TD 2 - Temporal point processes

Beyond Poisson processes

1 Basic exercises

Exercise 1. Let $X_1,\dots,X_k$ be independent random variables with respective hazard rate functions q_i, $i=1,\dots,k$. Let $Z=\min_{i=1,\dots,k} X_i$. Prove that the hazard rate function of Z is $q=\sum_{i=1}^k q_i$.

Exercise 2 (Life time k-sample)**.** Let $(\xi_1,\dots,\xi_k)$ be i.i.d. positive random variables with common hazard rate function q. Let $N=\{\xi_1,\dots,\xi_k\}$ denote the point process made of those k random variables. Prove that the intensity of N is

$$\lambda_t=(k-N_{t-})q(t).$$

Exercise 3 (Poisson contamination)**.** Let $h:\mathbb{R}_+\to\mathbb{R}_+$ and $(N^i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. Poisson processes with intensity $h(t)$. Let $N_t=\sum_{i=0}^\infty N^i_{t-i}$. In words, the point process N is the aggregation of the processes N^i where each of them is translated so that its starting time is $t=i$ instead of $t=0$.

1. What is the intensity of N ?
2. Consider the same construction as above except that the starting times $t=i$ are replaced by $t=\tilde{T}_i$ where $\tilde{N}=\{\tilde{T}_i, i\in\mathbb{N}\}$ is a point process independent of $(N^i)_{i\in\mathbb{N}}$. Formally, in this case we defined N as $N_t=\sum_{i=0}^\infty N^i_{t-\tilde{T}_i}$. What is the intensity of N in this case ?

Exercise 4 (Thinning simulation)**.** Let Π be a Poisson process with rate 1 on $\mathbb{R}_+^2$ and $\lambda_t=\lambda(t;N\cap[0,t))$ be a specified intensity. Remind that the thinning representation yields that the point process N defined by

$$N_t=\int_0^t\int \mathbf{1}_{[0,\lambda_s]}(z)\Pi(ds,dz),$$

admits λ_t as an intensity.

Use the idea of the thinning representation to write an algorithm which gives a simulation of a point process with the following intensities :

1. *renewal process*: the intensity depends on the time elapsed since the last point, i.e. $\lambda_t=q(t-T_{N_{t-}})$, with $q(x)=K/(a+x)$. K and a are positive real parameters.
2. *linear Wold process*: the intensity depends on the time elapsed since the last point and the interval between the last two points, i.e. $\lambda_t=\mu+\alpha_1(t-T_{N_{t-}})+\alpha_2(T_{N_{t-}}-T_{N_{t-}-1})$, with the convention that $T_{-1}=0$. μ, α_1 and α_2 are positive real parameters.

You may vary the values of the parameters to see the qualitative behavior of those processes. In each case, plot the graph of the intensity function as well as the points of the underlying Poisson process Π (those who are accepted and those who are rejected).
Careful: the intensity of the linear Wold process is not bounded a priori. Hence you must adapt the algorithm sketched during the course.

Exercise 5 (Change-time simulation)**.** Let Π be a Poisson process with rate 1 on $\mathbb{R}_+$. Let $\lambda_t = \lambda(t; \Pi \cap [0, \Lambda_t))$ be a specified intensity, where $\Lambda_t = \int_0^t \lambda_s ds$. Remind that the change-time representation yields that the point process N defined by

$$N_t = \Pi_{\Lambda_t}$$

admits λ_t as an intensity.

Use the idea of the change-time representation to write an algorithm which gives a simulation of a point process with the following intensities :

1. *renewal process*: the intensity depends on the time elapsed since the last point, i.e. $\lambda_t = q(t - T_{N_{t-}})$, with $q(x) = x^\alpha$ with $\alpha = -\frac{1}{2}, 0, \frac{1}{2}$ or 1 for instance.
2. *linear Wold process*: the intensity depends on the time elapsed since the last point and the interval between the last two points, i.e. $\lambda_t = \mu + \alpha_1(t - T_{N_{t-}}) + \alpha_2(T_{N_{t-}} - T_{N_{t-}-1})$, with the convention that $T_{-1} = 0$. μ, α_1 and α_2 are positive real parameters.

You may vary the values of the parameters to see the qualitative behavior of those processes. *Comparison can be made between the two methods with respect to the simulation of the linear Wold process.*

2 Intermediate exercises

Exercise 6 (Generalization of exercises 2 and 3)**.** Let N^1 and N^2 be two independent point processes with intensities λ^1 and λ^2. Prove that $N = N^1 \cup N^2$ is a point process with intensity $\lambda_t = \lambda_t^1 + \lambda_t^2$.

Exercise 7 (Thinning coupling)**.** Let $\lambda^1, \lambda^2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two measurable functions. Use the thinning representation to construct two Poisson processes N^1 and N^2 with respective intensities λ^1 and λ^2 such that, for all $t \geq 0$,

$$\mathbb{E}\left[\mathrm{Card}\left(\left(N^1 \Delta N^2\right) \cap [0,t]\right)\right] = \int_0^t |\lambda^1(s) - \lambda^2(s)| ds, \tag{1}$$

and

$$\mathbb{P}\left(\left(N^1 \Delta N^2\right) \cap [0,t] \neq \emptyset\right) = 1 - \exp\left(-\int_0^t |\lambda^1(s) - \lambda^2(s)| ds\right),$$

where $A \Delta B$ denotes the symmetric difference between two sets A and B, that is $A \Delta B = (A \cap B^c) \cup (A^c \cap B)$.

1. *Favorable case:* Assume that λ^2 is such that $|\lambda_t^1 - \lambda_t^2| \leq \varepsilon$ for all $t \geq 0$ and some $\varepsilon > 0$. Prove that, for all $t \geq 0$,
$$\mathbb{P}\left(\left(N^1 \Delta N^2\right) \cap [0,t] \neq \emptyset\right) \xrightarrow[\varepsilon \to 0]{} 0.$$
It means that N^2 (as a function of ε) converges to N^1 in total variation on all compact time intervals.
2. *Unfavorable case:* Let $\lambda_t^1 = \sum_{k=0}^{+\infty} \mathbf{1}_{[2k, 2k+1)}(t)$ and $\lambda_t^2 = \sum_{k=0}^{+\infty} \mathbf{1}_{[2k+1, 2k+2)}(t)$. Retrieve the fact that N^1 and N^2 are disjoints, in other words that $N^1 \Delta N^2 = N^1 \cup N^2$, from Equation (1).
Note that this fact is quite obvious since the supports of the intensities λ^1 and λ^2 are disjoint.
Moreover, prove that N^1 and N^2 are independent.

Exercise 8 (Change-time coupling)**.** Let $\lambda^1, \lambda^2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two measurable functions. Use the change-time representation to construct two Poisson processes N^1 and N^2 with respective intensities λ^1 and λ^2 such that, for all $t \geq 0$,

$$\mathbb{E}\left[|N_t^1 - N_t^2|\right] = \left|\int_0^t \lambda^1(s) - \lambda^2(s)ds\right|, \tag{2}$$

and

$$\mathbb{P}\left(N_t^1 \neq N_t^2\right) = 1 - \exp\left(-\left|\int_0^t \lambda^1(s) - \lambda^2(s)ds\right|\right).$$

1. Check that the favorable case of exercise 7 is also favorable here. However, check that the convergence result is very weak here.
2. Under the unfavorable case of exercise 7, check that the two processes are disjoints once again. However, prove that for all $t \geq 0$,

$$\mathbb{E}\left[|N_t^1 - N_t^2|\right] \leq 1 \quad \text{and} \quad \mathbb{P}\left(N_t^1 \neq N_t^2\right) \leq 1 - e^{-1} \approx 0,63.$$

   Moreover, prove that $N^2 = \{T+1, T \in N^1\}$.

3 Advanced exercises

Exercise 9 (Thinning and renewal)**.** Let $q^1, q^2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two left-continuous functions. Use the thinning representation to construct (like in Exercise 7) two renewal processes N^1 and N^2 with respective intensities $\lambda_t^1 = q^1(A_t^1)$ and $\lambda_t^2 = q^2(A_t^2)$, where A^1 and A^2 are the two age processes associated with N^1 and N^2, that is $A_t^i = t - T_{N_{t-}^i}$.

Assume that q^1, q^2 are such that $|q^1(x) - q^2(x)| \leq \varepsilon$ for all x. Prove that, for all $t \geq 0$,

$$\mathbb{P}\left(\left(N^1 \Delta N^2\right) \cap [0,t] \neq \emptyset\right) \leq 1 - \exp(-\varepsilon t).$$

Exercise 10 (Lebesgue-Stieltjes integral)**.** Let $[a,b] \subset \mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ be a non negative measurable function and $g : [a,b] \to \mathbb{R}$ be a right continuous non decreasing measurable function (a main example is when g is a cumulative distribution function). The objective is to give sense to the Lebesgue-Stieltjes integral,

$$\int_a^b f(x)dg(x),$$

and to prove an integration by parts formula for this integral.

1. Prove that there exists a measure μ_g on $[a,b]$ such that for all $x < y$, $\mu_g((x,y]) = g(y) - g(x)$.

   In the following, $\int_a^b f(x)dg(x)$ is defined as the standard Lebesgue integral $\int_a^b f(x)\mu_g(dx)$.

2. Check that this definition is consistent with the fundamental theorem of calculus when g is $\mathscr{C}^1$.
3. If both f and g are non negative, non decreasing and right-continuous, prove the integration by parts formula (a.k.a. the product formula):

$$f(b)g(b) = f(a)g(a) + \int_a^b f(x)dg(x) + \int_a^b g(y)df(y) - \sum_x \Delta f(x)\Delta g(x),$$

where $\Delta f(x) = f(x) - f(x-)$ and $\Delta g(x) = g(x) - g(x-)$ gives the jumps of f and g. Note that the product in the sum above is non zero for at most countably many x's which are the common discontinuities of f and g.
Hint: One could start from the Fubini formula

$$\int_{(a,b]^2} df(x)dg(y) = \left(\int_{(a,b]} df(x)\right)\left(\int_{(a,b]} dg(y)\right)$$

and decompose the set $(a,b]^2$ into three parts.

Exercise 11 (Optimal stopping time)**.** Let N be a Poisson process with intensity λ and $T > 0$ be a fixed time horizon. We will answer the following practical problem: The process N represents a flow of items produced, that is N_t is the number of items produced until time t. As soon as an item is produced it is placed into a warehouse until time T. For any item, the owner must pay a fee proportional to its sojourn time in the warehouse. Hence, the total fee is given by:

$$\int_0^T (T-t)N(dt) = \sum_{i, T_i < T} (T - T_i),$$

where T_i is the time of production of item i. However, there is a possibility for the owner to reduce this cost. He can (only once) remove all the items in the warehouse. If we denote τ the time at which he chooses to remove all the goods, then the fee reduction is

$$N_\tau (T - \tau).$$

Of course, the choice of τ cannot anticipate the future hence it must be a stopping time. The objective is to find the optimal stopping time τ.

Let $\mathscr{S}$ be the set of $\mathscr{F}_t^N$-stopping times bounded by T. The objective is to find an optimal τ^* in $\mathscr{S}$ in the sense that

$$\forall \tau \in \mathscr{S}, \mathbb{E}\left[N_\tau (T-\tau)\right] \leq \mathbb{E}\left[N_{\tau^*}(T-\tau^*)\right].$$

1. Use Lebesgue-Stieltjes' integration by parts formula (Exercise 10) to prove that

$$N_\tau (T-\tau) = \int_0^\tau (T-t)N(dt) - \int_0^\tau N_t dt.$$

2. Deduce that

$$\tau^* = \inf\left\{t \geq 0, \, (T-t)\lambda - N_t < 0\right\}.$$