

## Case 16

The minimal zero supports are given by (1,2),(1,3,4),(1,3,5),(2,4,6),(3,5,6),(4,5,6).

$$A = \begin{pmatrix} 1 & -1 & -\cos(\phi_2) & \cos(\phi_1 + \phi_2) & \cos(\phi_2 + \phi_3) & b_3 \\ -1 & 1 & b_1 & \cos(\phi_4 + \phi_5) & b_2 & -\cos(\phi_5) \\ -\cos(\phi_2) & b_1 & 1 & -\cos(\phi_1) & -\cos(\phi_3) & \cos(\phi_3 + \phi_6) \\ \cos(\phi_1 + \phi_2) & \cos(\phi_4 + \phi_5) & -\cos(\phi_1) & 1 & \cos(\phi_4 + \phi_6) & -\cos(\phi_4) \\ \cos(\phi_2 + \phi_3) & b_2 & -\cos(\phi_3) & \cos(\phi_4 + \phi_6) & 1 & -\cos(\phi_6) \\ b_3 & -\cos(\phi_5) & \cos(\phi_3 + \phi_6) & -\cos(\phi_4) & -\cos(\phi_6) & 1 \end{pmatrix}$$

The zeros are given by the columns:

$$(u_1, u_2, u_3, u_4, u_5, u_6) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_1) \\ 0 \\ \sin(\phi_1 + \phi_2) \\ \sin(\phi_2) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_3) \\ 0 \\ \sin(\phi_2 + \phi_3) \\ 0 \\ \sin(\phi_2) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_4) \\ 0 \\ \sin(\phi_5) \\ 0 \\ \sin(\phi_4 + \phi_5) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_6) \\ 0 \\ \sin(\phi_3 + \phi_6) \\ \sin(\phi_3) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(\phi_6) \\ \sin(\phi_4) \\ \sin(\phi_4 + \phi_6) \end{pmatrix}$$

$$1. Au_1 = \begin{pmatrix} 0 \\ 0 \\ b_1 - \cos(\phi_2) \\ \cos(\phi_1 + \phi_2) + \cos(\phi_4 + \phi_5) \\ b_2 + \cos(\phi_2 + \phi_3) \\ b_3 - \cos(\phi_5) \end{pmatrix} \Rightarrow \begin{cases} b_1 \geq \cos(\phi_2) \\ \cos(\phi_1 + \phi_2) + \cos(\phi_4 + \phi_5) \geq 0 \\ b_2 \geq -\cos(\phi_2 + \phi_3) \\ b_3 \geq \cos(\phi_5) \end{cases}$$

$$2. Au_2 = \begin{pmatrix} 0 \\ -\sin(\phi_1) + b_1 \sin(\phi_1 + \phi_2) + \sin(\phi_2) \cos(\phi_4 + \phi_5) \\ 0 \\ 0 \\ -\sin(\phi_2) \cos(\phi_1 - \phi_3) + \sin(\phi_2) \cos(\phi_4 + \phi_6) \\ b_3 \sin(\phi_1) - \sin(\phi_2) \cos(\phi_4) + \sin(\phi_1 + \phi_2) \cos(\phi_3 + \phi_6) \end{pmatrix} \Rightarrow \begin{cases} -\sin(\phi_1) + b_1 \sin(\phi_1 + \phi_2) + \sin(\phi_2) \cos(\phi_4 + \phi_5) \geq 0 \\ -\sin(\phi_2) \cos(\phi_1 - \phi_3) + \sin(\phi_2) \cos(\phi_4 + \phi_6) \geq 0 \\ b_3 \sin(\phi_1) - \sin(\phi_2) \cos(\phi_4) + \sin(\phi_1 + \phi_2) \cos(\phi_3 + \phi_6) \geq 0 \end{cases}$$

$$3. Au_3 = \begin{pmatrix} 0 \\ b_2 * \sin(\phi_2) - \sin(\phi_3) + b_1 * \sin(\phi_2 + \phi_3) \\ 0 \\ -\sin(\phi_2) \cos(\phi_1 - \phi_3) + \sin(\phi_2) \cos(\phi_4 + \phi_6) \\ 0 \\ b_3 \sin(\phi_3) + \sin(\phi_3) \cos(\phi_2 + \phi_3 + \phi_6) \end{pmatrix} \Rightarrow \begin{cases} b_2 * \sin(\phi_2) - \sin(\phi_3) + b_1 * \sin(\phi_2 + \phi_3) \geq 0 \\ -\sin(\phi_2) \cos(\phi_1 - \phi_3) + \sin(\phi_2) \cos(\phi_4 + \phi_6) \geq 0 \\ b_3 \geq -\cos(\phi_2 + \phi_3 + \phi_6) \end{cases}$$

$$4. Au_4 = \begin{pmatrix} -\sin(\phi_4) + b_3 \sin(\phi_4 + \phi_5) + \sin(\phi_5) \cos(\phi_1 + \phi_2) \\ 0 \\ b_1 \sin(\phi_4) - \sin(\phi_5) \cos(\phi_1) + \sin(\phi_4 + \phi_5) \cos(\phi_3 + \phi_6) \\ 0 \\ b_2 \sin(\phi_4) - \sin(\phi_4) \cos(\phi_5 - \phi_6) \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -\sin(\phi_4) + b_3 \sin(\phi_4 + \phi_5) + \sin(\phi_5) \cos(\phi_1 + \phi_2) \geq 0 \\ b_1 \sin(\phi_4) - \sin(\phi_5) \cos(\phi_1) + \sin(\phi_4 + \phi_5) \cos(\phi_3 + \phi_6) \geq 0 \\ b_2 \sin(\phi_4) - \sin(\phi_4) \cos(\phi_5 - \phi_6) \geq 0 \end{cases}$$

$$5. Au_5 = \begin{pmatrix} b_3 \sin(\phi_3) + \sin(\phi_3) \cos(\phi_2 + \phi_3 + \phi_6) \\ b_1 \sin(\phi_6) + b_2 \sin(\phi_3 + \phi_6) - \sin(\phi_3) \cos(\phi_5) \\ 0 \\ -\sin(\phi_6) \cos(\phi_1) + \sin(\phi_6) \cos(\phi_3 + \phi_4 + \phi_6) \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} b_3 \sin(\phi_3) + \sin(\phi_3) \cos(\phi_2 + \phi_3 + \phi_6) \geq 0 \\ b_1 \sin(\phi_6) + b_2 \sin(\phi_3 + \phi_6) - \sin(\phi_3) \cos(\phi_5) \geq 0 \\ -\sin(\phi_6) \cos(\phi_1) + \sin(\phi_6) \cos(\phi_3 + \phi_4 + \phi_6) \geq 0 \end{cases}$$

$$6. Au_6 = \begin{pmatrix} \sin(\phi_4) \cos(\phi_2 + \phi_3) + b_3 \sin(\phi_4 + \phi_6) + \sin(\phi_6) \cos(\phi_1 + \phi_2) \\ b_2 \sin(\phi_4) - \sin(\phi_4) \cos(\phi_5 - \phi_6) \\ -\sin(\phi_6) \cos(\phi_1) + \sin(\phi_6) \cos(\phi_3 + \phi_4 + \phi_6) \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \sin(\phi_4) \cos(\phi_2 + \phi_3) + b_3 \sin(\phi_4 + \phi_6) + \sin(\phi_6) \cos(\phi_1 + \phi_2) \geq 0 \\ b_2 \sin(\phi_4) - \sin(\phi_4) \cos(\phi_5 - \phi_6) \geq 0 \\ -\sin(\phi_6) \cos(\phi_1) + \sin(\phi_6) \cos(\phi_3 + \phi_4 + \phi_6) \geq 0 \end{cases}$$

Considering inequalities on  $\phi_i$  we get: 
$$\begin{cases} \phi_1 + \phi_2 + \phi_4 + \phi_5 \leq \pi \\ \phi_4 + \phi_6 \leq |\phi_1 - \phi_3| \\ \phi_3 + \phi_4 + \phi_6 \leq \phi_1 \end{cases} \Rightarrow \begin{cases} \phi_1 + \phi_2 + \phi_4 + \phi_5 \leq \pi \\ \phi_3 + \phi_4 + \phi_6 \leq \phi_1 \end{cases}$$

Case  $\phi_3 + \phi_4 + \phi_6 \geq \pi$  is unreal

$$b_1 = \cos(\phi_2),$$

$$b_2 = \cos(\phi_5 - \phi_6)$$

$$b_3 = \cos(\phi_5)$$

Other inequalities are fulfilled.

### Parametrization:

$$A = \begin{pmatrix} 1 & -1 & -\cos(\phi_2) & \cos(\phi_1 + \phi_2) & \cos(\phi_2 + \phi_3) & \cos(\phi_5) \\ -1 & 1 & \cos(\phi_2) & \cos(\phi_4 + \phi_5) & \cos(\phi_5 - \phi_6) & -\cos(\phi_5) \\ -\cos(\phi_2) & \cos(\phi_2) & 1 & -\cos(\phi_1) & -\cos(\phi_3) & \cos(\phi_3 + \phi_6) \\ \cos(\phi_1 + \phi_2) & \cos(\phi_4 + \phi_5) & -\cos(\phi_1) & 1 & \cos(\phi_4 + \phi_6) & -\cos(\phi_4) \\ \cos(\phi_2 + \phi_3) & \cos(\phi_5 - \phi_6) & -\cos(\phi_3) & \cos(\phi_4 + \phi_6) & 1 & -\cos(\phi_6) \\ \cos(\phi_5) & -\cos(\phi_5) & \cos(\phi_3 + \phi_6) & -\cos(\phi_4) & -\cos(\phi_6) & 1 \end{pmatrix}$$

$\phi_1 + \phi_2 + \phi_4 + \phi_5 \leq \pi, \phi_3 + \phi_4 + \phi_6 \leq \phi_1$

### Copositivity:

It's necessary to consider next sets for I:

$$(2, 3, 4), (2, 3, 5), (1, 4, 5), (2, 4, 5), (3, 4, 5), (2, 3, 4, 5), (1, 3, 6), (2, 3, 6), (1, 4, 6), (1, 5, 6), (2, 5, 6), (3, 4, 6)$$

1.  $I = (1, 5, 6) : b_3 \geq -\cos(\phi_2 + \phi_3 + \phi_6) \Rightarrow \phi_2 + \phi_3 + \phi_6 + \pi - \phi_2 - \phi_3 + \phi_6 \geq \pi \Rightarrow \phi_2 + \phi_3 + \phi_6 \geq \phi_2 + \phi_3 - \phi_6$  performs strictly
2.  $I = (1, 4, 6) : b_3 \geq \cos(\phi_5) \Rightarrow \phi_5 + \phi_1 + \phi_2 - \phi_4 \leq \pi$  performs strictly
3.  $I = (2, 3, 4) : b_2 = \cos(\phi_5 - \phi_6) \Rightarrow \phi_5 + \phi_4 + |\phi_5 - \phi_6| - \phi_1 \leq \pi$  performs strictly
4.  $I = (3, 4, 6) : \phi_1 + \phi_4 \geq \phi_3 + \phi_6$  performs strictly
5.  $I = (2, 5, 6) : b_2 = \cos(\phi_5 - \phi_6) \Rightarrow \phi_6 + \phi_5 \geq |\phi_5 - \phi_6|$  performs strictly
6.  $I = (1, 3, 6) : b_3 \geq -\cos(\phi_2 + \phi_3 + \phi_6) \Rightarrow \phi_2 + \phi_3 + \phi_6 + \phi_2 - \phi_3 - \phi_6 \geq 0 \Rightarrow \phi_2 \geq 0$  performs strictly
7.  $I = (2, 3, 6) : -\phi_2 + \phi_5 + \pi - \phi_3 - \phi_6 \geq 0$  performs strictly
8.  $I = (1, 4, 5) : -\phi_1 - \phi_2 + \pi - \phi_2 - \phi_3 + \pi - \phi_4 - \phi_6 \geq 0$  performs strictly
9.  $I = (2, 3, 4, 5), (3, 4, 5)(2, 3, 5) : u = e_3 + e_5, b_2 = \cos(\phi_5 - \phi_6) \Rightarrow \cos(\phi_2) + \cos(\phi_5 - \phi_6) \geq 0 \Rightarrow \phi_2 \leq \pi - |\phi_5 - \phi_6|$   
 $-\cos(\phi_1) + \cos(\phi_4 + \phi_6) \geq 0 \Rightarrow \phi_1 \geq \phi_4 + \phi_6$  performs strictly
10.  $I = (2, 4, 5) : b_2 = \cos(\phi_5 - \phi_6) \Rightarrow \pi - \phi_4 - \phi_5 + \pi - |\phi_5 - \phi_6| - \phi_4 - \phi_6 \geq 0 \Rightarrow 2\pi \geq 2\phi_4 + \phi_6 + |\phi_5 - \phi_6| + \phi_5$  performs strictly

That proves copositivity.

### Extremality:

$$X = FPF^T$$

$$X = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & * & * \\ b_{31} & * & b_{33} & b_{34} \\ b_{41} & * & b_{43} & b_{44} \end{pmatrix}, F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}, \begin{cases} u_{21}f_1 + u_{23}f_3 + u_{24}f_4 = 0 \\ f_1 = -f_2 \end{cases}, F = \begin{pmatrix} \sin(\phi_2) & 0 \\ -\sin(\phi_2) & 0 \\ 0 & \sin(\phi_2) \\ -\sin(\phi_1) & -\sin(\phi_1 + \phi_2) \end{pmatrix}$$

$$Y = GQG^T$$

$$Y = \begin{pmatrix} b_{11} & * & b_{13} & * & b_{15} & * \\ * & b_{22} & * & b_{24} & * & b_{26} \\ b_{31} & * & b_{33} & * & b_{35} & b_{36} \\ * & b_{42} & * & b_{44} & b_{45} & b_{46} \\ b_{51} & * & b_{53} & b_{54} & b_{55} & b_{56} \\ * & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{pmatrix}, G = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{pmatrix}, \begin{cases} u_{31}g_1 + u_{33}g_3 + u_{35}g_5 = 0 \\ u_{53}g_3 + u_{55}g_5 + u_{56}g_6 = 0 \\ u_{64}g_4 + u_{65}g_5 + u_{66}g_6 = 0 \\ u_{42}g_2 + u_{44}g_4 + u_{46}g_6 = 0 \end{cases},$$

$$G = \begin{pmatrix} \sin(\phi_2) & 0 \\ -\sin(\phi_3 - \phi_5 + \phi_6) & -\sin(\phi_2 + \phi_3 - \phi_5 + \phi_6) \\ 0 & \sin(\phi_2) \\ -\sin(\phi_3 + \phi_4 + \phi_6) & -\sin(\phi_2 + \phi_3 + \phi_4 + \phi_6) \\ -\sin(\phi_3) & -\sin(\phi_2 + \phi_3) \\ \sin(\phi_3 + \phi_6) & \sin(\phi_2 + \phi_3 + \phi_6) \end{pmatrix}$$

Necessary equalities, that are constrains on elements of matrices P and Q:

$$\begin{cases} x_{11} = y_{11} \\ x_{22} = y_{22} \\ x_{33} = y_{33} \\ x_{31} = y_{31} \\ x_{44} = y_{44} \end{cases}$$

$$\Rightarrow \begin{cases} p_{11} = q_{11} \\ \sin^2(\phi_2)p_{11} = \sin^2(\phi_3 - \phi_5 + \phi_6)q_{11} + 2\sin(\phi_3 - \phi_5 + \phi_6)\sin(\phi_2 + \phi_3 - \phi_5 + \phi_6)q_{21} + \sin^2(\phi_2 + \phi_3 - \phi_5 + \phi_6)q_{22} \\ p_{22} = q_{22} \\ p_{21} = q_{21} \\ \sin^2(\phi_3 + \phi_4 + \phi_6)p_{11} + 2\sin(\phi_3 + \phi_4 + \phi_6)\sin(\phi_2 + \phi_3 + \phi_4 + \phi_6)p_{12} + \sin^2(\phi_2 + \phi_3 + \phi_4 + \phi_6)p_{22} = \sin^2(\phi_1)q_{11} + \\ + 2\sin(\phi_1)\sin(\phi_1 + \phi_2)q_{12} + \sin^2(\phi_1 + \phi_2)q_{22} \end{cases}$$

Let  $b_{11} = b_{22} = 0 \Rightarrow p_{11} = q_{11} = 0$

$$\begin{pmatrix} 2\sin(\phi_3 - \phi_5 + \phi_6)\sin(\phi_2 + \phi_3 - \phi_5 + \phi_6) & \sin^2(\phi_2 + \phi_3 - \phi_5 + \phi_6) \\ -2\sin(\phi_3 + \phi_4 + \phi_6 - \phi_1)\sin(\phi_3 + \phi_4 + \phi_6 + \phi_1 + \phi_2) & -\sin(\phi_3 + \phi_4 + \phi_6 - \phi_1)\sin(\phi_3 + \phi_4 + \phi_6 + \phi_1 + 2\phi_2) \end{pmatrix}$$

There is no extremality in common case, when :  $\sin(\phi_2 + \phi_3 - \phi_5 + \phi_6) = 0$  or  $\sin(\phi_3 + \phi_4 + \phi_6 - \phi_1) = 0$  or  $\det = 0 \Rightarrow \sin(\phi_1 + \phi_2 + \phi_4 + \phi_5) = 0 \Rightarrow$

$$\begin{cases} \phi_1 + \phi_2 + \phi_4 + \phi_5 = \pi \\ \phi_2 + \phi_3 - \phi_5 + \phi_6 = 0, \pi \\ \phi_3 + \phi_4 + \phi_6 - \phi_1 = 0 \end{cases}$$

Consider case  $\phi_1 + \phi_2 + \phi_4 + \phi_5 = \pi$  Determinant of matrix behind is 0, let's prove it.

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_1) \\ 0 \\ \sin(\phi_1 + \phi_2) \\ \sin(\phi_2) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sin(\phi_3) \\ 0 \\ \sin(\phi_2 + \phi_3) \\ 0 \\ \sin(\phi_2) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\phi_4) \\ 0 \\ \sin(\phi_1 + \phi_2 + \phi_4) \\ 0 \\ \sin(\phi_1 + \phi_2) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \sin(\phi_6) \\ 0 \\ \sin(\phi_3 + \phi_6) \\ \sin(\phi_3) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(\phi_6) \\ \sin(\phi_4) \\ \sin(\phi_4 + \phi_6) \end{pmatrix}$$

$\det = (1) + (2) + (3)$

$$(1) = \sin(\phi_4)(\sin(\phi_1 + \phi_2)[\sin(\phi_6)\sin(\phi_2)\sin(\phi_3)] - \sin(\phi_2 + \phi_3)[\sin(\phi_2)[\sin(\phi_3 + \phi_6)\sin(\phi_4 + \phi_6) - \sin(\phi_3)\sin(\phi_4)] + \sin(\phi_6)[\sin^2(\phi_2)\sin(\phi_4 + \phi_6)])$$

$$(2) = -\sin(\phi_1)(\sin(\phi_2 + \phi_3)[\sin(\phi_1 + \phi_2 + \phi_4)(\sin(\phi_3 + \phi_6)\sin(\phi_4 + \phi_6) - \sin(\phi_3)\sin(\phi_4)) - \sin(\phi_6)\sin(\phi_1 + \phi_2)\sin(\phi_3 + \phi_6)] + \sin(\phi_6)[\sin(\phi_2)\sin(\phi_1 + \phi_2)\sin(\phi_6) - \sin(\phi_2)\sin(\phi_1 + \phi_2 + \phi_4)\sin(\phi_4 + \phi_6)])$$

$$(3) = \sin(\phi_3)(\sin(\phi_1 + \phi_2)[\sin(\phi_1 + \phi_2 + \phi_4)(\sin(\phi_3 + \phi_6)\sin(\phi_4 + \phi_6) - \sin(\phi_3)\sin(\phi_4)) - \sin(\phi_6)\sin(\phi_1 + \phi_2)\sin(\phi_3 + \phi_6)] + \sin(\phi_6)[- \sin(\phi_2)\sin(\phi_1 + \phi_2)\sin(\phi_4)])$$

After simplification:

$$(1) = \sin(\phi_2)\sin(\phi_6)\sin(\phi_3)\sin(\phi_4)[\sin(\phi_1 + \phi_2) - \sin(\phi_3 + \phi_2 + \phi_4 + \phi_6)]$$

$$(2) = -\sin(\phi_1)\sin(\phi_4)\sin(\phi_3)\sin(\phi_6)[\sin(\phi_1 + 2\phi_2 + \phi_3 + \phi_4 + \phi_6)]$$

$$(3) = \sin(\phi_3)\sin(\phi_4)\sin(\phi_1 + \phi_2)\sin(\phi_6)[\sin(\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_6) - \sin(\phi_2)]$$

$$\det = \sin(\phi_6)\sin(\phi_3)\sin(\phi_4) * 0 = 0$$

So, in this case there is no extremality.

Also, in the case  $\phi_1 + \phi_4 + \phi_6 = \phi_3$  zeros  $u_2, u_3, u_5, u_6$  are dependent and there is no extremality. The case  $\phi_2 + \phi_3 - \phi_5 + \phi_6 = \pi$  is unreal.

## Result

There are copositive extremal matrices on conditions  $\phi_i > 0, \phi_1 + \phi_2 + \phi_4 + \phi_5 < \pi, \phi_3 + \phi_4 + \phi_6 < \phi_1$ , excluding  $\phi_2 + \phi_3 - \phi_5 + \phi_6 = 0$