Case 21

The minimal zero supports are given by (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 6), (2, 4, 6), (3, 4, 6)

$$A = \begin{pmatrix} 1 & -cos(\phi_2) & -cos(\phi_1) & cos(\phi_2+\phi_3) & cos(\phi_2+\phi_6) & cos(\phi_1+\phi_4) \\ -cos(\phi_2) & 1 & cos(\phi_1+\phi_2) & -cos(\phi_3) & -cos(\phi_6) & cos(\phi_3+\phi_5) \\ -cos(\phi_1) & cos(\phi_1+\phi_2) & 1 & cos(\phi_4+\phi_5) & b_1 & -cos(\phi_4) \\ cos(\phi_2+\phi_3) & -cos(\phi_3) & cos(\phi_4+\phi_5) & 1 & b_2 & -cos(\phi_5) \\ cos(\phi_2+\phi_6) & -cos(\phi_6) & b_1 & b_2 & 1 & b_3 \\ cos(\phi_1+\phi_4) & cos(\phi_3+\phi_5) & -cos(\phi_4) & -cos(\phi_5) & b_3 & 1 \end{pmatrix}$$

The zeros are given by the columns:

$$(u_1,u_2,u_3,u_4,u_5,u_6) = \begin{pmatrix} sin(\phi_1+\phi_2) \\ sin(\phi_1) \\ sin(\phi_2) \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} sin(\phi_3) \\ sin(\phi_2+\phi_3) \\ 0 \\ sin(\phi_2) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} sin(\phi_6) \\ sin(\phi_2+\phi_6) \\ 0 \\ 0 \\ sin(\phi_2) \\ 0 \end{pmatrix} \begin{pmatrix} sin(\phi_4) \\ 0 \\ sin(\phi_1+\phi_4) \\ 0 \\ 0 \\ sin(\phi_1) \end{pmatrix} \begin{pmatrix} 0 \\ sin(\phi_5) \\ 0 \\ sin(\phi_3+\phi_5) \\ 0 \\ sin(\phi_3) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ sin(\phi_5) \\ sin(\phi_4) \\ 0 \\ sin(\phi_4+\phi_5) \end{pmatrix}$$

1. $$Au_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ sin(\phi_2)cos(\phi_1+\phi_2+\phi_3)+sin(\phi_2)cos(\phi_4+\phi_5) \\ b_1 sin(\phi_2)+sin(\phi_2)cos(\phi_1+\phi_2+\phi_6) \\ sin(\phi_1)cos(\phi_1+\phi_2+\phi_4)+sin(\phi_1)cos(\phi_3+\phi_5) \end{pmatrix} => \begin{cases} cos(\phi_1+\phi_2+\phi_3) \geq -cos(\phi_4+\phi_5) \\ b_1 \geq -cos(\phi_1+\phi_2+\phi_6) \\ cos(\phi_1+\phi_2+\phi_4) \geq -cos(\phi_3+\phi_5) \end{cases}$$

2. $$Au_2 = \begin{pmatrix} 0 \\ 0 \\ sin(\phi_2)cos(\phi_1+\phi_2+\phi_3)+sin(\phi_2)cos(\phi_4+\phi_5) \\ 0 \\ b_2 sin(\phi_2)-sin(\phi_2)cos(\phi_3-\phi_6) \\ sin(\phi_3)cos(\phi_1+\phi_4)+sin(\phi_3)cos(\phi_2+\phi_3+\phi_5) \end{pmatrix} => \begin{cases} cos(\phi_1+\phi_2+\phi_3) \geq -cos(\phi_4+\phi_5) \\ b_2 \geq cos(\phi_3-\phi_6) \\ cos(\phi_2+\phi_3+\phi_5) \geq -cos(\phi_1+\phi_4) \end{cases}$$

3. $$Au_3 = \begin{pmatrix} 0 \\ 0 \\ b_1 sin(\phi_2)+sin(\phi_2)cos(\phi_1+\phi_2+\phi_6) \\ b_2 sin(\phi_2)-sin(\phi_2)cos(\phi_3-\phi_6) \\ 0 \\ b_3 sin(\phi_2)+sin(\phi_6)cos(\phi_1+\phi_4)+sin(\phi_2+\phi_6)cos(\phi_3+\phi_5) \end{pmatrix}$$
$$=> \begin{cases} b_1 \geq -cos(\phi_1+\phi_2+\phi_6) \\ b_2 \geq cos(\phi_3-\phi_6) \\ b_3 sin(\phi_2)+sin(\phi_6)cos(\phi_1+\phi_4)+sin(\phi_2+\phi_6)cos(\phi_3+\phi_5) \geq 0 \end{cases}$$

4. $$Au_4 = \begin{pmatrix} 0 \\ sin(\phi_1)cos(\phi_1+\phi_2+\phi_4)+sin(\phi_1)cos(\phi_3+\phi_5) \\ 0 \\ sin(\phi_4)cos(\phi_2+\phi_3)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_5) \\ b_3 sin(\phi_1)+b_1 sin(\phi_1+\phi_4)+sin(\phi_4)cos(\phi_2+\phi_6) \\ 0 \end{pmatrix}$$
$$=> \begin{cases} cos(\phi_1+\phi_2+\phi_4) \geq -cos(\phi_3+\phi_5) \\ cos(\phi_2+\phi_3) \geq -cos(\phi_1+\phi_4+\phi_5) \\ b_3 sin(\phi_1)+b_1 sin(\phi_1+\phi_4)+sin(\phi_4)cos(\phi_2+\phi_6) \geq 0 \end{cases}$$

5. $$Au_5 = \begin{pmatrix} sin(\phi_3)cos(\phi_1+\phi_4)+sin(\phi_3)cos(\phi_2+\phi_3+\phi_5) \\ 0 \\ sin(\phi_5)cos(\phi_1+\phi_2)+sin(\phi_5)cos(\phi_3+\phi_4+\phi_5) \\ 0 \\ b_3 sin(\phi_3)+b_2 sin(\phi_3+\phi_5)-sin(\phi_5)cos(\phi_6) \\ 0 \end{pmatrix} => \begin{cases} cos(\phi_1+\phi_4) \geq -cos(\phi_2+\phi_3+\phi_5) \\ cos(\phi_1+\phi_2) \geq -cos(\phi_3+\phi_4+\phi_5) \\ b_3 sin(\phi_3)+b_2 sin(\phi_3+\phi_5)-sin(\phi_5)cos(\phi_6) \geq 0 \end{cases}$$

6. $Au_6 = \begin{pmatrix} sin(\phi_4)cos(\phi_2+\phi_3) + sin(\phi_4)cos(\phi_1+\phi_4+\phi_5) \\ sin(\phi_5)cos(\phi_1+\phi_2) + sin(\phi_5)cos(\phi_3+\phi_4+\phi_5) \\ 0 \\ 0 \\ b_2 sin(\phi_4) + b_1 sin(\phi_5) + b_3 sin(\phi_4+\phi_5) \\ 0 \end{pmatrix} => \begin{cases} cos(\phi_1+\phi_4+\phi_5) \geq -cos(\phi_2+\phi_3) \\ cos(\phi_3+\phi_4+\phi_5) \geq -cos(\phi_1+\phi_2) \\ b_2 sin(\phi_4) + b_1 sin(\phi_5) + b_3 sin(\phi_4+\phi_5) \geq 0 \end{cases}$

Consider inequalities on ϕ_i :
Because of cyclic structure of zeros 1,2,4,5,6: $\phi_1+\phi_2+\phi_3+\phi_4+\phi_5 < \pi$

Copositivity:

Sets $(1,3,4), (2,3,4), (1,3,5), (2,3,5), (1,4,5), (2,4,5), (3,4,5), (1,3,4,5), (2,3,4,5), (1,2,6),$ $(2,3,6), (1,4,6), (1,5,6), (2,5,6), (3,5,6), (2,3,5,6), (4,5,6), (1,4,5,6)$ have to be considered.

For above domain of parameters copositive inequalities for sets $(1,3,4), (2,3,4), (1,2,6), (2,3,6), (1,4,6)$ holds true.

Consider sets, that relates to b_i :
if $\phi_1+\phi_2+\phi_6 \leq \pi$

1. $I = (1,3,5) : b_1 \geq -cos(\phi_1+\phi_2+\phi_6) => \phi_1+\phi_2+\phi_6+\pi-\phi_2-\phi_6+\phi_1 \geq \pi$ holds true strictly
2. $I = (2,3,5) : b_1 \geq -cos(\phi_1+\phi_2+\phi_6) => \phi_1+\phi_2+\phi_6+\pi-\phi_1-\phi_2+\phi_6 \geq \pi$ holds true strictly
3. $I = (1,4,5) : b_2 \geq cos(\phi_3-\phi_6) => \pi-|\phi_3-\phi_6|+\pi-\phi_2-\phi_6+\pi-\phi_2-\phi_3 \geq \pi$ holds true strictly
4. $I = (2,4,5) : b_2 \geq cos(\phi_3-\phi_6) => \phi_3+\phi_6+\pi-|\phi_3-\phi_6| \geq \pi$ holds true strictly
5. $I = (3,4,5) : \phi_1+\phi_2+\phi_6+\pi-|\phi_3-\phi_6|+\pi-\phi_4-\phi_5 \geq \pi$
   $\phi_3 \geq \phi_6 : \phi_1+\phi_2+\phi_6+\pi-\phi_3+\phi_6-\phi_4-\phi_5 \geq 0, 0 \leq \phi_1+\phi_2+\phi_6+\pi-\phi_3+\phi_6-\phi_4-\phi_5 \leq \phi_1+\phi_2+\phi_6+\pi-\phi_4-\phi_5 \leq \phi_1+\phi_2+\phi_3+\pi-\phi_4-\phi_5, \phi_1+\phi_2+\phi_3+\phi_4+\phi_5 < \pi$
   $\phi_6 \geq \phi_3 : \phi_1+\phi_2+\phi_6+\pi+\phi_3-\phi_6-\phi_4-\phi_5 \geq 0, \phi_1+\phi_2+\phi_3+\phi_4+\phi_5 < \pi$ holds true strictly

Similar inequalities hold true if $\phi_1+\phi_2+\phi_6 \geq \pi$
Consider inequality conditions on b_3:

1. $I = (1,5,6) : b_3 = cos(\phi), \pi-\phi_2-\phi_6+\pi-\phi_1-\phi_4 \geq \phi =>$ if $\phi_2+\phi_6+\phi_1+\phi_4 \geq \pi : cos(\phi) \geq cos(\phi_2+\phi_6+\phi_1+\phi_4)$
   if $\phi_2+\phi_6+\phi_1+\phi_4 \leq \pi$ holds true strictly
2. $I = (3,5,6) : b_3 = cos(\phi), b_1 = -cos(\phi_1+\phi_2+\phi_6), \phi_2+\phi_6+\phi_1+\phi_4 \geq \phi, \phi_2+\phi_6+\phi_1 \leq \pi =>$ if $\phi_2+\phi_6+\phi_1+\phi_4 \leq \pi : b_3 = cos(\phi) \geq cos(\phi_2+\phi_6+\phi_1+\phi_4)$
   if $\phi_2+\phi_6+\phi_1+\phi_4 \geq \pi$ holds true strictly
3. $I = (4,5,6) : b_3 = cos(\phi), b_2 = cos(\phi_3-\phi_6), \pi-|\phi_3-\phi_6|+\phi_5 \geq \phi =>$ if $\phi_6 \geq \phi_3+\phi_5 : b_3 = cos(\phi) \geq -cos(\phi_3+\phi_5-\phi_6)$
   if $\phi_6 \leq \phi_3+\phi_5$ holds true strictly
4. $I = (2,5,6) : b_3 = cos(\phi), \pi-\phi_3-\phi_5+\phi_6 \geq \phi =>$ if $\phi_6 \leq \phi_3+\phi_5 : b_3 = cos(\phi) \geq -cos(\phi_3+\phi_5-\phi_6)$
   if $\phi_6 \geq \phi_3+\phi_5$ holds true strictly

It was gotten , that $b_3 \geq -cos(\phi_3+\phi_5-\phi_6), b_3 \geq cos(\phi_2+\phi_6+\phi_1+\phi_4)$

1. If $b_3 = \frac{-sin(\phi_2+\phi_6)cos(\phi_3+\phi_5)-sin(\phi_6)cos(\phi_1+\phi_4)}{sin(\phi_2)} \geq cos(\phi_2+\phi_6+\phi_1+\phi_4)$ from inequality 3.3
   $-cos(\phi_1+\phi_2+\phi_4) \geq cos(\phi_3+\phi_5) => \phi_1+\phi_2+\phi_3+\phi_4+\phi_5 \geq \pi$ - holds only if it's an equation
2. If $b_3 = \frac{-sin(\phi_4)cos(\phi_2+\phi_6)-b_1 sin(\phi_1+\phi_4)}{sin(\phi_1)} \geq cos(\phi_2+\phi_6+\phi_1+\phi_4)$ from inequality 4.3
   $-cos(\phi_1+\phi_2+\phi_6) \geq b_1$ - holds only if it's an equation
3. If $b_3 = \frac{sin(\phi_5)cos(\phi_6)-b_2 sin(\phi_3+\phi_5)}{sin(\phi_3)} \geq -cos(\phi_3+\phi_5-\phi_6)$ from inequality 4.3
   $cos(\phi_3-\phi_6) \geq b_2$ - holds only if it's an equation

It follows, that $b_3 = \frac{-b_2 sin(\phi_4) - b_1 sin(\phi_5)}{sin(\phi_4+\phi_5)}$ from inequality 6.3 $b_3 \geq -cos(\phi_3+\phi_5-\phi_6) => -b_1 sin(\phi_5) \geq -cos(\phi_3+\phi_5-\phi_6)sin(\phi_4+\phi_5) + cos(\phi_3-\phi_6)sin(\phi_4) = -sin(\phi_5)cos(\phi_4+\phi_5+\phi_3-\phi_6) => -cos(\phi_1+\phi_2+\phi_6) \leq cos(\phi_4+\phi_5+\phi_3-\phi_6)$
Consider domain of ϕ_i under gotten conditions:

1. $\pi \geq \phi_1+\phi_4+\phi_5+\phi_3+\phi_2, \phi_4+\phi_5+\phi_3 \geq \phi_6, \phi_1+\phi_2+\phi_6 \leq \pi$
2. $\pi \leq \phi_1+2\phi_6-\phi_5-\phi_3-\phi_4+\phi_2, \phi_4+\phi_5+\phi_3 \geq \phi_6, \phi_1+\phi_2+\phi_6 \geq \pi$ doesn't hold
3. $\pi \leq \phi_1+\phi_4+\phi_5+\phi_3+\phi_2, \phi_4+\phi_5+\phi_3 \leq \phi_6, \phi_1+\phi_2+\phi_6 \geq \pi$ doesn't hold
4. $\pi \geq \phi_1+2\phi_6-\phi_5-\phi_3-\phi_4+\phi_2, \phi_4+\phi_5+\phi_3 \leq \phi_6, \phi_1+\phi_2+\phi_6 \leq \pi$

$b_3 \geq cos(\phi_1+\phi_2+\phi_6+\phi_4) => -b_2 sin(\phi_4) \geq cos(\phi_1+\phi_2+\phi_6+\phi_4)sin(\phi_4+\phi_5) - cos(\phi_1+\phi_2+\phi_6)sin(\phi_5) = sin(\phi_4)cos(\phi_1+\phi_2+\phi_4+\phi_5+\phi_6) => -cos(\phi_1+\phi_2+\phi_4+\phi_5+\phi_6) \geq cos(\phi_3-\phi_6)$
Consider domain of ϕ_i under gotten conditions:

1. $\pi \leq \phi_1+\phi_4+\phi_5+\phi_3+\phi_2, \phi_3 \geq \phi_6$ doesn't hold
2. $\pi \leq \phi_1+\phi_4+\phi_5-\phi_3+\phi_2+2\phi_6, \phi_3 \leq \phi_6, \pi \geq \phi_1+\phi_2+\phi_4+\phi_5+\phi_6$
3. $\pi \geq \phi_1+\phi_4+\phi_5+\phi_3+\phi_2, \phi_3 \leq \phi_6, \pi \leq \phi_1+\phi_2+\phi_4+\phi_5+\phi_6$

Finally, consider 4-elem sets:

1. $I = (1,3,4,5) : u = e_1 + e_3$
2. $I = (2,3,4,5) : u = e_2 + e_4$
3. $I = (1,4,5,6) : u = sin(\phi_6-\phi_3)e_1 - sin(\phi_2+\phi_6)e_4 + sin(\phi_2+\phi_3)e_5 =>$

   $$Au = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \end{pmatrix},$$

   $a = sin(\phi_4+\phi_5)[sin(\phi_6-\phi_3)cos(\phi_1+\phi_4) + sin(\phi_2+\phi_6)cos(\phi_5)] + sin(\phi_2+\phi_3)[cos(\phi_1+\phi_2+\phi_6)sin(\phi_5) - cos(\phi_6-\phi_3)sin(\phi_4)]$,
   $cos(\phi_1+\phi_2+\phi_4+\phi_5+\phi_6) \leq -cos(\phi_3-\phi_6) => a \geq sin(\phi_4+\phi_5)[sin(\phi_6-\phi_3)cos(\phi_1+\phi_4) + sin(\phi_2+\phi_6)cos(\phi_5) + cos(\phi_1+\phi_2+\phi_4+\phi_6)sin(\phi_2+\phi_3)] = sin(\phi_4+\phi_5)sin(\phi_2+\phi_6)[cos(\phi_1+\phi_2+\phi_3+\phi_4) + cos(\phi_5)] \geq 0, \phi_1+\phi_2+\phi_3+\phi_4+\phi_5 \leq \pi$

There is the symmetry, acting as $\phi_2 <-> \phi_2, \phi_1 <-> \phi_3, \phi_4 <-> \phi_5, \phi_6 <-> \pi-\phi_2-\phi_6$, as result $1 <-> 2, 3 <-> 4, 5 <-> 5, 6 <-> 6 => 1456 <-> 2356$.
$b_1 = -cos(\phi_1+\phi_2+\phi_6), b_2 = cos(\phi_3-\phi_6), b_3 = \frac{-cos(\phi_3-\phi_6)sin(\phi_4)+cos(\phi_1+\phi_2+\phi_6)sin(\phi_5)}{sin(\phi_4+\phi_5)}$

Extremality

The submatrix A_{12346} is isomorphic to a T-matrix. Any B in the face of A is hence such that B_{12346} is proportional to A_{12346}. In particular, only the diagonal element B_{55} may differ from the other diagonal elements of B. But either of the zeros u_3 ensures that B_{55} equals the other diagonal elements. Hence B is proportional to A and A is extremal.

Result

There are extremal copositive matrices under conditions on parameters $\phi_i : \phi_1+\phi_2+\phi_3+\phi_4+\phi_5 < \pi$,
$\pi-\phi_1-\phi_4-\phi_5+\phi_3-\phi_2 < 2\phi_6 < \pi-\phi_1+\phi_5+\phi_3+\phi_4-\phi_2$

References

[1] Peter J.C. Dickinson. *A new certificate for copositivity. Optimization Online.*

[2] Peter J.C. Dickinson and Roland Hildebrand. *Considering copositivity locally.* J.Math. Anal.Appl., 437(2):1184–1195, 2016

.