Case 2

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The supports of the minimal zeros are given by $\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,6\},\{4,5,6\}$. This determines a number of off-diagonal elements, leading to matrices of the form

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & * \\ -1 & 1 & 1 & 1 & * & -1 \\ -1 & 1 & 1 & 1 & -\cos\phi_1 & -\cos\phi_2 \\ 1 & -1 & * & -\cos\phi_1 & 1 & -\cos\phi_3 \\ 1 & * & -1 & -\cos\phi_2 & -\cos\phi_3 & 1 \end{pmatrix}, \qquad \phi_1+\phi_2+\phi_3 = \pi.$$

The asterisk denotes still unknown elements. Here $\phi_k > 0$, and the minimal zero with support $\{4,5,6\}$ is given by $(0,0,0,\sin\phi_3,\sin\phi_2,\sin\phi_1)^T$.

For the pairs $(i,j) = (3,5)$ and $(i,j) = (2,6)$ there does not exist a minimal zero such that $u_iu_j > 0$. Hence there must exist a minimal zero such that $u_i + u_j > 0$ and $(Au)_i = (Au)_j = 0$. For $(i,j) = (3,5)$ this can be the zero with support $\{3,6\}$, in which case $A_{35} = \cos\phi_3$, or the zero with support $\{4,5,6\}$, in which case $A_{35} = \frac{\sin\phi_1 - \sin(\phi_1+\phi_2)}{\sin\phi_2}$. For $(i,j) = (2,6)$ this can be the zero with support $\{2,5\}$, in which case $A_{26} = \cos\phi_3$, or the zero with support $\{4,5,6\}$, in which case $A_{26} = \frac{\sin\phi_2 - \sin(\phi_1+\phi_2)}{\sin\phi_1}$. For the other zeros we would get $A_{ij} = -1$, which leads to an additional minimal zero with support $\{i,j\}$ and is hence not possible.

For angles $\xi,\zeta \geq 0$, $\xi+\zeta \leq \pi$ we have $(\sin\xi\cos\zeta + \cos\xi\sin\zeta)(\cos\zeta - 1) \leq 0$ and hence $\sin\xi - \sin(\xi+\zeta) \leq -\sin\zeta\cos(\xi+\zeta)$. It follows that

$$\begin{aligned} A_{35} &= \max(-\cos(\phi_1+\phi_2), \frac{\sin\phi_1 - \sin(\phi_1+\phi_2)}{\sin\phi_2}) = -\cos(\phi_1+\phi_2), \\ A_{26} &= \max(-\cos(\phi_1+\phi_2), \frac{\sin\phi_2 - \sin(\phi_1+\phi_2)}{\sin\phi_1}) = -\cos(\phi_1+\phi_2). \end{aligned}$$

We obtain

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & \cos\phi_3 \\ -1 & 1 & 1 & 1 & \cos\phi_3 & -1 \\ -1 & 1 & 1 & 1 & -\cos\phi_1 & -\cos\phi_2 \\ 1 & -1 & \cos\phi_3 & -\cos\phi_1 & 1 & -\cos\phi_3 \\ 1 & \cos\phi_3 & -1 & -\cos\phi_2 & -\cos\phi_3 & 1 \end{pmatrix}, \qquad \phi_k > 0,\ \phi_1+\phi_2+\phi_3 = \pi.$$

All principal submatrices of A not containing an off-diagonal 1 are positive semi-definite. Hence A is copositive by the criterion of Hoffman and Pereira for all ϕ_k.

Let B be a matrix in the face of A. Then the zero with support $\{1,2\}$ yields $B_{1i} + B_{2i} = 0$ for $i = 1,\dots,5$. Similarly, the zero with support $\{1,3\}$ yields $B_{1i} + B_{3i} = 0$ for $i = 1,2,3,4,6$, the zero with support $\{1,4\}$ yields $B_{1i} + B_{4i} = 0$ for $i = 1,2,3,4$, the zero with support $\{2,5\}$ yields $B_{2i} + B_{5i} = 0$ for $i = 1,2,5,6$, the zero with support $\{3,6\}$ yields $B_{3i} + B_{6i} = 0$ for $i = 1,3,5,6$, and the zero $u = (0,0,0,\sin\phi_3,\sin\phi_2,\sin\phi_1)^T$ yields $(Bu)_i = 0$ for $i = 4,5,6$. The only solution of this system of linear equations is A, up to multiplication by a constant. Hence A is also extremal for all $\phi_k > 0$, $\phi_1+\phi_2+\phi_3 = \pi$.