Case 6

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Basic structure

The minimal zero supports are given by $\{1,2\},\{1,3\},\{1,4,5\},\{2,4,6\},\{3,4,6\},\{4,5,6\}$. There is a symmetry exchanging indices 2 and 3. We may write a copositive matrix with this minimal zero support set as

$$A = \begin{pmatrix} 1 & -1 & -1 & \cos(\phi_2+\phi_5) & -\cos\phi_5 & b_3 \\ -1 & 1 & 1 & \cos(\phi_1+\phi_4) & b_4 & -\cos\phi_4 \\ -1 & 1 & 1 & \cos(\phi_1+\phi_3) & b_5 & -\cos\phi_3 \\ \cos(\phi_2+\phi_5) & \cos(\phi_1+\phi_4) & \cos(\phi_1+\phi_3) & 1 & -\cos\phi_2 & -\cos\phi_1 \\ -\cos\phi_5 & b_4 & b_5 & -\cos\phi_2 & 1 & \cos(\phi_1+\phi_2) \\ b_3 & -\cos\phi_4 & -\cos\phi_3 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & 1 \end{pmatrix},$$

where $\phi_j \in (0,\pi)$, $j = 1,\dots,5$; $\phi_1+\phi_j < \pi$, $j = 2,3,4$; $\phi_2+\phi_5 < \pi$.

The minimal zeros of A are given by the columns of

$$U = \begin{pmatrix} 1 & 1 & \sin\phi_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & \sin\phi_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \sin\phi_1 & 0 \\ 0 & 0 & \sin\phi_5 & \sin\phi_4 & \sin\phi_3 & \sin(\phi_1+\phi_2) \\ 0 & 0 & \sin(\phi_2+\phi_5) & 0 & 0 & \sin\phi_1 \\ 0 & 0 & 0 & \sin(\phi_1+\phi_4) & \sin(\phi_1+\phi_3) & \sin\phi_2 \end{pmatrix}.$$

First order conditions

Consider the conditions $(Au_i)_j \geq 0$. $(Au_1)_4 \geq 0$ is equivalent to $\phi_1+\phi_2+\phi_4+\phi_5 \leq \pi$, and $(Au_2)_4 \geq 0$ is equivalent to $\phi_1+\phi_2+\phi_3+\phi_5 \leq \pi$. The other conditions are either automatically satisfied, or involve the elements b_k. The latter type of conditions involves only a single element b_k each, and hence by the irreducibility condition with respect to $\mathcal{N}^6$ yields the following values for the b_k:

$$\begin{aligned} b_3 &= \max\left(\cos\phi_4, \cos\phi_3, -\cos(\phi_1+\phi_2+\phi_5), \frac{\sin\phi_1 - \cos(\phi_2+\phi_5)\sin\phi_4}{\sin(\phi_1+\phi_4)}, \frac{\sin\phi_1 - \cos(\phi_2+\phi_5)\sin\phi_3}{\sin(\phi_1+\phi_3)}\right) \\ &= \max(\cos\phi_3, \cos\phi_4), \\ b_4 &= \max\left(\cos\phi_5, \frac{\sin\phi_2 - \cos(\phi_1+\phi_4)\sin\phi_5}{\sin(\phi_2+\phi_5)}, -\cos(\phi_1+\phi_2+\phi_4)\right) = \cos\phi_5, \\ b_5 &= \max\left(\cos\phi_5, \frac{\sin\phi_2 - \cos(\phi_1+\phi_3)\sin\phi_5}{\sin(\phi_2+\phi_5)}, -\cos(\phi_1+\phi_2+\phi_3)\right) = \cos\phi_5. \end{aligned}$$

By possibly exchanging indices 2,3 we may assume $\phi_3 \leq \phi_4$, which determines $b_3 = \cos\phi_3$.

Note that now $(Au_2)_j = 0$ for all $j \neq 4$. However, $Au_2 = 0$ would prevent A from being extremal, and hence we may assume $\phi_1+\phi_2+\phi_3+\phi_5 < \pi$.

Parametrization

We arrive at the parametrization

$$A = \begin{pmatrix} 1 & -1 & -1 & \cos(\phi_2+\phi_5) & -\cos\phi_5 & \cos\phi_3 \\ -1 & 1 & 1 & \cos(\phi_1+\phi_4) & \cos\phi_5 & -\cos\phi_4 \\ -1 & 1 & 1 & \cos(\phi_1+\phi_3) & \cos\phi_5 & -\cos\phi_3 \\ \cos(\phi_2+\phi_5) & \cos(\phi_1+\phi_4) & \cos(\phi_1+\phi_3) & 1 & -\cos\phi_2 & -\cos\phi_1 \\ -\cos\phi_5 & \cos\phi_5 & \cos\phi_5 & -\cos\phi_2 & 1 & \cos(\phi_1+\phi_2) \\ \cos\phi_3 & -\cos\phi_4 & -\cos\phi_3 & -\cos\phi_1 & \cos(\phi_1+\phi_2) & 1 \end{pmatrix} \quad (1)$$

with $\phi_i \in (0,\pi)$, $\phi_1+\phi_2+\phi_3+\phi_5 < \pi$, $\phi_1+\phi_2+\phi_4+\phi_5 \le \pi$, $\phi_3 \le \phi_4$.

Copositivity / Absence of other minimal zeros

Copositivity of A will be checked by the criterion in Theorem 4.6 of [1]. For each index set $I \subset \{1,\dots,6\}$, of cardinality not smaller than 3 and not containing a known minimal zero support, we have to find a vector $u \in \mathbb{R}^6$ with at least one positive element such that $\mathrm{supp}(u) \subset I \subset \mathrm{supp}_{\geq 0}(Au)$ or show that the submatrix A_I is copositive. For index sets of cardinality three this reduces to checking an inequality on the corresponding angles. We obtain

1. $I = \{2,4,5\} : \pi - \phi_1 - \phi_4 + \pi - \phi_5 + \phi_2 \ge \pi \;\Leftrightarrow\; \pi \ge -\phi_2 + \phi_1 + \phi_4 + \phi_5$

2. $I = \{3,4,5\} : \pi - \phi_1 - \phi_3 + \pi - \phi_5 + \phi_2 \ge \pi \;\Leftrightarrow\; \pi \ge -\phi_2 + \phi_1 + \phi_3 + \phi_5$

3. $I = \{1,4,6\} : \pi - \phi_2 - \phi_5 + \pi - \phi_3 + \phi_1 \ge \pi \;\Leftrightarrow\; \pi \ge -\phi_1 + \phi_2 + \phi_3 + \phi_5$

4. $I = \{1,5,6\} : \pi - \phi_3 + \phi_5 + \pi - \phi_1 - \phi_2 \ge \pi \;\Leftrightarrow\; \pi \ge -\phi_5 + \phi_1 + \phi_3 + \phi_2$

5. $I = \{2,5,6\} : \pi - \phi_5 + \phi_4 + \pi - \phi_1 - \phi_2 \ge \pi \;\Leftrightarrow\; \pi \ge -\phi_4 + \phi_1 + \phi_5 + \phi_2$

6. $I = \{3,5,6\} : \pi - \phi_5 + \phi_3 + \pi - \phi_1 - \phi_2 \ge \pi \;\Leftrightarrow\; \pi \ge -\phi_3 + \phi_1 + \phi_5 + \phi_2$

7. $I = \{2,3,4\}; \{2,3,5\}; \{2,3,4,5\} \;:\; u = e_3 - e_2$

8. $I = \{2,3,5\}; \{2,3,6\}; \{2,3,5,6\} \;:\; u = e_2 - e_3$.

This proves copositivity.

All angle inequalities are satisfied strictly and the vectors u are not nonnegative. Hence there are no additional minimal zeros.

Extremality

We use the extremality criterion Theorem 17 point 5 in [2]. The matrix A is extremal whenever every matrix B satisfying $(Bu_i)_j = 0$ whenever $(Au_i)_j = 0$ is proportional to A. Let us consider the elements $(Au_i)_j$.

The following elements are always zero:

$$(Au_1)_{1,2,3,5},\ (Au_2)_{1,2,3,5,6},\ (Au_3)_{1,4,5},\ (Au_4)_{2,4,6},\ (Au_5)_{3,4,6},\ (Au_6)_{4,5,6}. \tag{2}$$

The following elements may become zero: If $\phi_3 = \phi_4$, then

$$(Au_1)_6 = (Au_4)_3 = (Au_5)_2 = 0.$$

If $\phi_1 + \phi_2 + \phi_4 + \phi_5 = \pi$, then

$$(Au_4)_1 = (Au_3)_2 = (Au_6)_2 = (Au_1)_4 = (Au_4)_5 = 0.$$

The following elements are always positive:

$$(Au_2)_4, (Au_3)_{3,6}, (Au_5)_{1,5}, (Au_6)_{1,3}.$$

We now use relations (2), which translate to corresponding relations on B. Consider the face of A. For every B in this face there exists a matrix $P \in \mathcal{S}_+^2$ such that

$$FPF^T = \begin{pmatrix} b_{11} & \star & \star & b_{14} & b_{15} & \star \\ \star & b_{22} & \star & b_{24} & \star & b_{26} \\ \star & \star & b_{33} & b_{34} & \star & b_{36} \\ b_{14} & b_{24} & b_{34} & b_{44} & b_{45} & b_{46} \\ b_{15} & \star & \star & b_{45} & b_{55} & b_{56} \\ \star & b_{26} & b_{36} & b_{46} & b_{56} & b_{66} \end{pmatrix}, \qquad (3)$$

where F is a 6×2 matrix of rank 2 such that $u_i^T F = 0,\ i = 3,4,5,6$. By means of the relations $(Bu_1)_j = 0$, $(Bu_2)_j = 0$ for appropriate j the missing elements b_{ij} are determined from the elements which are present in (3) by

$$b_{12} = -b_{11},\ b_{13} = -b_{11},\ b_{16} = -b_{36},\ b_{23} = -b_{13},\ b_{25} = -b_{15},\ b_{35} = -b_{15}. \qquad (4)$$

However, we obtain also the additional conditions $b_{11} = b_{22} = b_{33}$ on the elements present in (3) which translate to restrictions on P.

The first three rows of F have the left kernel vector

$$(\sin(\phi_3 - \phi_4), -\sin(\phi_1 + \phi_2 + \phi_3 + \phi_5), \sin(\phi_1 + \phi_2 + \phi_4 + \phi_5)),$$

and we may assume

$$F = \begin{pmatrix} 1 & 0 \\ \frac{\sin(\phi_3-\phi_4)}{\sin(\phi_1+\phi_2+\phi_3+\phi_5)} & \frac{\sin(\phi_1+\phi_2+\phi_4+\phi_5)}{\sin(\phi_1+\phi_2+\phi_3+\phi_5)} \\ 0 & 1 \\ \star & \star \\ \star & \star \\ \star & \star \end{pmatrix}.$$

Then $b_{11} = p_{11}$, $b_{33} = p_{22}$, $b_{22} = \frac{\sin^2(\phi_3-\phi_4)}{\sin^2(\phi_1+\phi_2+\phi_3+\phi_5)}p_{11} + 2\frac{\sin(\phi_3-\phi_4)\sin(\phi_1+\phi_2+\phi_4+\phi_5)}{\sin^2(\phi_1+\phi_2+\phi_3+\phi_5)}p_{12} + \frac{\sin^2(\phi_1+\phi_2+\phi_4+\phi_5)}{\sin^2(\phi_1+\phi_2+\phi_3+\phi_5)}p_{22}$, and the condition $b_{11} = b_{22} = b_{33}$ yields $p_{11} = p_{22}$ and a second linear condition

$$p_{11} = \frac{\sin^2(\phi_3 - \phi_4)p_{11} + 2\sin(\phi_3 - \phi_4)\sin(\phi_1 + \phi_2 + \phi_4 + \phi_5)p_{12} + \sin^2(\phi_1 + \phi_2 + \phi_4 + \phi_5)p_{22}}{\sin^2(\phi_1 + \phi_2 + \phi_3 + \phi_5)} \qquad (5)$$

on P.

Let us now consider the different cases. Note that the relations $\phi_3 = \phi_4$ and $\phi_1 + \phi_2 + \phi_4 + \phi_5 = \pi$ cannot hold simultaneously, because $\phi_1 + \phi_2 + \phi_3 + \phi_5 < \pi$.

Consider the case $\phi_3 < \phi_4$, $\phi_1+\phi_2+\phi_4+\phi_5 < \pi$: Here P is determined completely in dependence of p_{11}, because the coefficient at p_{12} in (5) is non-zero. Thus in this case there is no linearly independent solution B and A is extremal.

Consider the case $\phi_3 = \phi_4$, $\phi_1 + \phi_2 + \phi_4 + \phi_5 < \pi$: We get

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \star & \star \\ \star & \star \\ \star & \star \end{pmatrix}.$$

The equations $b_{11} = b_{22} = b_{33}$ are satisfied by the relation $p_{11} = p_{22}$ alone.

However, in (3) the element b_{23} appears in addition by each of the relations $(Bu_4)_3 = 0$, $(Bu_5)_2 = 0$, and by (4) this gives the additional condition $b_{11} = b_{23}$ between the elements of (3). Further, the relation $(Bu_1)_6 = 0$ yields $b_{16} = -b_{26}$, which translates to the relation $b_{26} = b_{36}$ between the elements of (3).

However, since the second and third row of F are now identical, these relations are satisfied automatically. Hence there remains the solution $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is linearly independent from the solution generated by A.

Since not all rows of F have zero elements, this matrix P gives rise to a non-zero solution B which is linearly independent of A, and A cannot be extremal.

Consider the case $\phi_3 < \phi_4$, $\phi_1 + \phi_2 + \phi_4 + \phi_5 = \pi$: The factor F takes the form

$$F = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ \star & \star \\ \star & \star \\ \star & \star \end{pmatrix}.$$

We now have also $u_1^T F = 0$.

The additional relation $(Bu_6)_2 = 0$ makes b_{25} appear at the respective place in (3). Now $(Bu_3)_2 = 0$ makes b_{12} appear, and $(Bu_4)_1 = 0$ makes b_{16} appear.

At this stage, both b_{16}, b_{26} are present in (3), and the condition $u_1^T F = 0$ leads to the relation $b_{16} + b_{26} = 0$. It follows that $\phi_3 = \phi_4$, a contradiction.

Result

In Case 6 the extremal matrices with unit diagonal are given by (1) with $\phi_i > 0$, $\phi_1 + \phi_2 + \phi_4 + \phi_5 < \pi$, $\phi_3 < \phi_4$, as well as those obtained by exchanging row and column indices 2 and 3 in (1).

References

[1] Peter J.C. Dickinson. *A new certificate for copositivity. Optimization Online.*

[2] Peter J.C. Dickinson and Roland Hildebrand. *Considering copositivity locally.* J.Math. Anal.Appl., 437(2):11841195, 2016