qquad (15)$$

$$- \frac{\sqrt{(\sin^2\phi_1 \sin^2\phi_8 - (\cos\phi_9 + \cos\phi_1\cos\phi_8)^2)(\sin^2\phi_5\sin^2\phi_8 - (\cos\phi_7 + \cos\phi_5\cos\phi_8)^2)}}{\sin^2\phi_8}.$$

Copositivity

Copositivity of A will be checked by the criterion in Theorem 4.6 of [1]. For each non-empty index set $I \subset \{1, \dots, 6\}$ we have to find a vector $v \in \mathbb{R}^6$ such that I contains the support of v and is contained in the nonnegative support of Av, or prove directly that A_I is copositive.

For I of size 1 or 2 we may take $v = \sum_{i \in I} e_i$. For I containing the support of a minimal zero u we may take $v = u$. Index sets which are contained in $\{1, 2, 3, 4, 5\}$ not be considered because the corresponding submatrix is already copositive. The same holds for subsets of $\{2, 3, 4, 6\}$. The other index sets I are

$$\{1,2,6\}, \{1,3,6\}, \{1,4,6\}, \{1,2,4,6\}, \{1,3,4,6\}, \{2,5,6\}, \{3,5,6\}, \{2,3,5,6\}, \{4,5,6\}, \{2,4,5,6\}. \qquad (16)$$

The submatrices of size 3 are copositive because

$$\phi_4 + (\pi - \phi_3 - \phi_6) + \phi_7 \geq 2\phi_4 + \pi > \pi, \; (\pi - \phi_4 - \phi_5) + (\pi - \phi_3 - \phi_6) + \phi_8 > -\phi_4 + \pi - \phi_3 - \phi_6 + \phi_7 \geq \pi,$$

$$(\pi - \phi_2 - \phi_3) + (\pi - \phi_3 - \phi_6) + \phi_9 \geq 3\pi - 2\phi_3 - 2\phi_6 > \pi, \; (\pi - \phi_3 - \phi_4) + \phi_7 + \phi_6 \geq \pi + 2\phi_6 > \pi,$$

$$(\pi - \phi_1 - \phi_2) + \phi_8 + \phi_6 > -\phi_2 + \phi_9 + \phi_6 \geq \pi, \; \phi_2 + \phi_9 + \phi_6 \geq \pi + 2\phi_2 > \pi.$$

For the index sets $\{1,2,4,6\}, \{2,4,5,6\}$ we may choose the vectors $e_1 + e_2$, $e_4 + e_5$, respectively, because $\cos(\phi_2+\phi_3)+a_{24} > \cos(\phi_2+\phi_3)+\cos(\phi_1+\phi_5) > 0$, $\cos(\phi_3+\phi_6) > \cos\phi_7$ and $a_{24}+\cos(\phi_3+\phi_4) > \cos(\phi_1 + \phi_5) + \cos(\phi_3 + \phi_4) > 0$, $\cos(\pi - \phi_6) > \cos\phi_9$.

Index set $\{1,3,4,6\}$*:* Set $\xi_1 = \pi - \phi_4 - \phi_5$, $\xi_2 = \pi - \phi_2 - \phi_3$, $\xi_3 = \pi - \phi_3 - \phi_6$. The other angles involved in the description of the off-diagonal entries of A_{1346} are ϕ_1, ϕ_8, ϕ_9. All six angles are in $(0, \pi)$. Further we have the inequalities

$$\phi_1 + \phi_8 + \phi_9 > \pi, \quad \xi_1 + \xi_3 + \phi_8 > \pi - \phi_3 - \phi_4 - \phi_6 + \phi_7 \geq \pi, \quad \xi_1 + \xi_2 = 2\pi - \phi_2 - \phi_3 - \phi_4 - \pi_5 > \pi,$$

$$\xi_1 + \phi_9 + \phi_8 > 2\pi - \phi_4 - \phi_5 - \phi_1 > \pi, \quad \xi_2 + \phi_9 \geq 2\pi - \phi_6 - \phi_3 > \pi, \quad \xi_1 + \xi_2 + \phi_9 + \phi_8 > 3\pi - \phi_2 - \phi_3 - \phi_4 - \phi_5 - \phi_1 > 2\pi.$$

These strict linear inequalities define the interior of a polytope with 32 extreme points, of which 28 can be obtained by increasing the elements of one of the four remaining extreme points. These four are given by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \phi_1 \\ \phi_9 \\ \phi_8 \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \\ 0 \\ 0 \\ \pi \\ 0 \end{pmatrix}, \begin{pmatrix} \pi \\ \pi \\ 0 \\ \pi \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi \\ 0 \\ 0 \\ 0 \\ \pi \end{pmatrix}, \begin{pmatrix} 0 \\ \pi \\ \pi \\ 0 \\ \pi \\ 0 \end{pmatrix}.$$

Therefore

$$A_{1346} = \begin{pmatrix} 1 & -\cos(\psi_1+\psi_2) & \cos\psi_1 & \cos(\psi_1+\psi_2+\psi_3) \\ -\cos(\psi_1+\psi_2) & 1 & -\cos\psi_2 & -\cos\psi_3 \\ \cos\psi_1 & -\cos\psi_2 & 1 & \cos(\psi_2+\psi_3) \\ \cos(\psi_1+\psi_2+\psi_3) & -\cos\psi_3 & \cos(\psi_2+\psi_3) & 1 \end{pmatrix} + N \qquad (17)$$

for some $\psi_1, \psi_2, \psi_3 > 0$ with $\psi_1+\psi_2+\psi_3 < \pi$, where N is a non-zero nonnegative matrix. The left matrix summand is PSD of rank 2 with kernel vectors $(0, \sin(\psi_2+\psi_3), \sin\psi_3, \sin\psi_2)^T$ and $(-\sin\psi_2, \sin\psi_1, \sin(\psi_1+\psi_2), 0)^T$. Hence A_{1346} is copositive.

In a similar way copositivity of A_{2356} is proven.

This proves copositivity of A.

Absence of other minimal zeros

By construction there are no further minimal zeros with supports of size 1 or 2. Index sets which contain a minimal zero support are excluded by definition. The submatrix A_{12345} contains only the zeros of the T-matrix component which do not contain $\{2, 4\}$ in their support, and these are exactly the first 4 zeros u_i. The submatrix A_{2346} does not have other zeros than u_6 by construction. Hence the index sets which remain to be checked are those in (16).

The inequalities on the angles proving copositivity of the submatrices corresponding to sets with cardinality 3 are strict, and hence there does not exist a zero with these supports.

The submatrix A_{1346} cannot have a positive kernel vector because the summand N in (17) is non-zero. In a similar way there does not exist a zero with support $\{2, 3, 5, 6\}$.

Thus there do not exist additional minimal zeros.

Extremality

We use the extremality criterion Theorem 17 point 5 in [2]. The matrix A is extremal whenever every matrix B satisfying $(Bu_i)_j = 0$ whenever $(Au_i)_j = 0$ is proportional to A. Let us consider the elements $(Au_i)_j$.

The following elements are always zero:

$$(Au_1)_{3,4,5},\ (Au_2)_{1,4,5},\ (Au_3)_{1,2,5},\ (Au_4)_{1,2,3},\ (Au_5)_{1,5,6},\ (Au_6)_{2,3,4,6}. \qquad (18)$$

The following elements may become zero: If $\phi_7 = \phi_3 + \phi_4 + \phi_6$, then

$$(Au_3)_6 = (Au_5)_2 = 0.$$

If $\phi_9 = \pi + \phi_2 - \phi_6$, then

$$(Au_2)_6 = (Au_5)_4 = 0.$$

The following elements are always positive:

$$(Au_1)_{1,2,6}, (Au_2)_{2,3}, (Au_3)_{3,4}, (Au_4)_{4,5,6}, (Au_5)_3, (Au_6)_{1,5}.$$

We now consider these cases separately.

Case $\phi_7 > \phi_3 + \phi_4 + \phi_6$, $\phi_9 > \pi + \phi_2 - \phi_6$: In this case there are only 19 linear relations on the 21 elements of B, and the solution space is at least 2-dimensional. Hence in this case A is not extremal.

Case $\phi_7 = \phi_3 + \phi_4 + \phi_6$, $\phi_9 > \pi + \phi_2 - \phi_6$: Let us construct the face of A in the copositive cone. Let F be a full rank 6×3 matrix such that $u_1^T F = u_4^T F = u_6^T F = 0$ and let G be a full rank 6×2 matrix such that $u_2^T G = u_3^T G = u_5^T G = 0$. Then for every B in the face of A there exist positive definite matrices $P \in \mathcal{S}_+^3$, $Q \in \mathcal{S}_+^2$ such that

$$FPF^T = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \star & \star & \star \\ b_{12} & b_{22} & b_{23} & b_{24} & \star & b_{26} \\ b_{13} & b_{23} & b_{33} & b_{34} & b_{35} & b_{36} \\ \star & b_{24} & b_{34} & b_{44} & b_{45} & b_{46} \\ \star & \star & b_{35} & b_{45} & b_{55} & \star \\ \star & b_{26} & b_{36} & b_{46} & \star & b_{66} \end{pmatrix}, \qquad GQG^T = \begin{pmatrix} b_{11} & b_{12} & \star & b_{14} & b_{15} & b_{16} \\ b_{12} & b_{22} & \star & \star & b_{25} & b_{26} \\ \star & \star & \star & \star & \star & \star \\ b_{14} & \star & \star & b_{44} & b_{45} & \star \\ b_{15} & b_{25} & \star & b_{45} & b_{55} & b_{56} \\ b_{16} & b_{26} & \star & \star & b_{56} & b_{66} \end{pmatrix}. \qquad (19)$$

The products have the elements $11, 12, 22, 26, 44, 45, 55, 66$ in common, which yields 8 relations on the 9 elements of P, Q. The matrix of the system in the elements $p_{11}, p_{12}, \dots, q_{22}$ is given by

$$\begin{pmatrix} f_{11}^2 & 2f_{11}f_{12} & 2f_{11}f_{13} & f_{12}^2 & 2f_{12}f_{13} & f_{13}^2 & -g_{11}^2 & -2g_{11}g_{12} & -g_{12}^2 \\ f_{11}f_{21} & f_{11}f_{22} + f_{21}f_{12} & f_{11}f_{23} + f_{21}f_{13} & f_{12}f_{22} & f_{12}f_{23} + f_{22}f_{13} & f_{13}f_{23} & -g_{11}g_{21} & -g_{11}g_{22} - g_{21}g_{12} & -g_{12}g_{22} \\ f_{21}^2 & 2f_{21}f_{22} & 2f_{21}f_{23} & f_{22}^2 & 2f_{22}f_{23} & f_{23}^2 & -g_{21}^2 & -2g_{21}g_{22} & -g_{22}^2 \\ f_{21}f_{61} & f_{21}f_{62} + f_{61}f_{22} & f_{21}f_{63} + f_{61}f_{23} & f_{22}f_{62} & f_{22}f_{63} + f_{62}f_{23} & f_{23}f_{63} & -g_{21}g_{61} & -g_{21}g_{62} - g_{61}g_{22} & -g_{22}g_{62} \\ f_{41}^2 & 2f_{41}f_{42} & 2f_{41}f_{43} & f_{42}^2 & 2f_{42}f_{43} & f_{43}^2 & -g_{41}^2 & -2g_{41}g_{42} & -g_{42}^2 \\ f_{41}f_{51} & f_{41}f_{52} + f_{51}f_{42} & f_{41}f_{53} + f_{51}f_{43} & f_{42}f_{52} & f_{42}f_{53} + f_{52}f_{43} & f_{43}f_{53} & -g_{41}g_{51} & -g_{41}g_{52} - g_{51}g_{42} & -g_{42}g_{52} \\ f_{51}^2 & 2f_{51}f_{52} & 2f_{51}f_{53} & f_{52}^2 & 2f_{52}f_{53} & f_{53}^2 & -g_{51}^2 & -2g_{51}g_{52} & -g_{52}^2 \\ f_{61}^2 & 2f_{61}f_{62} & 2f_{61}f_{63} & f_{62}^2 & 2f_{62}f_{63} & f_{63}^2 & -g_{61}^2 & -2g_{61}g_{62} & -g_{62}^2 \end{pmatrix}.$$

A possible choice of F, G is

$$F = \begin{pmatrix} -\frac{u_{42}}{u_{41}} & -\frac{u_{43}}{u_{41}} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{u_{13}}{u_{15}} & -\frac{u_{14}}{u_{15}} \\ -\frac{u_{62}}{u_{66}} & -\frac{u_{63}}{u_{66}} & -\frac{u_{64}}{u_{66}} \end{pmatrix} = \begin{pmatrix} -\frac{\sin(\phi_4+\phi_5)}{\sin\phi_5} & -\frac{\sin\phi_4}{\sin\phi_5} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{\sin\phi_2}{\sin\phi_1} & -\frac{\sin(\phi_1+\phi_2)}{\sin\phi_1} \\ -\frac{u_{62}}{u_{66}} & -\frac{u_{63}}{u_{66}} & -\frac{u_{64}}{u_{66}} \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 \\ -\frac{u_{31}}{u_{32}} & -\frac{u_{35}}{u_{32}} \\ 0 & 0 \\ -\frac{u_{21}}{u_{24}} & -\frac{u_{25}}{u_{24}} \\ 0 & 1 \\ -\frac{u_{51}}{u_{56}} & -\frac{u_{55}}{u_{56}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\sin(\phi_3+\phi_4)}{\sin\phi_3} & -\frac{\sin\phi_4}{\sin\phi_3} \\ 0 & 0 \\ -\frac{\sin\phi_2}{\sin\phi_3} & -\frac{\sin(\phi_2+\phi_3)}{\sin\phi_3} \\ 0 & 1 \\ -\frac{\sin\phi_6}{\sin\phi_3} & -\frac{\sin(\phi_3+\phi_6)}{\sin\phi_3} \end{pmatrix}.$$

The coefficient matrix becomes

$$\begin{pmatrix} f_{11}^2 & 2f_{11}f_{12} & 0 & f_{12}^2 & 0 & 0 & -1 & 0 & 0 \\ f_{11} & f_{12} & 0 & 0 & 0 & 0 & -g_{21} & -g_{22} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -g_{21}^2 & -2g_{21}g_{22} & -g_{22}^2 \\ f_{61} & f_{62} & f_{63} & 0 & 0 & 0 & -g_{21}g_{61} & -g_{21}g_{62} - g_{61}g_{22} & -g_{22}g_{62} \\ 0 & 0 & 0 & 0 & 0 & 1 & -g_{41}^2 & -2g_{41}g_{42} & -g_{42}^2 \\ 0 & 0 & 0 & 0 & f_{52} & f_{53} & 0 & -g_{41} & -g_{42} \\ 0 & 0 & 0 & f_{52}^2 & 2f_{52}f_{53} & f_{53}^2 & 0 & 0 & -1 \\ f_{61}^2 & 2f_{61}f_{62} & 2f_{61}f_{63} & f_{62}^2 & 2f_{62}f_{63} & f_{63}^2 & -g_{61}^2 & -2g_{61}g_{62} & -g_{62}^2 \end{pmatrix}. \qquad (20)$$

The solution pair (P, Q) producing the initial matrix A is positive definite. In particular, the diagonal elements of P, Q are non-zero. We look for a second linearly independent solution. We may hence assume that for this solution $q_{22} = 0$. We need to check whether the left 8×8 submatrix of the coefficient matrix has zero determinant. We shall look for a left kernel vector x.

From the first six columns we obtain $x_4 = -2f_{61}x_8$, $x_2 = -2f_{11}x_1$, $x_3 = f_{11}^2x_1 + f_{61}^2x_8$, $x_7 = -\frac{f_{12}^2x_1+f_{62}^2x_8}{f_{52}^2}$, $x_6 = -\frac{2f_{52}f_{53}x_7+2f_{62}f_{63}x_8}{f_{52}} = \frac{2f_{53}f_{12}^2x_1+2f_{53}f_{62}^2x_8-2f_{52}f_{62}f_{63}x_8}{f_{52}^2}$, $x_5 = -f_{53}x_6 - f_{53}^2x_7 - f_{63}^2x_8 = \frac{-f_{53}^2f_{12}^2x_1-f_{53}^2f_{62}^2x_8+2f_{52}f_{53}f_{62}f_{63}x_8-f_{52}^2f_{63}^2x_8}{f_{52}^2}$. The remaining two columns give

$$x_1 + g_{21}x_2 + g_{21}^2x_3 + g_{21}g_{61}x_4 + g_{41}^2x_5 + g_{61}^2x_8 = 0,$$

$$\frac{1}{2}(g_{22}x_2 + 2g_{21}g_{22}x_3 + g_{21}g_{62}x_4 + g_{61}g_{22}x_4 + 2g_{41}g_{42}x_5 + g_{41}x_6 + 2g_{61}g_{62}x_8) = 0.$$

Collecting the coefficients at x_1 and x_8 in these expressions and multiplying by f_{52}^2 we obtain the matrix

$$\begin{pmatrix} f_{52}^2(1-g_{21}f_{11})^2-g_{41}^2f_{53}^2f_{12}^2 & f_{52}^2(g_{21}f_{61}-g_{61})^2-g_{41}^2(f_{53}f_{62}-f_{52}f_{63})^2 \\ f_{12}^2f_{53}g_{41}(1-g_{42}f_{53})-g_{22}f_{11}f_{52}^2(1-g_{21}f_{11}) & f_{52}^2(g_{21}f_{61}-g_{61})(g_{22}f_{61}-g_{62})+g_{41}(f_{53}f_{62}-f_{52}f_{63})(f_{62}-g_{42}f_{53}f_{62}+g_{42}f_{52}f_{63}) \end{pmatrix}.$$

We get

$$f_{52}^2(1-g_{21}f_{11})^2 - g_{41}^2f_{53}^2f_{12}^2 = \frac{\sin^2\phi_2\sin^2\phi_4(\sin^2(\phi_3+\phi_4+\phi_5)-\sin^2(\phi_1+\phi_2))}{\sin^2\phi_1\sin^2\phi_3\sin^2\phi_5}$$

$$f_{12}^2f_{53}g_{41}(1-g_{42}f_{53})-g_{22}f_{11}f_{52}^2(1-g_{21}f_{11}) = \frac{\sin^2\phi_2\sin^2\phi_4(\sin(\phi_3+\phi_4+\phi_5)\sin(\phi_4+\phi_5)-\sin(\phi_1+\phi_2+\phi_3)\sin(\phi_1+\phi_2))}{\sin^2\phi_1\sin^2\phi_3\sin^2\phi_5}$$

$$f_{52}^2(g_{21}f_{61}-g_{61})^2-g_{41}^2(f_{53}f_{62}-f_{52}f_{63})^2 = \frac{\sin^2\phi_2}{\sin^2\phi_1\sin^2\phi_3}\left((\sin(\phi_3+\phi_4)f_{61}-\sin\phi_6)^2-(\sin(\phi_1+\phi_2)f_{62}-\sin\phi_2f_{63})^2\right)$$

$$f_{52}^2(g_{21}f_{61}-g_{61})(g_{22}f_{61}-g_{62})+g_{41}(f_{53}f_{62}-f_{52}f_{63})(f_{62}-g_{42}f_{53}f_{62}+g_{42}f_{52}f_{63}) =$$
$$= \frac{\sin^2\phi_2((\sin(\phi_3+\phi_4)f_{61}-\sin\phi_6)(\sin\phi_4f_{61}-\sin(\phi_3+\phi_6))-(\sin(\phi_1+\phi_2)f_{62}-\sin\phi_2f_{63})(\sin(\phi_1+\phi_2+\phi_3)f_{62}-\sin(\phi_2+\phi_3)f_{63}))}{\sin^2\phi_1\sin^2\phi_3}$$

Thus the coefficient matrix is degenerate if and only if

$$\big((\sin^2(\phi_3+\phi_4+\phi_5)-\sin^2(\phi_1+\phi_2))(\sin\phi_4f_{61}-\sin(\phi_3+\phi_6))-(\sin(\phi_3+\phi_4+\phi_5)\sin(\phi_4+\phi_5)-\sin(\phi_1+\phi_2+\phi_3)\sin(\phi_1+\phi_2))(\sin(\phi_3+\phi_4)f_{61}-\sin\phi_6)\big)(\sin(\phi_3+\phi_4)f_6$$

$$=\big((\sin^2(\phi_3+\phi_4+\phi_5)-\sin^2(\phi_1+\phi_2))(\sin(\phi_1+\phi_2+\phi_3)f_{62}-\sin(\phi_2+\phi_3)f_{63})-(\sin(\phi_3+\phi_4+\phi_5)\sin(\phi_4+\phi_5)-\sin(\phi_1+\phi_2+\phi_3)\sin(\phi_1+\phi_2))(\sin(\phi_1+\phi_2)f_{62}-\sin\phi_2f$$

This simplifies to

$$((\sin(\phi_1+\phi_2+\phi_3+\phi_4)\sin(\phi_1+\phi_2)-\sin\phi_5\sin(\phi_3+\phi_4+\phi_5))f_{61}+(\sin(\phi_1+\phi_2-\phi_6)\sin(\phi_1+\phi_2)-\sin(\phi_3+\phi_4+\phi_5+\phi_6)\sin(\phi_3+\phi_4+\phi_5)))(\sin(\phi_3+\phi_4)f_{61}-\sin\phi_6)=$$

$$=(\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5)\sin(\phi_3+\phi_4+\phi_5)f_{62}+(\sin\phi_1\sin(\phi_1+\phi_2)-\sin(\phi_2+\phi_3+\phi_4+\phi_5)\sin(\phi_3+\phi_4+\phi_5))f_{63})(\sin(\phi_1+\phi_2)f_{62}-\sin\phi_2f_{63}).$$

Hence the matrix A is not extremal if and only if

$$(\sin(\phi_1+\phi_2+\phi_3+\phi_4)\sin(\phi_1+\phi_2)-\sin\phi_5\sin(\phi_3+\phi_4+\phi_5))\sin(\phi_3+\phi_4)u_{62}^2+(-\sin(\phi_3+\phi_4)\sin(\phi_1+\phi_2-\phi_6)\sin(\phi_1+\phi_2)+\sin\phi_6\sin($$

On the other hand, by (7) we have

$$u_{62}^2 - 2\cos(\phi_3+\phi_4+\phi_6)u_{62}u_{66} + u_{66}^2 = u_{63}^2 - 2\cos\phi_1u_{63}u_{64} + u_{64}^2.$$

Set $\lambda = \frac{\sin(\phi_1+\phi_2)\cos(\phi_1+\phi_2)+\sin(\phi_3+\phi_4+\phi_5)\cos(\phi_3+\phi_4+\phi_5)}{2}$ and subtract λ times this equation from the condition above. Multiplying by 2 we obtain the equivalent condition

$$\begin{pmatrix} u_{62} \\ u_{66} \end{pmatrix}^T \begin{pmatrix} -\frac{1}{2}\sin 2(\phi_1+\phi_2+\phi_3+\phi_4)-\frac{1}{2}\sin 2\phi_5 & \cos(\phi_5+\phi_6-\phi_1-\phi_2)\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) \\ \cos(\phi_5+\phi_6-\phi_1-\phi_2)\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) & -\frac{1}{2}\sin 2(\phi_3+\phi_4+\phi_5+\phi_6)-\frac{1}{2}\sin 2(\phi_1+\phi_2-\phi_6) \end{pmatrix} \begin{pmatrix} u_{62} \\ u_{66} \end{pmatrix} =$$
$$= \begin{pmatrix} u_{63} \\ u_{64} \end{pmatrix}^T \begin{pmatrix} -\frac{1}{2}\sin 2(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) & \cos(\phi_2+\phi_3+\phi_4+\phi_5)\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) \\ \cos(\phi_2+\phi_3+\phi_4+\phi_5)\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) & -\frac{1}{2}\sin 2(\phi_2+\phi_3+\phi_4+\phi_5)-\frac{1}{2}\sin 2\phi_1 \end{pmatrix} \begin{pmatrix} u_{63} \\ u_{64} \end{pmatrix}. \tag{21}$$

The difference of the left and right-hand side turns out to be of different sign at different points of the manifold of matrices A in the above parametrization. Hence there is an open dense subset of matrices which are extremal, and a submanifold, given by the equation (21) above, of matrices which are not extremal. We may express the condition entirely by the angles $\phi_1,\dots,\phi_9$ by noting that the zero u_6 has elements

$$\begin{aligned} u_{62} &= w_1 \sin^2\phi_8, \\ u_{63} &= w_1(\cos\phi_5 + \cos\phi_7\cos\phi_8) + w_5(\cos\phi_1 + \cos\phi_8\cos\phi_9), \\ u_{64} &= w_5 \sin^2\phi_8, \\ u_{66} &= w_1(\cos\phi_7 + \cos\phi_5\cos\phi_8) + w_5(\cos\phi_9 + \cos\phi_1\cos\phi_8), \end{aligned}$$

where

$$\begin{aligned} w_1 &= \sqrt{\sin^2\phi_1\sin^2\phi_8 - (\cos\phi_9 + \cos\phi_1\cos\phi_8)^2}, \\ w_5 &= \sqrt{\sin^2\phi_5\sin^2\phi_8 - (\cos\phi_7 + \cos\phi_5\cos\phi_8)^2}. \end{aligned}$$

Case $\phi_7 > \phi_3 + \phi_4 + \phi_6$, $\phi_9 = \pi + \phi_2 - \phi_6$*:* This case is obtained from the previous one by applying the symmetry. There is again an open dense subset of extremal matrices and a submanifold of non-extremal matrices, which is given by the relation

$$\begin{pmatrix} u_{64} \\ u_{66} \end{pmatrix}^T \begin{pmatrix} -\frac12\sin 2(\phi_2+\phi_3+\phi_4+\phi_5)-\frac12\sin 2\phi_1 & -\cos(-\phi_1+\phi_3+\phi_4+\phi_5+\phi_6)\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) \\ -\cos(-\phi_1+\phi_3+\phi_4+\phi_5+\phi_6)\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) & \frac12\sin 2(\phi_1+\phi_2-\phi_6)+\frac12\sin 2(\phi_3+\phi_4+\phi_5+\phi_6) \end{pmatrix} \begin{pmatrix} u_{64} \\ u_{66} \end{pmatrix} = \tag{22}$$

$$= \begin{pmatrix} u_{63} \\ u_{62} \end{pmatrix}^T \begin{pmatrix} -\frac12\sin 2(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) & \cos(\phi_1+\phi_2+\phi_3+\phi_4)\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) \\ \cos(\phi_1+\phi_2+\phi_3+\phi_4)\sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5) & -\frac12\sin 2(\phi_1+\phi_2+\phi_3+\phi_4)-\frac12\sin 2\phi_5 \end{pmatrix} \begin{pmatrix} u_{63} \\ u_{62} \end{pmatrix}.$$

Case $\phi_7 = \phi_3 + \phi_4 + \phi_6$, $\phi_9 = \pi + \phi_2 - \phi_6$*:* In this case the additional linear constraints $(Bu_2)_6 = (Bu_5)_4 = 0$ on B can be expressed as the element $(4,6)$ in the matrix GQG^T in (19) being equal to b_{46}. The coefficient matrix (20) then has an additional row corresponding to this element, namely

$$(f_{41}f_{61}, f_{41}f_{62}+f_{61}f_{42}, f_{41}f_{63}+f_{61}f_{43}, f_{42}f_{62}, f_{42}f_{63}+f_{62}f_{43}, f_{43}f_{63}, -g_{41}g_{61}, -g_{41}g_{62}-g_{61}g_{42}, -g_{42}g_{62}) =$$

$$= (0, 0, f_{61}, 0, f_{62}, f_{63}, -g_{41}g_{61}, -g_{41}g_{62} - g_{61}g_{42}, -g_{42}g_{62})$$

Consider this coefficient matrix without the rows corresponding to the elements b_{26}, b_{46}. It is given by

$$\begin{pmatrix} f_{11}^2 & 2f_{11}f_{12} & 0 & f_{12}^2 & 0 & 0 & -1 & 0 & 0 \\ f_{11} & f_{12} & 0 & 0 & 0 & 0 & -g_{21} & -g_{22} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -g_{21}^2 & -2g_{21}g_{22} & -g_{22}^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -g_{41}^2 & -2g_{41}g_{42} & -g_{42}^2 \\ 0 & 0 & 0 & 0 & f_{52} & f_{53} & 0 & -g_{41} & -g_{42} \\ 0 & 0 & 0 & f_{52}^2 & 2f_{52}f_{53} & f_{53}^2 & 0 & 0 & -1 \\ f_{61}^2 & 2f_{61}f_{62} & 2f_{61}f_{63} & f_{62}^2 & 2f_{62}f_{63} & f_{63}^2 & -g_{61}^2 & -2g_{61}g_{62} & -g_{62}^2 \end{pmatrix}. \tag{23}$$

Its rows are linearly independent. Indeed, let x be a left kernel vector of this matrix. Then the third column implies that $x_7 = 0$. Columns $1, 2, 4, 5, 6$ allow to express all elements of x as a linear function of x_1. The last three columns then after simplification yield the conditions

$$\begin{aligned} (\sin(\phi_3+\phi_4+\phi_5) - \sin(\phi_1+\phi_2))x_1 &= 0, \\ (\sin(\phi_3+\phi_4+\phi_5)\sin(\phi_4+\phi_5) - \sin(\phi_1+\phi_2+\phi_3)\sin(\phi_1+\phi_2))x_1 &= 0, \\ (\sin(\phi_4+\phi_5) - \sin(\phi_1+\phi_2+\phi_3))x_1 &= 0. \end{aligned}$$

By $\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 < \pi$ this has only the solution $x_1 = 0$ and hence $x = 0$.

It follows that A is not extremal if and only if both rows corresponding to the elements b_{26}, b_{46} can be expressed as a linear combination of the rows in (23). Equivalently, both relations (21),(22) hold.

Result

In Case 23 the extremal matrices with unit diagonal are given by

$$A = \begin{pmatrix} 1 & -\cos\phi_4 & \cos(\phi_4+\phi_5) & \cos(\phi_2+\phi_3) & -\cos\phi_3 & \cos(\phi_3+\phi_6) \\ -\cos\phi_4 & 1 & -\cos\phi_5 & a_{24} & \cos(\phi_3+\phi_4) & -\cos\phi_7 \\ \cos(\phi_4+\phi_5) & -\cos\phi_5 & 1 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & -\cos\phi_8 \\ \cos(\phi_2+\phi_3) & a_{24} & -\cos\phi_1 & 1 & -\cos\phi_2 & -\cos\phi_9 \\ -\cos\phi_3 & \cos(\phi_3+\phi_4) & \cos(\phi_1+\phi_2) & -\cos\phi_2 & 1 & -\cos\phi_6 \\ \cos(\phi_3+\phi_6) & -\cos\phi_7 & -\cos\phi_8 & -\cos\phi_9 & -\cos\phi_6 & 1 \end{pmatrix},$$

where a_{24} is given by (15) and the angles $\phi_1, \dots, \phi_9 \in (0, \pi)$ satisfy

$$\min(\phi_7 - \phi_3 - \phi_4 - \phi_6, \phi_9 - \pi + \phi_6 - \phi_2) = 0$$

and (14). If $\phi_7 = \phi_3 + \phi_4 + \phi_6$, then (21) does not hold, if $\phi_9 = \pi - \phi_6 + \phi_2$, then relation (22) does not hold. The submanifolds defined by these relations consist of non-extremal copositive matrices. The set of extremal matrices corresponding to this case is hence parameterized by 8 angles.

References

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