TD 1 - Temporal point processes

Poisson process - Correction

1 Basic exercises

Exercise 1. Let T be a positive continuous random variable.

Step 1. Assume that $T \sim \mathscr{E}(\lambda)$. Its cumulative distribution function is $F(t) = 1 - e^{-\lambda t}$ and its survival function is $\overline{F}(t) = e^{-\lambda t}$. By definition of conditional probability, $\mathbb{P}(T > t+s \mid T > t) = \overline{F}(t+s)/\overline{F}(t)$. Hence, it suffices to check that

$$\frac{\overline{F}(t+s)}{\overline{F}(t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \overline{F}(s).$$

Step 2. Assume that T has the memoryless property and denote $\overline{F}$ its survival function. Using again the fact that $\mathbb{P}(T > t+s \mid T > t) = \overline{F}(t+s)/\overline{F}(t)$ we get that the memoryless property is equivalent to the exponentiation identity:

$$\forall t,s \geq 0, \quad \overline{F}(t+s) = \overline{F}(t)\overline{F}(s).$$

Then, it is basic functional analysis to know that all continuous functions satisfying the exponentiation identity are of the form $\overline{F}(t) = e^{\mu t}$, $\mu \in \mathbb{R}$. Since $\overline{F}$ is a survival function, it satisfies $\overline{F}(t) \to 0$ as $t \to +\infty$ so that μ must be negative. Writing $\lambda = -\mu$, we have $\overline{F}(t) = e^{-\lambda t}$, that is the survival function of the distribution $\mathscr{E}(\lambda)$.

We conclude by using the fact that the survival function (like the cumulative distribution function) characterizes the distribution of a random variable.

Exercise 2. Let T be a positive random variable with density. Let f denote its density, $\overline{F}$ its survival function and q its hazard rate.

Step 1. Assume that $T \sim \mathscr{E}(\lambda)$. Then, we have $f(t) = \lambda e^{-\lambda t}$, $\overline{F}(t) = e^{-\lambda t}$ and

$$q(t) = \frac{f(t)}{\overline{F}(t)} = \lambda,$$

for all $t > 0$.

Step 2. Assume that $q(t) = \lambda$ for all $t > 0$. Hence, the density satisfies $f(t) = \lambda \overline{F}(t)$ and is in particular continuous on $\mathbb{R}_+^*$. In turn, the fundamental Theorem of calculus states that $\overline{F}$ is $\mathscr{C}^1$ and that $\overline{F}'(t) = -f(t)$ for all $t > 0$. Then, for all $t > 0$, we have

$$-(\ln \overline{F})'(t) = -\frac{\overline{F}'(t)}{\overline{F}(t)} = \lambda.$$

Hence $\ln \overline{F}$ is the primitive of $-\lambda$ with the initial condition that $\ln \overline{F}(0) = 0$. Hence, $\overline{F}(t) = \exp(-\lambda t)$ and we conclude because the survival function characterizes the distribution.

Exercise 3. Without loss of generality, let us restrict the study to one day, that is the interval $[0,24)$ expressed in hours. Let us denote N the non homogeneous Poisson process studied in this exercise.

1. The intensity function is
$$\lambda(t) = \begin{cases} 1, & \text{if } t \in [0,8] \cup [20,24), \\ 2, & \text{else.} \end{cases}$$

2. It is the probability that $N_1 = 0$. Since $N_1 \sim \mathscr{P}(1)$, we have $\mathbb{P}(N_1 = 0) = e^{-1}$.

3. It is the probability
$$\mathbb{P}(N_{14} - N_{13} = 2 \,|\, N_9 - N_8 = 4).$$
Since $[8,9)$ and $[13,14)$ are disjoint, the two random variables above are independent so that
$$\mathbb{P}(N_{14} - N_{13} = 2 \,|\, N_9 - N_8 = 4) = \mathbb{P}(N_{14} - N_{13} = 2) = \frac{2^2}{2!} e^{-2} = 2e^{-2}.$$

4. It is the probability
$$\mathbb{P}(N_{10} - N_8 = 5 \,|\, N_9 - N_8 = 4) = \mathbb{P}(N_{10} - N_9 = 1 \,|\, N_9 - N_8 = 4).$$
Since $[8,9)$ and $[9,10)$ are disjoint, the two random variables above are independent so that
$$\mathbb{P}(N_{10} - N_9 = 1 \,|\, N_9 - N_8 = 4) = \mathbb{P}(N_{10} - N_9 = 1) = \frac{2}{1!} e^{-2} = 2e^{-2}.$$

5. It is the probability
$$\mathbb{P}(N_{14} - N_{13} = 0 \,|\, N_{20} - N_8 = 10) = \frac{\mathbb{P}(N_{14} - N_{13} = 0, N_{20} - N_8 = 10)}{\mathbb{P}(N_{20} - N_8 = 10)}.$$
Let us denote $A = [8,13) \cup [14,20)$.
On the one hand,
$$\mathbb{P}(N_{14} - N_{13} = 0, N_{20} - N_8 = 10) = \mathbb{P}(N_{14} - N_{13} = 0, N(A) = 10).$$
Since $[13,14)$ and A are disjoint, the two random variables above are independent so that (remark that the cumulative intensity on the set A is $2 \times 11 = 22$)
$$\mathbb{P}(N_{14} - N_{13} = 0, N(A) = 10) = \mathbb{P}(N_{14} - N_{13} = 0)\,\mathbb{P}(N(A) = 10) = e^{-2} \times \frac{22^{10}}{10!} e^{-22} = \frac{22^{10}}{10!} e^{-24}.$$
On the other hand, $\mathbb{P}(N_{20} - N_8 = 10) = \frac{24^{10}}{10!} e^{-24}$.
Finally, one gets $\mathbb{P}(N_{14} - N_{13} = 0 \,|\, N_{20} - N_8 = 10) = (22/24)^{10} = (11/12)^{10}$. Remark that it is the probability that a binomial $\mathscr{B}(10, 1/12)$ random variable has value 0. This is consistent with the fact that conditionally on $N_{20} - N_8 = 10$, those ten points are uniformly distributed inside the time interval $[8,20)$.

6. It is the probability

$$\mathbb{P}(N_9 - N_7 = 1 \,|\, N_8 - N_6 = 1) = \frac{\sum_{k=0}^{1} \mathbb{P}(N_7 - N_6 = k, N_8 - N_7 = 1-k, N_9 - N_8 = k)}{\mathbb{P}(N_8 - N_6 = 1)}.$$

On the one hand, for $k = 0, 1$, we have

$$\mathbb{P}(N_7 - N_6 = k, N_8 - N_7 = 1-k, N_9 - N_8 = k) = \frac{2^k}{k!}\frac{2^{1-k}}{(1-k)!}\frac{2^k}{k!}e^{-6} = \begin{cases} 2e^{-6}, & k=0 \\ 4e^{-6}, & k=1. \end{cases}$$

On the other hand, $\mathbb{P}(N_8 - N_6 = 1) = 4e^{-4}$. Finally, one gets $\mathbb{P}(N_9 - N_7 = 2 \,|\, N_8 - N_6 = 1) = \frac{3}{2}e^{-2}$.

Exercise 4. Let N be a non homogeneous Poisson process with intensity $\lambda(t)$.

1. Assume that $\lambda(t) = 0$ on $[s,u]$. In particular, $\int_s^u \lambda(t)dt = 0$. Hence, by definition of the Poisson process, $N([s,u]) \sim \mathscr{P}(0)$. Yet, the Poisson distribution with parameter 0 is the Dirac mass at 0 by convention. In turn, it means that $N([s,u]) = 0$ almost surely.

2. Let $t \geq 0$ such that λ is right continuous at t. Let $h > 0$ and denote $\mu_h = \int_t^{t+h} \lambda(u)du$. Since λ is right continuous at t, it is easy to prove that $\mu_h \sim_{h\to 0^+} \lambda(t)h$. Then, by definition of the Poisson process,

$$\mathbb{P}(N([t,t+h]) \geq 1) = 1 - e^{-\mu_h} = \mu_h + o(\mu_h) \sim_{h\to 0^+} \lambda(t)h.$$

Remark: we also have $\mathbb{P}(N([t,t+h]) = 1) = \mu_h e^{-\mu_h} \sim_{h\to 0^+} \lambda(t)h$.

2 Intermediate exercises

Exercise 5. See the Julia notebook.

Exercise 6. See the Julia notebook.

Exercise 7. See the Julia notebook.

Exercise 8. Let $N = \{T_1 < \cdots < T_k < \dots\}$ be a Poisson process with intensity $\lambda > 0$. Let $n \in \mathbb{N}^*$, $0 < t_1 < \cdots < t_n < t_{n+1} = T$ and $h_1, \dots, h_n > 0$ such that $t_k + h_k < t_{k+1}$.

1. We have

$$\begin{aligned} &\{t_1 < T_1 \leq t_1 + h_1 < t_2 < \cdots < t_n < T_n \leq t_n + h_n\} \\ &\qquad = \left\{N_{t_1} = 0, N_{t_1+h_1} - N_{t_1} = 1, \dots, N_{t_n} - N_{t_{n-1}+h_{n-1}} = 0, N_{t_n+h_n} - N_{t_n} = 1\right\}. \end{aligned}$$

2. Thanks to the previous question and using the independence property of the Poisson process, we compute $\mathbb{P}(t_1 < T_1 \leq t_1 + h_1 < t_2 < \dots T_n \leq t_n + h_n)$ as

$$\begin{aligned} & \mathbb{P}\left(N_{t_1} = 0, N_{t_1+h_1} - N_{t_1} = 1, \dots, N_{t_n} - N_{t_{n-1}+h_{n-1}} = 0, N_{t_n+h_n} - N_{t_n} = 1\right) \\ = \; & \mathbb{P}(N_{t_1} = 0) \times \cdots \times \mathbb{P}(N_{t_n+h_n} - N_{t_n} = 1) \\ = \; & e^{-\lambda t_1} \times \lambda h_1 e^{-\lambda h_1} \times \cdots \times \lambda h_n e^{-\lambda h_n} = \lambda^n h_1 \dots h_n e^{-\lambda(t_n+h_n)}. \end{aligned}$$

Hence,

$$\frac{\mathbb{P}(t_1 < T_1 \leq t_1 + h_1 < t_2 < \dots T_n \leq t_n + h_n)}{h_1 \dots h_n} \xrightarrow[h_1,\dots,h_n \to 0]{} \lambda^n e^{-\lambda t_n}.$$

3. As claimed in the exercise, we assume that the previous question implies that $(T_1,\dots,T_n)$ admits the density $f(t_1,\dots,t_n)=\lambda^n e^{-\lambda t_n}$. By definition of the inter event intervals, we have $(S_1,\dots,S_n)=(T_1,T_2-T_1,\dots,T_n-T_{n-1})$. In other words, they are obtained through the bijective change of variables $g(t_1,\dots,t_n)=(t_1,t_2-t_1,\dots,t_n-t_{n-1})$. The Jacobian matrix of g is
$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix},$$
and so its determinant equals 1. In turn, it means that the density of $(S_1,\dots,S_n)$ is
$$f_{(S_1,\dots,S_n)}(s_1,\dots,s_n)=\lambda^n e^{-\sum_{i=1}^n s_i}.$$
We recognize the product of the density of the $\mathscr{E}(\lambda)$ distribution and so the desired result follows.

Exercise 9. Let $(S_n)_{n\geq 1}$ be a sequence of i.i.d. r.v. distributed according to $\mathscr{E}(\lambda)$. Denote $T_n=\sum_{i=1}^N S_i$ and $N=\{T_n,n\in\mathbb{N}\}$ the associated point process.

1. By assumption, we know that the density of $(S_1,\dots,S_n)$ is
$$f_{(S_1,\dots,S_n)}(s_1,\dots,s_n)=\lambda^n e^{-\sum_{i=1}^n s_i}.$$
It suffices then to follow the same lines as Exercise 8 Question 3 using the inverse of g as a change of variable.

2. Let $0\leq s\leq t$ and $k,\ell\in\mathbb{N}$. We have
$$\{N_s=k,N_t-N_s=\ell\}=\{T_1<\dots<T_k\leq s<T_{k+1}<\dots<T_{k+\ell}\leq t\}.$$

3. Thanks to the previous question, we have
$$\begin{aligned}\mathbb{P}(N_s=k,N_t-N_s=\ell) &= \mathbb{P}(T_1<\dots<T_k\leq s<T_{k+1}<\dots<T_{k+\ell}\leq t<T_{k+\ell+1}) \\ &= \int \lambda^{k+\ell+1}e^{-\lambda t_{k+\ell+1}}\mathbf{1}_{0<t_1<\dots<t_k\leq s<t_{k+1}<\dots<t_{k+\ell}\leq t<t_{k+\ell+1}}dt_1\dots dt_{k+\ell+1} \\ &= \int \lambda^{k+\ell}e^{-\lambda t}\mathbf{1}_{0<t_1<\dots<t_k\leq s<t_{k+1}<\dots<t_{k+\ell}\leq t}dt_1\dots dt_{k+\ell} \\ &= e^{-\lambda t}\left(\lambda^k\frac{s^k}{k!}\right)\left(\lambda^\ell\frac{(t-s)^\ell}{\ell!}\right).\end{aligned}$$
We recognize the product of the probability mass functions of $\mathscr{P}(\lambda s)$ and $\mathscr{P}(\lambda(t-s))$ and so the desired result follows.

Exercise 10. Let N be a non homogeneous Poisson process with right continuous intensity $\lambda(t)$ on $[0,T]$. Denote $\Lambda(t)=\int_0^t\lambda(s)ds$.

1. Following the lines of Exercise 8, we have
$$\begin{aligned}&\mathbb{P}(N_T=n,t_1<T_1\leq t_1+h_1<t_2<\dots T_n\leq t_n+h_n) \\ =\ &\mathbb{P}\left(N_{t_1}=0,N_{t_1+h_1}-N_{t_1}=1,\dots,N_{t_n}-N_{t_{n-1}+h_{n-1}}=0,N_{t_n+h_n}-N_{t_n}=1,N_T-N_{t_n+h_n}=0\right) \\ =\ &e^{-\Lambda(t_1)}\times(\Lambda(t_1+h_1)-\Lambda(t_1))e^{-(\Lambda(t_1+h_1)-\Lambda(t_1))}\times\dots\times e^{-(\Lambda(T)-\Lambda(t_n+h_n))} \\ =\ &\prod_{i=1}^n(\Lambda(t_i+h_i)-\Lambda(t_i))e^{-\Lambda(T)}.\end{aligned}$$

Using the fact that λ is right continuous, it is easy to check that for all $i = 1, \dots, n$, $(\Lambda(t_i + h_i) - \Lambda(t_i))/h_i \to \lambda(t_i)$ as $h_i \to 0$. In turn, it implies that

$$\frac{\mathbb{P}(N_T = n, t_1 < T_1 \le t_1 + h_1 < t_2 < \dots T_n \le t_n + h_n)}{h_1 \dots h_n} \xrightarrow[h_1,\dots,h_n \to 0]{} \prod_{i=1}^{n} \lambda(t_i) e^{-\Lambda(T)}. \quad (1)$$

2. Since N is a Poisson process, we have $\mathbb{P}(N_T = n) = \Lambda(T)^n / n! e^{-\Lambda(T)}$. Then, by the definition of the conditional probability, we get

$$\frac{\mathbb{P}(t_1 < T_1 \le t_1 + h_1 < t_2 < \dots T_n \le t_n + h_n | N_T = n)}{h_1 \dots h_n} \xrightarrow[h_1,\dots,h_n \to 0]{} \frac{\prod_{i=1}^{n} \lambda(t_i) e^{-\Lambda(T)}}{\Lambda(T)^n / n! e^{-\Lambda(T)}} = n! \prod_{i=1}^{n} f(t_i).$$

Exercise 11. Let N^1 and N^2 be two independent Poisson processes with intensities $\lambda^1(t)$ and $\lambda^2(t)$. Let $N_t = N_t^1 + N_t^2$, $\Lambda^i(t) = \int_0^t \lambda^i(s) ds$, for $i = 1, 2$, and $\Lambda(t) = \Lambda^1(t) + \Lambda^2(t)$.

1. Let $0 < s < t$ and $k, \ell \in \mathbb{N}$. Let us denote $\Lambda^i_(s,t) = \Lambda^i(t) - \Lambda^i(s)$ for $i = 1, 2$. We have

$$\begin{aligned}
\mathbb{P}(N_s = k, N_t - N_s = \ell) &= \sum_{k_1=0}^{k} \sum_{\ell_1=0}^{\ell} \mathbb{P}\left(N_s^1 = k_1, N_s^2 = k - k_1, N_t^1 - N_s^1 = \ell_1, N_t^2 - N_s^2 = \ell - \ell_1\right) \\
&= \sum_{k_1=0}^{k} \sum_{\ell_1=0}^{\ell} \left(\frac{\Lambda^1(s)^{k_1}}{k_1!} e^{-\Lambda^1(s)}\right) \left(\frac{\Lambda^2(s)^{k-k_1}}{(k-k_1)!} e^{-\Lambda^2(s)}\right) \\
&\qquad \times \left(\frac{\Lambda^1(s,t)^{\ell_1}}{\ell_1!} e^{-\Lambda^1(s,t)}\right) \left(\frac{\Lambda^2(s,t)^{\ell-\ell_1}}{(\ell-\ell_1)!} e^{-\Lambda^2(s,t)}\right) \\
&= \left(\frac{1}{k!} \sum_{k_1=0}^{k} \binom{k}{k_1} \Lambda^1(s)^{k_1} \Lambda^2(s)^{k-k_1}\right) e^{-\Lambda(s)} \\
&\qquad \times \left(\frac{1}{\ell!} \sum_{\ell_1=0}^{\ell} \binom{\ell}{\ell_1} \Lambda^1(s,t)^{\ell_1} \Lambda^2(s,t)^{\ell-\ell_1}\right) e^{-(\Lambda(t)-\Lambda(s))} \\
&= \frac{\Lambda(s)^k}{k!} e^{-\Lambda(s)} \times \frac{(\Lambda(t) - \Lambda(s))^\ell}{\ell!} e^{-(\Lambda(t)-\Lambda(s))}
\end{aligned}$$

2. We recognize the product of the probability mass functions of $\mathscr{P}(\Lambda(s))$ and $\mathscr{P}(\Lambda(t) - \Lambda(s))$ and so the desired result follows.

Exercise 12. Let N be a Poisson process with rate λ and $p \in]0,1[$. Let $(\varepsilon_n)_n$ be a sequence of i.i.d. random variables distributed as $\mathscr{B}(p)$ which is furthermore independent of N. Let

$$N^0 = \{T_i \in N, \, \varepsilon_i = 0\} \text{ and } N^1 = \{T_i \in N, \, \varepsilon_i = 1\}.$$

1. Let $t \ge 0$, $k, \ell \in \mathbb{N}$ and denote $n = k + \ell$. It is clear that $\{N_t^0 = k, N_t^1 = \ell\}$ implies that $N_t = n$. More precisely, we have

$$\{N_t^0 = k, N_t^1 = \ell\} = \{N_t = n, \sum_{i=1}^{n} \varepsilon_i = \ell\}.$$

Hence,

$$\mathbb{P}\left(N_t^0 = k, N_t^1 = \ell\right) = \mathbb{P}\left(N_t = n, \sum_{i=1}^{n} \varepsilon_i = \ell\right) = \mathbb{P}(N_t = n) \, \mathbb{P}\left(\sum_{i=1}^{n} \varepsilon_i = \ell\right),$$

by independence between $(\varepsilon_i)_i$ and N. Since the ε_i's are i.i.d. and $\mathscr{B}(p)$, we know that $\sum_{i=1}^n \varepsilon_i$ is binomial distributed so that

$$\mathbb{P}\left(N_t^0 = k, N_t^1 = \ell\right) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \times \binom{n}{\ell} p^\ell (1-p)^k = \frac{[\lambda(1-p)]^k}{k!} e^{-\lambda(1-p)t} \times \frac{(\lambda p)^\ell}{\ell!} e^{-\lambda p t}.$$

We recognize the product of the probability mass functions of $\mathscr{P}((1-p)\lambda t)$ and $\mathscr{P}(p\lambda t)$ and so the desired result follows.

3 Advanced exercises

Exercise 13. Let N be a Poisson mixture with random rate $\tilde{\lambda}$ (with distribution $\tilde{P}$).

$$\mathbb{P}(N_t = k) = \int_\lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} d\tilde{P}(\lambda).$$

1. Since everything is non negative, we can apply Fubini Theorem without verifying any integrability condition. It gives

$$\mathbb{E}[N_t] = \sum_{k=0}^\infty k \mathbb{P}(N_t = k) = \int_\lambda \sum_{k=0}^\infty \frac{(\lambda t)^k}{(k-1)!} e^{-\lambda t} d\tilde{P}(\lambda) = \int_\lambda \lambda t \sum_{k=1}^\infty \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} d\tilde{P}(\lambda) = \int_\lambda \lambda t d\tilde{P}(\lambda).$$

   which is the desired result.

2. Following the same ideas, we have

$$\mathbb{E}\left[N_t^2\right] = \sum_{k=0}^\infty k^2 \mathbb{P}(N_t = k) = \int_\lambda \lambda t \sum_{k=1}^\infty k \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} d\tilde{P}(\lambda).$$

   Then, we use the fact that $k\frac{(\lambda t)^{k-1}}{(k-1)!} = \lambda t \frac{(\lambda t)^{k-2}}{(k-2)!} + \frac{(\lambda t)^{k-1}}{(k-1)!}$ to get

$$\mathbb{E}\left[N_t^2\right] = \int_\lambda (\lambda t)^2 + \lambda t \, d\tilde{P}(\lambda) = \mathbb{E}\left[\tilde{\lambda}^2\right] t^2 + \mathbb{E}\left[\tilde{\lambda}\right] t.$$

   Then, we use the facts that $\mathrm{Var}(N_t) = \mathbb{E}\left[N_t^2\right] - \mathbb{E}[N_t]^2$ and $\mathrm{Var}(\tilde{\lambda}) = \mathbb{E}\left[\tilde{\lambda}^2\right] - \mathbb{E}\left[\tilde{\lambda}\right]^2$ to get the desired equality. The inequality is trivial since $\mathrm{Var}(\tilde{\lambda})t^2 \geq 0$. Finally, the equality case corresponds to the case when $t = 0$ or $\mathrm{Var}(\tilde{\lambda}) = 0$, that is $\tilde{\lambda}$ is a.s. constant.

Exercise 14. Hints:

- The value of X_t when $N_t = 0$ is not explicitly defined in the exercise statement.
- A nice way to represent X_t is for instance $X_t = \sum_{n=0}^{+\infty} \left(\sum_{i=1}^n Y_i\right) \mathbf{1}_{N_t = n}$. How to use this idea for z^{X_t} ?