1 Case 15

The minimal zero supports are given by (1,2),(1,3,4),(1,3,5),(2,4,6),(3,4,6),(4,5,6).

$$A = \begin{pmatrix} 1 & -1 & -cos(\phi_2) & cos(\phi_1+\phi_2) & cos(\phi_2+\phi_3) & b_3 \\ -1 & 1 & b_1 & cos(\phi_4+\phi_5) & b_2 & -cos(\phi_5) \\ -cos(\phi_2) & b_1 & 1 & -cos(\phi_1) & -cos(\phi_3) & cos(\phi_1+\phi_4) \\ cos(\phi_1+\phi_2) & cos(\phi_4+\phi_5) & -cos(\phi_1) & 1 & cos(\phi_4+\phi_6) & -cos(\phi_4) \\ cos(\phi_2+\phi_3) & b_2 & -cos(\phi_3) & cos(\phi_4+\phi_6) & 1 & -cos(\phi_6) \\ b_3 & -cos(\phi_5) & cos(\phi_1+\phi_4) & -cos(\phi_4) & -cos(\phi_6) & 1 \end{pmatrix}$$

The zeros are given by the columns:

$$(u_1,u_2,u_3,u_4,u_5,u_6) = \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}sin(\phi_1)\\0\\sin(\phi_1+\phi_2)\\sin(\phi_2)\\0\\0\end{pmatrix}\begin{pmatrix}sin(\phi_3)\\0\\sin(\phi_2+\phi_3)\\0\\sin(\phi_2)\\0\end{pmatrix}\begin{pmatrix}0\\sin(\phi_4)\\0\\sin(\phi_5)\\0\\sin(\phi_4+\phi_5)\end{pmatrix}\begin{pmatrix}0\\0\\sin(\phi_4)\\sin(\phi_1+\phi_4)\\0\\sin(\phi_1)\end{pmatrix}\begin{pmatrix}0\\0\\0\\sin(\phi_6)\\sin(\phi_4)\\sin(\phi_4+\phi_6)\end{pmatrix}$$

1. $$Au_1 = \begin{pmatrix}0\\0\\b_1-cos(\phi_2)\\cos(\phi_1+\phi_2)+cos(\phi_4+\phi_5)\\b_2+cos(\phi_2+\phi_3)\\b_3-cos(\phi_5)\end{pmatrix} => \begin{cases}b_1 \geq cos(\phi_2)\\cos(\phi_1+\phi_2)+cos(\phi_4+\phi_5)\geq 0\\b_2\geq -cos(\phi_2+\phi_3)\\b_3\geq cos(\phi_5)\end{cases}$$

2. $$Au_2 = \begin{pmatrix}0\\-sin(\phi_1)+b_1sin(\phi_1+\phi_2)+sin(\phi_2)cos(\phi_4+\phi_5)\\0\\0\\-sin(\phi_2)cos(\phi_1-\phi_3)+sin(\phi_2)cos(\phi_4+\phi_6)\\b_3sin(\phi_1)+sin(\phi_1)cos(\phi_1+\phi_2+\phi_4)\end{pmatrix} => \begin{cases}-sin(\phi_1)+b_1sin(\phi_1+\phi_2)+sin(\phi_2)cos(\phi_4+\phi_5)\geq 0\\-sin(\phi_2)cos(\phi_1-\phi_3)+sin(\phi_2)cos(\phi_4+\phi_6)\geq 0\\b_3sin(\phi_1)+sin(\phi_1)cos(\phi_1+\phi_2+\phi_4)\geq 0\end{cases}$$

3. $$Au_3 = \begin{pmatrix}0\\b_2*sin(\phi_2)-sin(\phi_3)+b_1*sin(\phi_2+\phi_3)\\0\\-sin(\phi_2)cos(\phi_1-\phi_3)+sin(\phi_2)cos(\phi_4+\phi_6)\\0\\b_3sin(\phi_3)+sin(\phi_2+\phi_3)cos(\phi_1+\phi_4)-sin(\phi_2)cos(\phi_6)\end{pmatrix} => \begin{cases}b_2*sin(\phi_2)-sin(\phi_3)+b_1*sin(\phi_2+\phi_3)\geq 0\\-sin(\phi_2)cos(\phi_1-\phi_3)+sin(\phi_2)cos(\phi_4+\phi_6)\geq 0\\b_3sin(\phi_3)+sin(\phi_2+\phi_3)cos(\phi_1+\phi_4)-sin(\phi_2)cos(\end{cases}$$

4. $$Au_4 = \begin{pmatrix}-sin(\phi_4)+b_3sin(\phi_4+\phi_5)+sin(\phi_5)cos(\phi_1+\phi_2)\\0\\b_1sin(\phi_4)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_5)\\0\\b_2sin(\phi_4)-sin(\phi_4)cos(\phi_5-\phi_6)\\0\end{pmatrix} => \begin{cases}-sin(\phi_4)+b_3sin(\phi_4+\phi_5)+sin(\phi_5)cos(\phi_1+\phi_2)\geq 0\\b_1sin(\phi_4)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_5)\geq 0\\b_2sin(\phi_4)-sin(\phi_4)cos(\phi_5-\phi_6)\geq 0\end{cases}$$

5. $$Au_5 = \begin{pmatrix}b_3sin(\phi_1)+sin(\phi_1)cos(\phi_1+\phi_2+\phi_4)\\b_1sin(\phi_4)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_5)\\0\\0\\-sin(\phi_4)cos(\phi_3)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_6)\\0\end{pmatrix} => \begin{cases}b_3sin(\phi_1)+sin(\phi_1)cos(\phi_1+\phi_2+\phi_4)\geq 0\\b_1sin(\phi_4)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_5)\geq 0\\-sin(\phi_4)cos(\phi_3)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_6)\geq 0\end{cases}$$

6. $$Au_6 = \begin{pmatrix}sin(\phi_4)cos(\phi_2+\phi_3)+b_3sin(\phi_4+\phi_6)+sin(\phi_6)cos(\phi_1+\phi_2)\\b_2sin(\phi_4)-sin(\phi_4)cos(\phi_5-\phi_6)\\-sin(\phi_4)cos(\phi_3)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_6)\\0\\0\\0\end{pmatrix} => \begin{cases}sin(\phi_4)cos(\phi_2+\phi_3)+b_3sin(\phi_4+\phi_6)+sin(\phi_6)\\b_2sin(\phi_4)-sin(\phi_4)cos(\phi_5-\phi_6)\geq 0\\-sin(\phi_4)cos(\phi_3)+sin(\phi_4)cos(\phi_1+\phi_4+\phi_6)\geq\end{cases}$$

Considering inequalities on ϕ_i we get: $\begin{cases}\phi_1+\phi_2+\phi_4+\phi_5 \le \pi \\ \phi_4+\phi_6 \le |\phi_1-\phi_3| \\ \phi_1+\phi_4+\phi_6 \le \phi_3\end{cases} => \begin{cases}\phi_1+\phi_2+\phi_4+\phi_5 \le \pi \\ \phi_1+\phi_4+\phi_6 \le \phi_3\end{cases}$

Case $\phi_1+\phi_4+\phi_6 \ge \pi isunreal$
$b_1 = cos(\phi_2), b_2 = \max(-cos(\phi_2+\phi_3), cos(\phi_5-\phi_6))$,
$b_3 = \max(cos(\phi_5), \frac{-sin(\phi_6)cos(\phi_1+\phi_2)-sin(\phi_4)cos(\phi_2+\phi_3)}{sin(\phi_4+\phi_6)}, \frac{-sin(\phi_2+\phi_3)cos(\phi_1+\phi_4)+sin(\phi_2)cos(\phi_6)}{sin(\phi_3)})$
All other inequalities are fulfilled.

Copositivity:

It's necessery to consider next sets for I:
$(2,3,4),(2,3,5),(1,4,5),(2,4,5),(3,4,5),(2,3,4,5),(1,3,6),(2,3,6),(1,4,6),(1,5,6),(2,5,6),(3,5,6),(2,3,5,6)$

Consider set (1,5,6): let $b_3 = cos(\phi) => \phi \le \pi-\phi_2-\phi_3+\phi_6 => cos(\phi) \ge -cos(\phi_2+\phi_3-\phi_6)$ if $b_3 = cos(\phi_5) => \phi_5+\phi_2 \le \pi-\phi_3+\phi_6$,
if $b_3 = \frac{-sin(\phi_6)cos(\phi_1+\phi_2)-sin(\phi_4)cos(\phi_2+\phi_3)}{sin(\phi_4+\phi_6)} => \phi_3 \le \phi_1+\phi_4+\phi_6$,is fulfilled only if $\phi_3 = \phi_1+\phi_4+\phi_6$
if $b_3 = \frac{-sin(\phi_2+\phi_3)cos(\phi_1+\phi_4)+sin(\phi_2)cos(\phi_6)}{sin(\phi_3)} => \phi_3 \le \phi_1+\phi_4+\phi_6$,is fulfilled only if $\phi_3 = \phi_1+\phi_4+\phi_6$
Consider when $cos(\phi_5) \ge \frac{-sin(\phi_6)cos(\phi_1+\phi_2)-sin(\phi_4)cos(\phi_2+\phi_3)}{sin(\phi_4+\phi_6)} => cos(\phi_5)sin(\phi_4+\phi_6)+sin(\phi_6)cos(\phi_1+\phi_2)+sin(\phi_4)cos(\phi_2+\phi_3) \ge 0$
$cos(\phi_5) \ge -cos(\phi_2+\phi_3-\phi_6)$
$sin(\phi_6)(-cos(\phi_2+\phi_3-\phi_4-\phi_6)+cos(\phi_1+\phi_2)) \ge 0 => \phi_1+\phi_4+\phi_6 \le \phi_3$ Performs
when $cos(\phi_5) \ge \frac{-sin(\phi_2+\phi_3)cos(\phi_1+\phi_4)+sin(\phi_2)cos(\phi_6)}{sin(\phi_3)} => sin(\phi_2+\phi_3)cos(\phi_1+\phi_4)-sin(\phi_2)cos(\phi_6)+cos(\phi_5)sin(\phi_3) \ge 0$
$cos(\phi_1+\phi_4) \le cos(\phi_3-\phi_6), cos(\phi_5) \ge -cos(\phi_2+\phi_3-\phi_6) => 0 \ge 0$ Perfroms
$=> b_3 = cos(\phi_5)$
Consider other sets:

1. $I = (1,4,6): b_3 = cos(\phi_5) => -\phi_5+\pi-\phi_1-\phi_2+\phi_4 \le 0$ performs strictly
2. $I = (2,3,4),(2,3,4,5): u = e_3+e_4 => cos(\phi_2) \ge -cos(\phi_4+\phi_5) => \phi_4+\phi_5+\phi_2 \le \pi$, $\phi_4+\phi_6 \le \phi_3$ performs
3. $I = (2,3,5): -\phi_2+\phi_3+\phi_2+\phi_3 \ge 0$ performs strictly
4. $I = (2,5,6),(3,5,6),(2,3,5,6): u = e_5+e_6 => cos(\phi_5-\phi_6) \ge cos(\phi_5)$, $cos(\phi_1+\phi_4) \ge cos(\phi_3) = \phi-1+\phi_4 \le \phi_3$ performs strictly
5. $I = (1,3,6): b_3 = cos(\phi_5) => \phi_2+\pi-\phi_1-\phi_4-\phi_5 \ge 0$ performs strictly
6. $I = (2,3,6): -\phi_2+\phi_5+\pi-\phi_1-\phi_4 \ge 0$ performs strictly
7. $I = (1,4,5): -\phi_1-\phi_2+\pi-\phi_2-\phi_3+\pi-\phi_4-\phi_6 \ge 0$ performs strictly
8. $I = (2,4,5): b_2 \ge cos(\phi_5-\phi_6) => \pi-\phi_4-\phi_5+\pi-|\phi_5-\phi_6|-\phi_4-\phi_6 \ge 0 => 2\pi \ge 2\phi_4+\phi_6+|\phi_5-\phi_6|+\phi_5$ performs strictly

That proves copositivity.

1.1 Parametrization:

$$A = \begin{pmatrix} 1 & -1 & -cos(\phi_2) & cos(\phi_1+\phi_2) & cos(\phi_2+\phi_3) & cos(\phi_5) \\ -1 & 1 & cos(\phi_2) & cos(\phi_4+\phi_5) & b_2 & -cos(\phi_5) \\ -cos(\phi_2) & cos(\phi_2) & 1 & -cos(\phi_1) & -cos(\phi_3) & cos(\phi_1+\phi_4) \\ cos(\phi_1+\phi_2) & cos(\phi_4+\phi_5) & -cos(\phi_1) & 1 & cos(\phi_4+\phi_6) & -cos(\phi_4) \\ cos(\phi_2+\phi_3) & b_2 & -cos(\phi_3) & cos(\phi_4+\phi_6) & 1 & -cos(\phi_6) \\ cos(\phi_5) & -cos(\phi_5) & cos(\phi_1+\phi_4) & -cos(\phi_4) & -cos(\phi_6) & 1 \end{pmatrix}$$

$b_2 = \max(-cos(\phi_2+\phi_3), cos(\phi_5-\phi_6))$,
with conditions : $\phi_i \in (0,\pi), \phi_1+\phi_2+\phi_4+\phi_5 < \pi, \phi_1+\phi_4+\phi_6 \le \phi_3, \phi_5+\phi_2+\phi_3-\phi_6 \le \pi$

Extremality:

$X = FPF^T$

$$X = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{15} \\ b_{21} & b_{22} & * & * \\ b_{31} & * & b_{33} & b_{35} \\ b_{51} & * & b_{53} & b_{55} \end{pmatrix}, F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_5 \end{pmatrix}, \begin{cases} u_{31}f_1 + u_{33}f_3 + u_{35}f_5 = 0 \\ f_1 = -f_2 \end{cases}, F = \begin{pmatrix} sin(\phi_2) & 0 \\ -sin(\phi_2) & 0 \\ 0 & sin(\phi_2) \\ -sin(\phi_3) & -sin(\phi_2+\phi_3) \end{pmatrix}$$

$Y = GQG^T$

$$Y = \begin{pmatrix} b_{11} & * & b_{13} & b_{14} & * & * \\ * & b_{22} & * & b_{24} & * & b_{26} \\ b_{31} & * & b_{33} & b_{34} & * & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ * & * & * & b_{54} & b_{55} & b_{56} \\ * & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{pmatrix}, G = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{pmatrix}, \begin{cases} u_{21}g_1 + u_{23}g_3 + u_{24}g_4 = 0 \\ u_{53}g_3 + u_{54}g_4 + u_{56}g_6 = 0 \\ u_{42}g_2 + u_{44}g_4 + u_{46}g_6 = 0 \\ u_{64}g_4 + u_{65}g_5 + u_{66}g_6 = 0 \end{cases},$$

$$G = \begin{pmatrix} sin(\phi_2) & 0 \\ -sin(\phi_1+\phi_4+\phi_5) & -sin(\phi_1+\phi_2+\phi_4+\phi_5) \\ 0 & sin(\phi_2) \\ -sin(\phi_1) & -sin(\phi_1+\phi_2) \\ -sin(\phi_1+\phi_4+\phi_6) & -sin(\phi_1+\phi_2+\phi_4+\phi_6) \\ sin(\phi_1+\phi_4) & sin(\phi_1+\phi_2+\phi_5) \end{pmatrix}$$

Necessary equalities, that are constrains on elements of matrices P and Q:

$$\begin{cases} x_{11} = y_{11} \\ x_{22} = y_{22} \\ x_{33} = y_{33} \\ x_{31} = y_{31} \\ x_{55} = y_{55} \end{cases}$$

$$=> \begin{cases} p_{11} = q_{11} \\ sin^2(\phi_2)p_{11} = sin^2(\phi_1+\phi_4+\phi_5)q_{11} + 2sin(\phi_1+\phi_4+\phi_5)sin(\phi_1+\phi_2+\phi_4+\phi_5)q_{21} + sin^2(\phi_1+\phi_2+\phi_4+\phi_5)q_{22} \\ p_{22} = q_{22} \\ p_{21} = q_{21} \\ sin^2(\phi_3)p_{11} + 2sin(\phi_3)sin(\phi_2+\phi_3)p_{12} + sin^2(\phi_2+\phi_3)p_{22} = sin^2(\phi_1+\phi_4+\phi_6)q_{11} + \\ +2sin(\phi_1+\phi_4+\phi_6)sin(\phi_1+\phi_4+\phi_6+\phi_2)q_{12} + sin^2(\phi_1+\phi_4+\phi_6+\phi_2)q_{22} \end{cases}$$

Let $b_{11} = b_{22} = 0 => p_{11} = q_{11} = 0$

$$\begin{pmatrix} 2sin(\phi_1+\phi_4+\phi_5)sin(\phi_1+\phi_2+\phi_4+\phi_5) & sin^2(\phi_1+\phi_2+\phi_4+\phi_5) \\ 2sin(\phi_3)sin(\phi_2+\phi_3) - 2sin(\phi_1+\phi_4+\phi_6)sin(\phi_1+\phi_4+\phi_6+\phi_2) & sin^2(\phi_2+\phi_3) - sin^2(\phi_1+\phi_4+\phi_6+\phi_2) \end{pmatrix} =$$

$$\begin{pmatrix} 2sin(\phi_1+\phi_4+\phi_5)sin(\phi_1+\phi_2+\phi_4+\phi_5) & sin^2(\phi_1+\phi_2+\phi_4+\phi_5) \\ -2sin(\phi_1+\phi_4+\phi_6-\phi_3)sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_6) & -sin(\phi_1+\phi_4+\phi_6-\phi_3)sin(\phi_1+2\phi_2+\phi_3+\phi_4+\phi_6) \end{pmatrix} =>$$

There is no extremality in common case, when : $sin(\phi_1+\phi_2+\phi_4+\phi_5) = 0$ or $sin(\phi_1+\phi_4+\phi_6-\phi_3) = 0$ or $det = 0 => sin(\phi_2+\phi_3-\phi_5+\phi_6) = 0 =>$

$$\begin{cases} \phi_1+\phi_2+\phi_4+\phi_5 = \pi \\ \phi_1+\phi_4+\phi_6-\phi_3 = 0 \\ \phi_2+\phi_3-\phi_5+\phi_6 = 0, \pi \end{cases}$$

Consider case $\phi_2+\phi_3-\phi_5+\phi_6 = \pi$. Determinant of matrix behind is 0, let's prove it.

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} sin(\phi_1) \\ 0 \\ sin(\phi_1+\phi_2) \\ sin(\phi_2) \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} sin(\phi_3) \\ 0 \\ sin(\phi_2+\phi_3) \\ 0 \\ sin(\phi_2) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ sin(\phi_4) \\ 0 \\ -sin(\phi_2+\phi_3+\phi_6) \\ 0 \\ -sin(\phi_4+\phi_2+\phi_3+\phi_6) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ sin(\phi_4) \\ sin(\phi_1+\phi_4) \\ 0 \\ sin(\phi_1) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ sin(\phi_6) \\ sin(\phi_4) \\ sin(\phi_4+\phi_6) \end{pmatrix}$$

$det = (1) + (2) + (3)$

$(1) = sin(\phi_4)(sin(\phi_1+\phi_2)[-sin(\phi_1+\phi_4)sin(\phi_2)sin(\phi_4+\phi_6)+sin(\phi_6)sin(\phi_2)sin(\phi_1)] - sin(\phi_2+\phi_3)[-sin(\phi_2)sin(\phi_1)sin(\phi_4)] + sin(\phi_4)[sin^2(\phi_2)sin(\phi_4+\phi_6)])$

$(2) = -sin(\phi_1)(sin(\phi_2+\phi_3)[sin(\phi_2+\phi_3+\phi_6)sin(\phi_1) - sin(\phi_2+\phi_3+\phi_4+\phi_6)sin(\phi_1+\phi_4)]sin(\phi_4) + sin(\phi_4)[sin(\phi_2+\phi_3+\phi_6)sin(\phi_2)sin(\phi_4+\phi_6) - sin(\phi_6)sin(\phi_2)sin(\phi_2+\phi_3+\phi_4+\phi_6)])$

$(3) = sin(\phi_3)(sin(\phi_1+\phi_2)[sin(\phi_2+\phi_3+\phi_6)sin(\phi_1)-sin(\phi_2+\phi_3+\phi_4+\phi_6)sin(\phi_1+\phi_4)]sin(\phi_4)+sin(\phi_4)[sin(\phi_2)sin(\phi_2+\phi_3+\phi_4+\phi_6)sin(\phi_4)])$
After simplification:
$(1) = sin(\phi_2)sin(\phi_1)sin^2(\phi_4)[sin(\phi_2+\phi_3) - sin(\phi_1+\phi_2+\phi_4+\phi_6)]$
$(2) = -sin(\phi_1)sin^2(\phi_4)sin(\phi_2+\phi_3)[-sin(\phi_1+\phi_2+\phi_3+\phi_4+\phi_6) + sin(\phi_2)]$
$(3) = -sin(\phi_3)sin^2(\phi_4)sin(\phi_1)sin(\phi_1+2\phi_2+\phi_3+\phi_4+\phi_6)$
$det = sin(\phi_1)sin^2(\phi_4) * 0 = 0$

In the case $\phi_1+\phi_4+\phi_6 = \phi_3$ zeros are dependent too.

Result:

There are copositive extreme matrices on conditions:$\phi_i > 0, \phi_2+\phi_3 < \pi, \phi_2+\phi_3+\phi_5 < \pi+\phi_6, \phi_1+\phi_4+\phi_6 < \phi_3$, excluding $\phi_2+\phi_3+\phi_6 = \phi_5, \phi_2+\phi_3+\phi_6 = \pi+\phi_5$