Microscopic approach of a time elapsed neural model

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Action potential: brief and stereotyped phenomenon (*spike*).

Physiological constraint: refractory period.
Age structured equations (K. Pakdaman, B. Perthame, D. Salort, 2010)

- **Age** = delay since last spike.
- \( n(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the population with age } s \text{ at time } t. \end{cases} \)

\[
\begin{align*}
\frac{\partial n(t,s)}{\partial t} + \frac{\partial n(t,s)}{\partial s} + p(s, X(t)) n(t,s) &= 0 \\
m(t) &= n(t,0) = \int_0^{+\infty} p(s, X(t)) n(t,s) ds
\end{align*}
\]

(PPS)

**Parameters**

- Rate function \( p \). For example, \( p(s, X) = 1_{\{s > \sigma(X)\}} \).

\[
X(t) = \int_0^t d(x)m(t-x)dx \quad \text{(global neural activity)}
\]

- Propagation time.
- \( d \) = delay function. For example, \( d(x) = e^{-\tau x} \).
Microscopic modelling

- The spiking times are the relevant information.

**Microscopic modelling**

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Microscopic modelling

Time point processes = random countable sets of times (points of \( \mathbb{R} \) or \( \mathbb{R}_+ \)).

- \( N \) is a random countable set of points of \( \mathbb{R} \) (or \( \mathbb{R}_+ \)) locally finite a.s.
- Denote \( \cdots < T_{-1} < T_0 \leq 0 < T_1 < \cdots \) the ordered sequence of points of \( N \).
- Point measure: \( N(dt) = \sum_{i \in \mathbb{Z}} \delta_{T_i}(dt) \). Hence, \( \int f(t)N(dt) = \sum_{i \in \mathbb{Z}} f(T_i) \).
Age process

- Age = delay since last spike.

Microscopic age

- We consider the continuous to the left (hence predictable) version of the age.
Dichotomy of the behaviour of $N$ with respect to time 0:

- $N_\sim = N \cap (-\infty, 0]$ is a point process with distribution $\mathbb{P}_0$ (initial condition).
  Suppose that $N_\sim \neq \emptyset$ so that the age at time 0 is finite.
Dichotomy of the behaviour of $N$ with respect to time 0:

- $N_\neg = N \cap (-\infty, 0]$ is a point process with distribution $\mathbb{P}_0$ (initial condition).
  Suppose that $N_\neg \neq \emptyset$ so that the age at time 0 is finite.
- $N_+ = N \cap (0, +\infty)$ is a point process admitting some intensity $\lambda(t, \mathcal{F}_{t_\neg})$. 
Dichotomy of the behaviour of $N$ with respect to time $0$:

- $N_- = N \cap (-\infty, 0]$ is a point process with distribution $\mathbb{P}_0$ (initial condition). Suppose that $N_- \neq \emptyset$ so that the age at time $0$ is finite.
- $N_+ = N \cap (0, +\infty)$ is a point process admitting some intensity $\lambda(t, \mathcal{F}_t^N)$.

**Stochastic intensity**

- Heuristically,

\[
\lambda(t, \mathcal{F}_t^N) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\left(N([t, t + \Delta t]) = 1 \mid \mathcal{F}_t^N\right),
\]

where $\mathcal{F}_t^N$ denotes the history of $N$ before time $t$.

- Local behaviour: probability to find a new point (spike).
- May depend on the past (e.g. refractory period, aftershocks).
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\[
p(s, X(t)) \leftrightarrow \lambda(t, \mathcal{F}^N_{t-})
\]
Some classical point processes in neuroscience

- Poisson process: $\lambda(t, F_{t-}^N) = \lambda(t) = \text{deterministic function}$.
- Renewal process: $\lambda(t, F_{t-}^N) = f(S_{t-}) \iff \text{i.i.d. ISIs}$.
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- Renewal process: $\lambda(t, \mathcal{F}_{t^-}) = f(S_{t^-}) \iff \text{i.i.d. ISIs.}$
- Hawkes process: $\lambda(t, \mathcal{F}_{t^-}) = \mu + \int_{-\infty}^{t^-} h(t-x)N(dx). \quad h \geq 0$
Some classical point processes in neuroscience

- **Poisson process**: \( \lambda(t, \mathcal{F}_t^N) = \lambda(t) = \) deterministic function.

- **Renewal process**: \( \lambda(t, \mathcal{F}_t^N) = f(S_t) \leftrightarrow \text{i.i.d. ISIs.} \)

- **Hawkes process**: \( \lambda(t, \mathcal{F}_t^N) = \mu + \int_{-\infty}^{t-} h(t-x)N(dx) \quad h \geq 0 \)
  \[ = \mu + \sum_{V \in N, V < t} h(t-V). \]

\[ \int_{-\infty}^{t-} h(t-x)N(dx) \quad \text{convolution integrals} \quad \int_0^t d(x)m(t-x)dx = X(t). \]
Technical construction

- (PPS) system: $n(t,.)$ is the probability density of the age at time $t$.
- One neuron scale: at fixed time $t$, the distribution is a Dirac mass at $S_t$. 

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Microscopic measure

- We construct an ad hoc random measure $U$ which satisfies a system of SDEs driven by Poisson noise similar to (PPS). Ogata’s thinning (1981)

\[ U(t,ds) = \delta_{S_t}(ds) \]
Technical construction

- (PPS) system: \( n(t,.) \) is the probability density of the age at time \( t \).
- One neuron scale: at fixed time \( t \), the distribution is a Dirac mass at \( S_t^- \).

### Microscopic measure

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### Macroscopic measure

- We consider the expectation measure \( u(dt, ds) = \mathbb{E}[U(dt, ds)] \).

\( u(t,.) \) is the distribution of \( S_t^- \)
Theorem

Let \((\lambda(t, \mathcal{F}_t^-))_{t>0}\) be some non negative predictable process which is \(L^1_{loc}\) in expectation.

The measure \(u\) satisfies the following system,

\[
\begin{cases}
\frac{\partial}{\partial t} u(dt, ds) + \frac{\partial}{\partial s} u(dt, ds) + \rho_{\lambda, \mathbb{P}_0}(t, s) u(dt, ds) = 0, \\
u(dt, 0) = \int_{s \in \mathbb{R}_+} \rho_{\lambda, \mathbb{P}_0}(t, s) u(t, ds) \ dt, \\
u(0, ds) = u^{in}(s) ds, \quad u^{in} \in L^1,
\end{cases}
\]

in the weak sense where \(\rho_{\lambda, \mathbb{P}_0}(t, s) = \mathbb{E}[\lambda(t, \mathcal{F}_t^-) | S_{t^-} = s]\) for almost every \(t\).

The initial condition \(u^{in}\) is the distribution of \(-T_0\).
Theorem

Let \((\lambda(t, \mathcal{F}_{t-}^N))_{t>0}\) be some non negative predictable process which is \(L^1_{\text{loc}}\) in expectation.

The measure \(u\) satisfies the following system,

\[
\begin{aligned}
\frac{\partial}{\partial t} u(dt, ds) + \frac{\partial}{\partial s} u(dt, ds) + \rho_{\lambda, P_0}(t, s) u(dt, ds) &= 0, \\
\int_{s \in \mathbb{R}_+} \rho_{\lambda, P_0}(t, s) u(t, ds) dt &= 0, \\
\int_0^{\infty} u(0, ds) &= u^{\text{in}}(s) ds, \quad u^{\text{in}} \in L^1,
\end{aligned}
\]

in the weak sense where \(\rho_{\lambda, P_0}(t, s) = \mathbb{E}[\lambda(t, \mathcal{F}_{t-}^N) | S_{t-} = s]\) for almost every \(t\).

The initial condition \(u^{\text{in}}\) is the distribution of \(-T_0\).

Corollary (LLN)

The empirical measure \(\frac{1}{n} \sum_{i=1}^n \delta_{S_i^t} (ds)\) of the age processes associated to some i.i.d. point processes converges (in some weak sense) as \(n \to \infty\) towards the mean measure \(u\).
The system in expectation

\[
\begin{align*}
\frac{\partial}{\partial t} u(dt, ds) + \frac{\partial}{\partial s} u(dt, ds) + \rho_{\lambda, P_0}(t, s) u(dt, ds) &= 0, \\
u(dt, 0) &= \int_{s \in \mathbb{R}_+} \rho_{\lambda, P_0}(t, s) u(t, ds) \, dt, \\
u(0, ds) &= u^{in}(s) ds, \quad u^{in} \in L^1,
\end{align*}
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where \( \rho_{\lambda, P_0}(t, s) = \mathbb{E} \left[ \lambda(t, \mathcal{F}^N_t) \bigg| S_{t-} = s \right] \).

- This result may seem OK, but \( \rho \) is not explicit.
- In particular, this system may seem linear, but it is non-linear in general.
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- This result may seem OK, but \( \rho \) is not explicit.
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- \( \lambda(t, \mathcal{F}_{t-}^N) = f(t, S_{t-}) \) (Poisson, renewal).

\[ \rightarrow \rho_{\lambda, P_0}(t, s) = f(t, s) \] and the system admits a unique solution.
Review of the examples

The system in expectation

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\begin{cases}
\frac{\partial}{\partial t} u(dt, ds) + \frac{\partial}{\partial s} u(dt, ds) + \rho_{\lambda, P_0}(t, s) u(dt, ds) = 0, \\
\int_{s \in \mathbb{R}^+} \rho_{\lambda, P_0}(t, s) u(t, ds) dt = u(dt, 0), \\
u(0, ds) = u^{in}(s) ds, \quad u^{in} \in L^1,
\end{cases}
\]

where \( \rho_{\lambda, P_0}(t, s) = \mathbb{E} \left[ \lambda(t, \mathcal{F}^N_{t-}) | S_{t-} = s \right] \).

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\( \rightarrow \) \( \rho_{\lambda, P_0}(t, s) = f(t, s) \) and the system admits a unique solution.

\( \rightarrow \) \( \rho_{\lambda, P_0} \) is much more complex.

- Hawkes process.
The system in expectation

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\begin{aligned}
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u(0, ds) &= u^{in}(s) ds, \quad u^{in} \in L^1,
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where \( \rho_{\lambda, P_0}(t, s) = \mathbb{E} \left[ \lambda(t, F_{t-}^N) \mid S_{t-} = s \right] \).

- This result may seem OK, but \( \rho \) is not explicit.
- In particular, this system may seem linear, but it is non-linear in general.

- \( \lambda(t, F_{t-}^N) = f(t, S_{t-}) \) (Poisson, renewal).
  \[\rightarrow \rho_{\lambda, P_0}(t, s) = f(t, s) \text{ and the system admits a unique solution.} \]
- Hawkes process.
  \[\rightarrow \rho_{\lambda, P_0} \text{ is much more complex.} \]

The integral \( v(t, s) := \int_s^{+\infty} u(t, d\sigma) \) satisfies a closed system.
Linear Hawkes process

\[
\lambda(t, \mathcal{F}_{t-}^N) = \mu + \int_{-\infty}^{t-} h(t-x)N(dx) \quad \text{where } N_{-} \sim \mathbb{P}_0
\]
**Linear Hawkes process**

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\]

**Technical difficulties**

- The conditional expectation \( \rho_{\lambda, \mathbb{P}_0}(t, s) = \mathbb{E}[\lambda(t, \mathcal{F}_t^N) | S_t^- = s] \).
- No closed PDE system on \( u \).
Linear Hawkes process

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\lambda(t, \mathcal{F}_{t-}^N) = \mu + \int_{-\infty}^{t-} h(t-x) N(dx) \quad \text{where } N_- \sim P_0
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Technical difficulties

- The conditional expectation \( \rho_{\lambda, P_0}(t, s) = \mathbb{E}[\lambda(t, \mathcal{F}_{t-}^N) \mid S_{t-} = s] \).
- No closed PDE system on \( u \).

- The integral of \( u \) given by \( v(t, s) := \int_s^{+\infty} u(t, d\sigma) \) satisfies a closed system:

There exists \( \Phi_{P_0}^{\mu, h} \) (depending only on \( \mu, h \) and \( P_0 \)) such that

\[
\begin{cases}
\frac{\partial}{\partial t} v(t, s) + \frac{\partial}{\partial s} v(t, s) + \Phi_{P_0}^{\mu, h}(t, s)v(t, s) = 0, \\
v(t, 0) = 1, \quad v(t = 0, s) = v^{in}(s),
\end{cases}
\tag{PPS-\( v \)}
\]

where \( v^{in} \) is the survival function of \( -T_0 \).
Summary

- Microscopic system.
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- Population-based version. No dependence between neurons.
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- Population-based version. No dependence between neurons.
- PDE analysis \( \leadsto \) density (and regularity) for the age distribution.
- Point processes \( \leadsto \) PDE parameters.
Summary

- Microscopic system.
- System in expectation.
- Population-based version. No dependence between neurons.
- PDE analysis $\rightarrow$ density (and regularity) for the age distribution.
- Point processes $\rightarrow$ PDE parameters.

- Outlook:
  - Regularity of the mean measure $u$.
  - Mean field limit: multivariate Hawkes processes with weak interaction.
References

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