Case 4

Peter J.C. Dickinson (peter.dickinson@cantab.net)

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This is the case when we consider a matrix A with minimal zero support set:

$$\{1,2\},\quad \{1,3\},\quad \{1,4\},\quad \{2,5,6\},\quad \{3,5,6\},\quad \{4,5,6\} \tag{1}$$

From [Toolbox, Corollary 2.10] without loss of generality we have $\mathcal{V}^A_{\min} = \mathbb{R}_{++}\mathcal{W}$, where $\mathcal{W} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ and

$$\mathbf{v}_1 = \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}1\\0\\1\\0\\0\\0\end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix}0\\\sin(\theta_0)\\0\\0\\\sin(\theta_1)\\\sin(\theta_0+\theta_1)\end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix}0\\0\\\sin(\theta_0)\\0\\\sin(\theta_2)\\\sin(\theta_0+\theta_2)\end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix}0\\0\\0\\\sin(\theta_0)\\\sin(\theta_3)\\\sin(\theta_0+\theta_3)\end{pmatrix}. \tag{2}$$

and $0 < \theta_3 \leq \theta_2 \leq \theta_1 < \pi - \theta_0 < \pi$.

From [Toolbox, Lemma 3.1], in order for the matrix A to be irreducible with respect to $\mathcal{S}^6_+$, these vectors must be linearly independent. If $\theta_1 = \theta_2$ then $\mathbf{0} = \sin(\theta_0)\,\mathbf{v}_1 - \sin(\theta_0)\,\mathbf{v}_2 - \mathbf{v}_4 + \mathbf{v}_5$, and thus A would be reducible with respect to $\mathcal{S}^6_+$. Similar if $\theta_2 = \theta_3$ then $\mathbf{0} = \sin(\theta_0)\,\mathbf{v}_2 - \sin(\theta_0)\,\mathbf{v}_3 - \mathbf{v}_5 + \mathbf{v}_6$, and thus A would be reducible with respect to $\mathcal{S}^6_+$.

From now on we will consider when $0 < \theta_3 < \theta_2 < \theta_1 < \pi - \theta_0 < \pi$ and we shall also consider the following matrix:

$$B = \begin{pmatrix} 1 & -1 & -1 & -1 & -\cos(\theta_0+\theta_1) & \cos(\theta_3) \\ -1 & 1 & 1 & 1 & \cos(\theta_0+\theta_1) & -\cos(\theta_1) \\ -1 & 1 & 1 & 1 & \cos(\theta_0+\theta_2) & -\cos(\theta_2) \\ -1 & 1 & 1 & 1 & \cos(\theta_0+\theta_3) & -\cos(\theta_3) \\ -\cos(\theta_0+\theta_1) & \cos(\theta_0+\theta_1) & \cos(\theta_0+\theta_2) & \cos(\theta_0+\theta_3) & 1 & -\cos(\theta_0) \\ \cos(\theta_3) & -\cos(\theta_1) & -\cos(\theta_2) & -\cos(\theta_3) & -\cos(\theta_0) & 1 \end{pmatrix}. \tag{3}$$

We will begin by looking at the following technical results on the set of zeros of B.

Lemma 1. *For $0 < \theta_3 < \theta_2 < \theta_1 < \pi - \theta_0 < \pi$ we have $\mathcal{V}^B_{\min} = \mathbb{R}_{++}\mathcal{W}$.*

Proof. There are trivially no zeros of B with support of cardinality one.

From [Toolbox, Lemma 2.5], up to multiplication by a positive scalar, the zeros of B with support of cardinality two are exactly those given in $\mathcal{W}$.

From [Toolbox, Lemma 2.4], if we wish to find minimal zeros of B whose support have cardinality strictly greater than two, we need only consider the maximal principle submatrices of B with no off-diagonal entries equal to plus or minus one. For $i = 1,2,3$ these are the principle submatrices

$$\begin{aligned} B_{\{i+1,5,6\}} &= \begin{pmatrix} 1 & \cos(\theta_0+\theta_i) & -\cos(\theta_i) \\ \cos(\theta_0+\theta_i) & 1 & -\cos(\theta_0) \\ -\cos(\theta_i) & -\cos(\theta_0) & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \cos(\theta_0+\theta_i) \\ -\cos(\theta_i) \end{pmatrix} \begin{pmatrix} 1 \\ \cos(\theta_0+\theta_i) \\ -\cos(\theta_i) \end{pmatrix}^{\mathsf{T}} + \begin{pmatrix} 0 \\ \sin(\theta_0+\theta_i) \\ -\sin(\theta_i) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\theta_0+\theta_i) \\ -\sin(\theta_i) \end{pmatrix}^{\mathsf{T}}. \end{aligned}$$

and the principle submatrix

$$B_{\{1,5,6\}} = \begin{pmatrix} 1 & -\cos(\theta_0+\theta_1) & \cos(\theta_3) \\ -\cos(\theta_0+\theta_1) & 1 & -\cos(\theta_0) \\ \cos(\theta_3) & -\cos(\theta_0) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ \cos(\theta_0+\theta_1) \\ -\cos(\theta_1) \end{pmatrix} \begin{pmatrix} -1 \\ \cos(\theta_0+\theta_1) \\ -\cos(\theta_1) \end{pmatrix}^{\mathsf{T}} + \begin{pmatrix} 0 \\ \sin(\theta_0+\theta_1) \\ -\sin(\theta_1) \end{pmatrix} \begin{pmatrix} 0 \\ \sin(\theta_0+\theta_1) \\ -\sin(\theta_1) \end{pmatrix}^{\mathsf{T}} + \begin{pmatrix} 0 & 0 & \cos(\theta_3)-\cos(\theta_1) \\ 0 & 0 & 0 \\ \cos(\theta_3)-\cos(\theta_1) & 0 & 0 \end{pmatrix}.$$

Now noting that $\cos(\theta_3) > \cos(\theta_1)$, the required result immediately follows. □

Lemma 2. *For $0 < \theta_3 < \theta_2 < \theta_1 < \pi - \theta_0 < \pi$ we have*

$$\begin{aligned} &\operatorname{supp}(B\mathbf{v}_1) = \{6\}, && \operatorname{supp}(B\mathbf{v}_2) = \{5,6\}, && \operatorname{supp}(B\mathbf{v}_3) = \{5\}, \\ &\operatorname{supp}(B\mathbf{v}_4) = \{1,3,4\}, && \operatorname{supp}(B\mathbf{v}_5) = \{1,2,4\}, && \operatorname{supp}(B\mathbf{v}_6) = \{1,2,3\}. \end{aligned}$$

Proof. Using basic trigonometric relations, this is trivial but tedious to show. □

Lemma 3. *For $0 < \theta_3 < \theta_2 < \theta_1 < \pi - \theta_0 < \pi$ we have*

$$\mathcal{V}^B = \mathbb{R}_{++}\left(\{\mathbf{v}_4\} \cup \{\mathbf{v}_5\} \cup \{\mathbf{v}_6\} \cup \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\right).$$

Proof. This follows immediately from Lemmas 1 and 2 and [Toolbox, Lemma 4.4]. □

We shall now consider matrices with such a set of zeros.

Lemma 4. *For $0 < \theta_3 < \theta_2 < \theta_1 < \pi - \theta_0 < \pi$ and $A \in \mathcal{S}_1^6$ the following are equivalent:*

1. *$A \in \mathcal{COP}^n$ and $\mathcal{W} \subseteq \mathcal{V}^A$,*
2. *For all $i, j = 1, \ldots, n$ with $i \le j$ we have*

$$\begin{aligned} a_{ij} &= b_{ij} && \text{if } (i,j) \neq (1,5),\ (1,6), \\ a_{ij} &\ge b_{ij} && \text{if } (i,j) = (1,5),\ (1,6). \end{aligned}$$

Furthermore for such an A with these conditions holding we have:

a. *$\mathcal{V}^A = \mathcal{V}^B$,*

b. *A does not give an exposed ray of the copositive cone,*

c. *if $A \neq B$ then A is reducible with respect to $\mathcal{N}^n$ and thus does not give an extreme ray of the copositive cone.*

Proof. The equivalence follows directly from [Toolbox, Lemmas 1.2, 2.5, 2.8 and 4.1].

Using the explicit description of $\mathcal{V}^B$ from Lemma 3 it can be seen that for all $\mathbf{v} \in \mathcal{V}^B$ we have $\mathbf{v}^T A \mathbf{v} = \mathbf{v}^T B \mathbf{v} = 0$, and thus $\mathcal{V}^B \subseteq \mathcal{V}^A$. Furthermore, we have $A - B \in \mathcal{N}^6$ and thus $\mathcal{V}^A \subseteq \mathcal{V}^B$.

The statement on being an exposed ray follows from [Toolbox, Theorem 5.1].

The final statement on being reducible is trivial. □

We have thus shown that the only candidate for giving an extreme ray in this case is B (although it would not give an exposed ray). Now we shall now show that B does indeed give an extreme of the copositive cone.

Theorem 5. *If $0 < \theta_3 < \theta_2 < \theta_1 < \pi - \theta_0 < \pi$ then B gives an extreme ray of the copositive cone.*

Proof. Suppose for the sake of contradiction that B does not give an extreme ray of the copositive cone. From [Toolbox, Theorem 5.2], there exists $C \in \mathcal{COP}^6 \setminus \{\alpha B \mid \alpha \in \mathbb{R}\}$ such that $\mathcal{V}^C_{\min} = \mathcal{V}^B_{\min}$ and $\operatorname{supp}(C\mathbf{v}) = \operatorname{supp}(B\mathbf{v})$

for all $\mathbf{v} \in \mathcal{V}^B_{\min}$. Considering Lemma 1 we thus have $c_{ii} > 0$ for all i, and without loss of generality $c_{11} = 1$. Using Lemma 2 to consider the condition on the supports, we observe that there exist $a, b, c, d, e, f, g, h, i \in \mathbb{R}$ such that

$$C = \begin{pmatrix} 1 & -1 & -1 & -1 & -a & -b \\ -1 & 1 & 1 & 1 & a & c \\ -1 & 1 & 1 & 1 & d & e \\ -1 & 1 & 1 & 1 & f & b \\ -a & a & d & f & g & h \\ -b & c & e & b & h & i \end{pmatrix},$$

$$\mathbf{0} = \begin{pmatrix} \sin\theta_0 \\ 0 \\ 0 \\ \sin\theta_0 \\ 0 \\ 0 \\ \sin\theta_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sin\theta_1 & 0 & \sin(\theta_0+\theta_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\theta_0 & 0 & 0 & 0 & 0 & 0 & \sin\theta_1 & \sin(\theta_0+\theta_1) & 0 \\ 0 & 0 & \sin\theta_0 & 0 & 0 & 0 & 0 & \sin\theta_1 & \sin(\theta_0+\theta_1) \\ 0 & 0 & 0 & \sin\theta_2 & \sin(\theta_0+\theta_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta_0 & 0 & 0 & \sin\theta_2 & \sin(\theta_0+\theta_2) & 0 \\ 0 & 0 & 0 & 0 & \sin\theta_0 & 0 & 0 & \sin\theta_2 & \sin(\theta_0+\theta_2) \\ 0 & \sin(\theta_0+\theta_3) & 0 & 0 & 0 & \sin\theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin\theta_0 & \sin\theta_3 & \sin(\theta_0+\theta_3) & 0 \\ 0 & \sin\theta_0 & 0 & 0 & 0 & 0 & 0 & \sin\theta_3 & \sin(\theta_0+\theta_3) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{pmatrix}. \tag{4}$$

One solution to (4) would correspond to $C = B$. Therefore, in order to have $C \neq B$, we require

$$\begin{aligned} 0 &= \begin{vmatrix} \sin\theta_1 & 0 & \sin(\theta_0+\theta_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\theta_0 & 0 & 0 & 0 & 0 & 0 & \sin\theta_1 & \sin(\theta_0+\theta_1) & 0 \\ 0 & 0 & \sin\theta_0 & 0 & 0 & 0 & 0 & \sin\theta_1 & \sin(\theta_0+\theta_1) \\ 0 & 0 & 0 & \sin\theta_2 & \sin(\theta_0+\theta_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta_0 & 0 & 0 & \sin\theta_2 & \sin(\theta_0+\theta_2) & 0 \\ 0 & 0 & 0 & 0 & \sin\theta_0 & 0 & 0 & \sin\theta_2 & \sin(\theta_0+\theta_2) \\ 0 & \sin(\theta_0+\theta_3) & 0 & 0 & 0 & \sin\theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin\theta_0 & \sin\theta_3 & \sin(\theta_0+\theta_3) & 0 \\ 0 & \sin\theta_0 & 0 & 0 & 0 & 0 & 0 & \sin\theta_3 & \sin(\theta_0+\theta_3) \end{vmatrix} \\ &= 2\sin^3\theta_0 \Big(\sin\theta_1\ \sin^2\theta_2 \sin(\theta_0+\theta_1)\sin^2(\theta_0+\theta_3) + \sin\theta_2\ \sin^2\theta_3 \sin(\theta_0+\theta_2)\sin^2(\theta_0+\theta_1) \\ &\qquad + \sin\theta_3\ \sin^2\theta_1 \sin(\theta_0+\theta_3)\sin^2(\theta_0+\theta_2) - \sin\theta_1\ \sin^2\theta_3 \sin(\theta_0+\theta_1)\sin^2(\theta_0+\theta_2) \\ &\qquad - \sin\theta_2\ \sin^2\theta_1 \sin(\theta_0+\theta_2)\sin^2(\theta_0+\theta_3) - \sin\theta_3\ \sin^2\theta_2 \sin(\theta_0+\theta_3)\sin^2(\theta_0+\theta_1) \Big) \\ &= -2\sin^6\theta_0\ \sin(\theta_1-\theta_2)\ \sin(\theta_1-\theta_3)\ \sin(\theta_2-\theta_3) \\ &< 0. \end{aligned}$$

This is a contradiction, and thus completes the proof. □