Pricing Parisian options using numerical inversion of Laplace transforms

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Plan

Definition

- barrier options counting the time spent in a row above (resp. below) a fixed level (the *barrier*). If this time is longer than a fixed value (the window width), the option is activated ("In") or canceled ("Out").
- Parisian options are less sensitive to influential agent on the market than standard barrier options.
- Parisian options naturally appears in the analysis of structured products such as re-callable convertible bond: the owner wants to recall its bond if ever the underlying stock has been traded out of a given range for a while.
- Some attempts to use Parisian options to price credit risk derivatives.

Definition (single barrier)

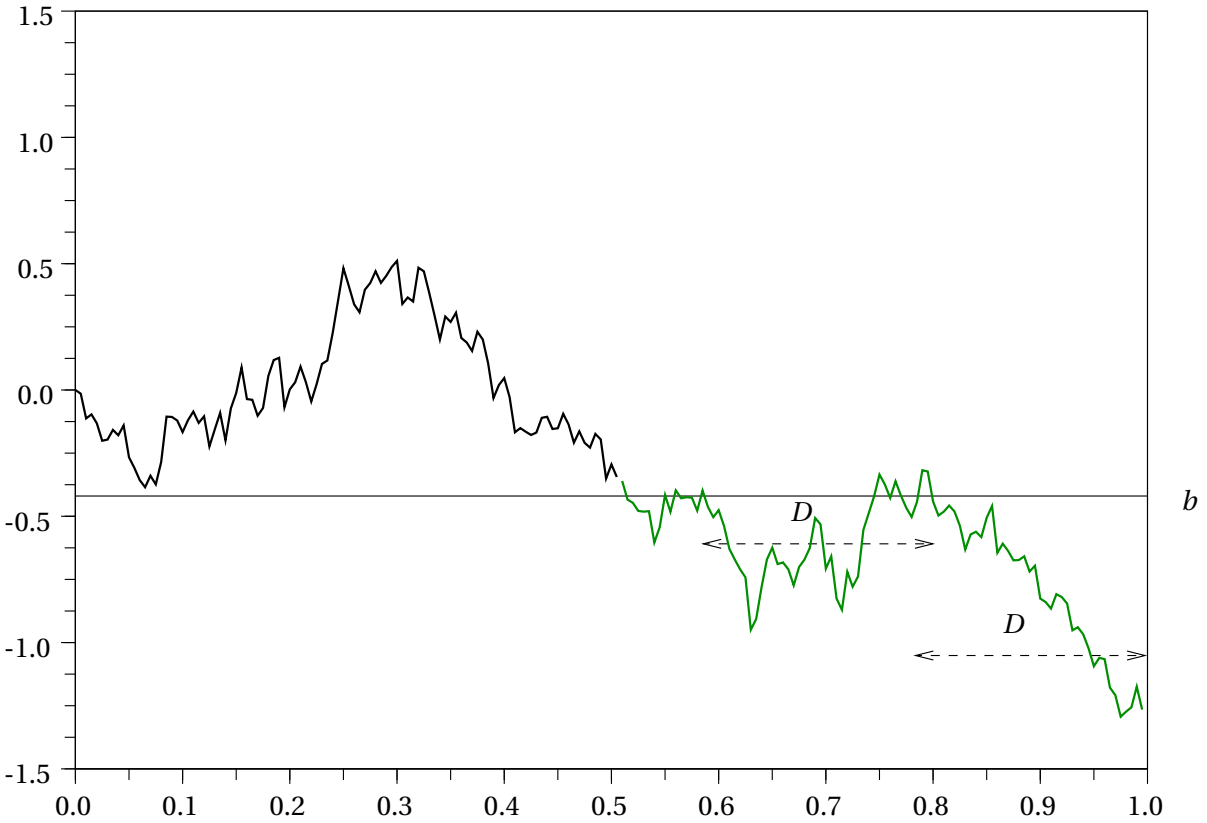


FIG.: single barrier Parisian option

Definition (single barrier)

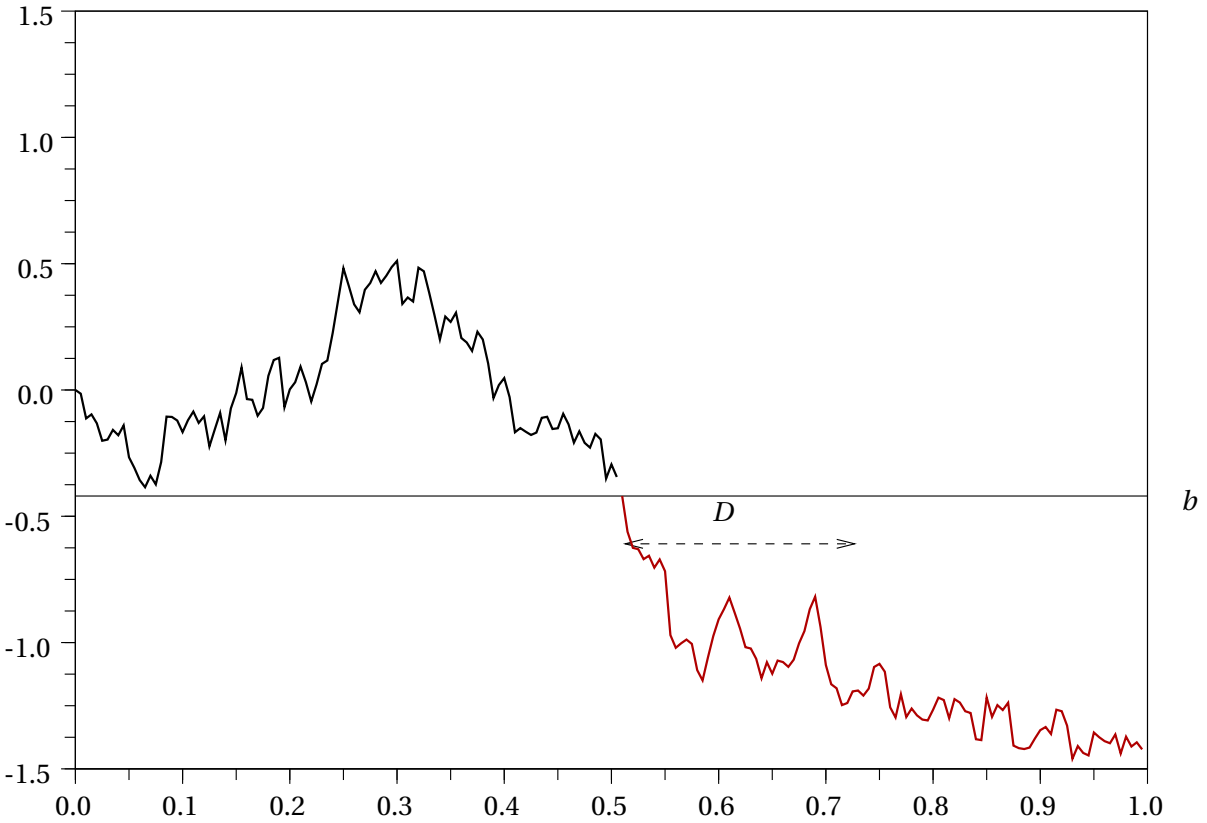


FIG.: single barrier Parisian option

Definition (double barrier)

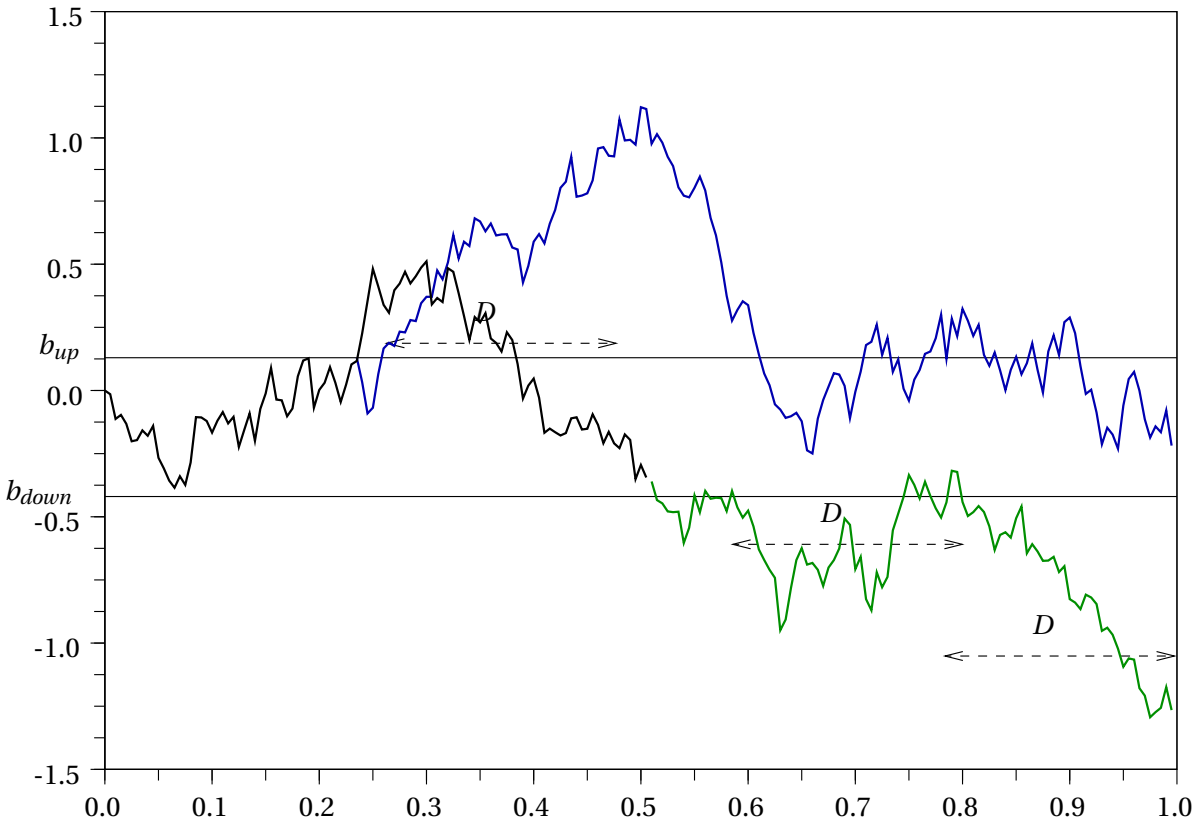


FIG.: double barrier Parisian option

Payoff

- Parisian Down In Call,

$$(S_T - K)_+ \mathbf{1}_{\left\{ \substack{\exists\, 0 \le t_1 < t_2 \le T \\ t_2 - t_1 \ge D} \text{, s.t. } \forall u \in [t_1, t_2]\ S_u \le L \right\}}.$$

  L is the barrier and D the option window.

- Parisian Up Out Call,

$$(S_T - K)_+ \mathbf{1}_{\left\{ \substack{\forall\, 0 \le t_1 < t_2 \le T \\ t_2 - t_1 \ge D} \text{, s.t. } \exists u \in [t_1, t_2]\ S_u < L \right\}}.$$

- Parisian Double Out Call

$$(S_T - K)_+ \mathbf{1}_{\left\{ \substack{\forall\, 0 \le t_1 < t_2 \le T \\ t_2 - t_1 \ge D} , \exists u \in [t_1, t_2] \text{ s.t. } S_u < L_{\text{up}} \text{ and } S_u > L_{\text{down}} \right\}}.$$

Definition

- The price of a Parisian Down In Call (PDIC) is given by

$$f(T) = \mathrm{e}^{-(r+\frac{m^2}{2})T}\underbrace{\mathbb{E}_{\mathbb{P}}\left(\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T}-K)_+\mathbf{1}_{\left\{\substack{\exists\, 0\le t_1<t_2\le T\\ t_2-t_1\ge D}\text{, s.t. } \forall u\in[t_1,t_2]\, Z_u\le b\right\}}\right)}_{\text{"star" price}},$$

where $b=\frac{1}{\sigma}\log\left(\frac{L}{x}\right)$ and Z is a $\mathbb{P}-$B.M.

- Let $W=(W_t, t\ge 0)$ be a B.M on $(\Omega,\mathscr{F},\mathbb{Q})$, with $\mathscr{F}=\sigma(W)$. Assume that

$$S_t = x\mathrm{e}^{(r-\delta-\frac{\sigma^2}{2})t+\sigma W_t}.$$

- We set $m=\dfrac{r-\delta-\frac{\sigma^2}{2}}{\sigma}$. We can introduce $\mathbb{P}\sim\mathbb{Q}$ s.t.

$$\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}(\phi(S_t, t\le T)) = \mathrm{e}^{-(r+\frac{m^2}{2})T}\mathbb{E}_{\mathbb{P}}(\mathrm{e}^{mZ_T}\phi(x\mathrm{e}^{Z_t}, t\le T))$$

where Z is $\mathbb{P}$-B.M.

Brownian Excursions I

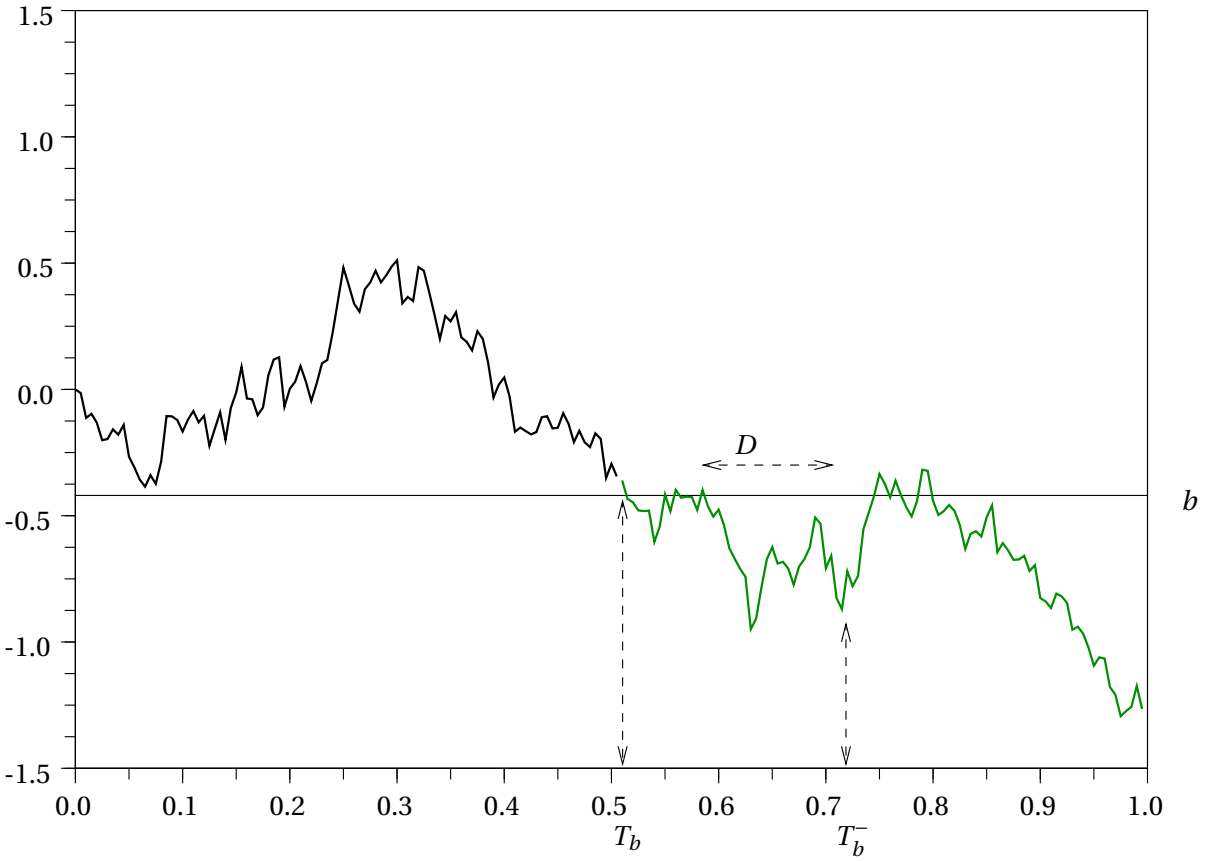


FIG.: Brownian excursions

Brownian Excursions II

$$\begin{aligned} g_t^b &= \sup\{u \le t \mid Z_u = b\}, \\ T_b^- &= \inf\{t > 0 \mid (t - g_t^b)\,\mathbf{1}_{\{Z_t < b\}} > D\}, \\ T_b^+ &= \inf\{t > 0 \mid (t - g_t^b)\,\mathbf{1}_{\{Z_t > b\}} > D\}. \end{aligned}$$

T_b^- : first time the B.M. Z stays below b longer than D.
Price of a PDIC :

$$f(T) = \mathrm{e}^{-(r+\frac{m^2}{2})T}\,\mathbb{E}_{\mathbb{P}}\Big(\underbrace{\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)_+}_{\text{Call payoff}}\ \underbrace{\mathbf{1}_{\{T_b^- < T\}}}_{\text{Parisian part}}\Big).$$

Price of a Double Parisian Out Call :

$$f(T) = \mathrm{e}^{-(r+\frac{m^2}{2})T}\,\mathbb{E}_{\mathbb{P}}\Big(\underbrace{\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T} - K)_+}_{\text{Call payoff}}\ \underbrace{\mathbf{1}_{\{T_{b_{low}}^- > T\}}\mathbf{1}_{\{T_{b_{up}}^+ > T\}}}_{\text{Parisian part}}\Big).$$

Some parity relationships

Same kind of parity relationships as for standard barrier options. For instance,

$$P\left\{\begin{matrix} D \\ U \end{matrix}\right\}\left\{\begin{matrix} O \\ I \end{matrix}\right\} P(x,T;K,L;r,\delta) = xK\, P\left\{\begin{matrix} U \\ D \end{matrix}\right\}\left\{\begin{matrix} O \\ I \end{matrix}\right\} C\left(\frac{1}{x},T;\frac{1}{K},\frac{1}{L};\delta,r\right).$$

$$PDOP(x,T;K,L;r,\delta) = \mathbb{E}\left(\mathrm{e}^{mZ_T}(K - x\mathrm{e}^{\sigma Z_T})^+ \mathbf{1}_{\{T_b^- > T\}}\right)\mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T}.$$

By introducing the new B.M. $W = -Z$, we can rewrite

$$\mathbb{E}\left(\mathrm{e}^{-mW_T}(K - x\mathrm{e}^{-\sigma W_T})^+ \mathbf{1}_{\{T_{-b}^+ > T\}}\right)\mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T} =$$

$$xK\,\mathbb{E}\left(\mathrm{e}^{-(m+\sigma)W_T}\left(\frac{1}{x}\mathrm{e}^{\sigma W_T} - \frac{1}{K}\right)^+ \mathbf{1}_{\{T_{-b}^+ > T\}}\right)\mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T}.$$

Link between single and double barrier Parisian options

Consider a Double Parisian Out Call (DPOC)

$$DPOC(x,T;K,L_d,L_u;r,\delta) = e^{-(\frac{m^2}{2}+r)T}\mathbb{E}\left[e^{mZ_T}(S_T-K)_+\mathbf{1}_{\{T^-_{b_d}>T\}}\mathbf{1}_{\{T^+_{b_u}>T\}}\right].$$

Rewrite the two indicators

$$\mathbf{1}_{\{T^-_{b_d}>T\}}\mathbf{1}_{\{T^+_{b_u}>T\}} = \underbrace{1}_{\text{Call}} - \underbrace{\mathbf{1}_{\{T^-_{b_d}<T\}}}_{PDIC(L=L_d)} - \underbrace{\mathbf{1}_{\{T^+_{b_u}<T\}}}_{PUIC(L=L_u)} + \underbrace{\mathbf{1}_{\{T^-_{b_d}<T\}}\mathbf{1}_{\{T^+_{b_u}<T\}}}_{\text{new term}}.$$

Plan

What is known about Parisian options

- There is no explicit formula for the law of T_b^- : we only know its Laplace transform[1].
- We know that the r.v. T_b^- and $Z_{T_b^-}$ are independent and we know the density of $Z_{T_b^-}$.
- Chesney et al. have shown that it is possible to compute the Laplace transforms (w.r.t. maturity time) of the Parisian option prices.

[1] [Chesney et al., 1997]

Monte Carlo approach

- Crude Monte Carlo simulations perform badly because of the time discretisation.
  Attempt to improve the crude Monte Carlo by Baldi, Caramellino and Iovino.
- Recover the density of T_b^- by numerically inverting its the Laplace transform.

$$\mathbb{E}_{\mathbb{P}}(\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma Z_T}-K)_+\mathbf{1}_{\{T_b^-<T\}}) = \mathbb{E}_{\mathbb{P}}(\mathrm{e}^{mZ_{T_b^-}}\,\mathrm{e}^{mW_{T-T_b^-}}(x\mathrm{e}^{\sigma Z_{T_b^-}}\,\mathrm{e}^{\sigma W_{T-T_b^-}}-K)_+\mathbf{1}_{\{T_b^-<T\}}),$$

where $W \perp\!\!\!\perp Z$.

$$f^\star(T) = \int_0^\infty \int_{\mathbb{R}} \mathrm{e}^{my}\,\mathrm{e}^{\frac{m^2}{2}(T-t)}\,C(x\mathrm{e}^{\sigma y}, T-t)\mathbf{1}_{\{t<T\}}\,d_{T_b^-\otimes Z_{T_b^-}}(t,y),$$

where $C(z,t)$ is the price of a Call option with maturity t and spot z.
Actually $d_{T_b^-\otimes Z_{T_b^-}} = d_{T_b^-}\times d_{Z_{T_b^-}}$.

EDP approach

- EDP for the price $V(t,x)$ of a vanilla call option

$$\partial_t V + \frac{1}{2}\sigma^2 x^2 \partial_{xx} V + rx\partial_x V - rV \quad = \quad 0 \quad (t,x) \in [0,T) \times \mathbb{R}_+^*.$$

- The payoff is not Markovian. Need to introduce a second state variable $\Longrightarrow$ solve a 2-dimensional EDP[2].

$$\tau = t - \sup\{u \leq t : S_u \geq L\}.$$

$$\begin{cases} \partial_t V + \frac{1}{2}\sigma^2 x^2 \partial_{xx} V + rx\partial_x V - rV & = 0, \quad \text{on } [0,T) \times (L,\infty), \\ \partial_t V + \frac{1}{2}\sigma^2 x \partial_{xx} V + rx\partial_x V - rV + \partial_\tau V & = 0, \quad \text{on } [0,T) \times [0,L), \end{cases}$$

continuity condition : $V(L,t,\tau) = V(L,t,0)$.

[2] [Haber et al., 1999]

Laplace transform approach

- Use Laplace transforms as suggested by Chesney, Jeanblanc and Yor[3]. Few numerical computations but not straightforward to implement.
- We have managed to find "closed" formulae for the Laplace transforms of the Parisian (single and double barrier) option prices.

[3] [Chesney et al., 1997]

Example of a Laplace transform

For all $\lambda > \frac{(m+\sigma)^2}{2}$ and $K > L$, the Laplace transform of a Parisian Down In call price is given by

$$\widehat{f}^{\star}(\lambda) = \frac{\psi(-\theta\sqrt{D})\,\mathrm{e}^{2b\theta}}{\theta\psi(\theta\sqrt{D})} K\,\mathrm{e}^{(m-\theta)k}\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right),$$

with

$$\psi(z) \triangleq \int_0^{+\infty} x e^{-\frac{x^2}{2}+zx} dx = 1 + z\sqrt{2\pi} e^{\frac{z^2}{2}} \mathcal{N}(z),$$

$$\mathcal{N}(z) \triangleq \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{u^2}{2}}\, du.$$

Plan

Fourier series representation

Fact

Let f be a continuous function defined on $\mathbb{R}^+$ and α a positive number. Assume that the function $f(t)\,\mathrm{e}^{-\alpha t}$ is integrable. Then, given the Laplace transform $\hat{f}$, f can be recovered from the contour integral

$$f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \mathrm{e}^{st}\hat{f}(s)\,ds, \quad t>0.$$

Problem : the Laplace transforms have been computed for real values of the parameter λ.
$\Longrightarrow$ We have to prove that they are analytic in a complex half plane and find their abscissa of convergence.

Analytical prolongation

Proposition 1 (abscissa of convergence)

The abscissa of convergence of the Laplace transforms of the star prices of Parisian options is smaller than $\frac{(m+\sigma)^2}{2}$. *All these Laplace transforms are analytic on the complex half plane* $\{z \in \mathbb{C} : \mathscr{R}\mathrm{e}(z) > \frac{(m+\sigma)^2}{2}\}$.

Proof : It is sufficient to notice that the star price of a Parisian option is bounded by $\mathbb{E}(\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma W_T}+K))$.

$$\mathbb{E}(\mathrm{e}^{mZ_T}(x\mathrm{e}^{\sigma W_T}+K)) \leq K\mathrm{e}^{\frac{m^2}{2}T} + x\mathrm{e}^{\frac{(m+\sigma)^2}{2}T} = \mathscr{O}(\mathrm{e}^{\frac{(m+\sigma)^2}{2}T}).$$

Hence, [Widder, 1941, Theorem 2.1] yields that the abscissa of convergence of the Laplace transforms of the star prices is smaller that $\frac{(m+\sigma)^2}{2}$. The second part of the proposition ensues from [Widder, 1941, Theorem 5.a]. ▲

Analytical prolongation

Lemma 1 (Analytical prolongation of $\mathcal{N}$)

The unique analytic prolongation of the normal cumulative distribution function on the complex plane is defined by

$$\mathcal{N}(x+iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{(v+iy)^2}{2}}\, dv.$$

Proof : It is sufficient to notice that the function defined above is holomorphic on the complex plane (and hence analytic) and that it coincides with the normal cumulative distribution function on the real axis. ▲

Trapezoidal rule I

$$f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \mathrm{e}^{st}\hat{f}(s)\,ds.$$

A trapezoidal discretisation of step $h = \frac{\pi}{t}$ leads to

$$f_{\frac{\pi}{t}}(t) = \frac{e^{\alpha t}}{2t}\widehat{f}(\alpha) + \frac{e^{\alpha t}}{t}\sum_{k=1}^{\infty}(-1)^k \mathscr{R}\mathrm{e}\left(\widehat{f}\left(\alpha + i\frac{k\pi}{t}\right)\right).$$

Proposition 2 (adapted from [Abate et al., 1999])

If f is a continuous bounded function satisfying $f(t) = 0$ for $t < 0$, we have

$$\left|e_{\frac{\pi}{t}}(t)\right| \stackrel{\Delta}{=} \left|f(t) - f_{\frac{\pi}{t}}(t)\right| \leq \|f\|_\infty \frac{e^{-2\alpha t}}{1 - e^{-2\alpha t}}.$$

Proposition 2 actually holds for $h < 2\frac{\pi}{t}$.

Trapezoidal rule II

We want to compute numerically

$$f_{\frac{\pi}{t}}(t) \triangleq \frac{e^{\alpha t}}{2t}\widehat{f}(\alpha) + \frac{e^{\alpha t}}{t}\sum_{k=1}^{\infty}(-1)^k \mathscr{R}\mathrm{e}\left(\widehat{f}\left(\alpha + i\frac{k\pi}{t}\right)\right).$$

We still need to approximate the series.

$$s_p(t) \triangleq \frac{e^{\alpha t}}{2t}\widehat{f}(\alpha) + \frac{e^{\alpha t}}{t}\sum_{k=1}^{p}(-1)^k \mathscr{R}\mathrm{e}\left(\widehat{f}\left(\alpha + i\frac{\pi k}{t}\right)\right).$$

very slow convergence of $s_p(t) \Longrightarrow$ need of an acceleration technique.

Euler summation

For $p, q > 0$, we set

$$E(q,p,t) = \sum_{k=0}^{q} C_q^k 2^{-q} s_{p+k}(t).$$

Proposition 3

Let $f \in \mathscr{C}^{q+4}$ such that there exists $\epsilon > 0$ s.t. $\forall k \leq q+4$, $f^{(k)}(s) = \mathscr{O}(e^{(\alpha-\epsilon)s})$, where α is the abscissa of convergence of $\hat{f}$. Then,

$$\left| f_{\frac{\pi}{t}}(t) - E(q,p,t) \right| \leq \frac{te^{\alpha t}\left|f'(0) - \alpha f(0)\right|}{\pi^2} \frac{p!\,(q+1)!}{2^q\,(p+q+2)!} + \mathscr{O}\left(\frac{1}{p^{q+3}}\right),$$

when p goes to infinity.

Regularity of the Parisian option prices

Proposition 4

Let f be the "star" price of a Parisian option of maturity t. f is of class $\mathscr{C}^\infty$ and for all $k \geq 0$, $f^{(k)}(t) = \mathscr{O}\left(\mathrm{e}^{\frac{(m+\sigma)^2}{2}t}\right)$ when t goes to infinity.

Proof :

$$f(t) = \mathbb{E}\left[e^{mZ_t}(\mathrm{e}^{\sigma Z_t} - K)_+ \mathbf{1}_{\{T_b^- < t\}}\right].$$

Relying on the strong Markov property,

$$f(t) = \mathbb{E}\left(\mathbf{1}_{\{T_b^- < t\}} \mathbb{E}\left[(xe^{\sigma(W_{t-\tau}+z)} - K)^+ e^{m(W_{t-\tau}+z)}\right]_{|z=Z_{T_b^-},\, \tau=T_b^-}\right),$$

where $W \perp\!\!\!\perp \mathscr{F}_{T_b^-}$.

$$f(t) = \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw\, (xe^{\sigma(w\sqrt{\tau}+z)} - K)^+ e^{m(w\sqrt{\tau}+z)} p(w)\nu(z)\mu(t-\tau).$$

▲

Density for a Parisian time

Proposition 5

T_b^- has a density μ w.r.t the Lebesgue measure. μ is of class $\mathscr{C}^\infty$ and for all $k \geq 0$, $\mu^{(k)}(0) = \mu^{(k)}(\infty) = 0$.

Proof :

$$\mathbb{E}\left(\mathrm{e}^{-\frac{\lambda^2}{2}T_b^-}\right) = \frac{\mathrm{e}^{\lambda b}}{\psi(\lambda\sqrt{D})} \quad \text{for } \lambda \in \mathbb{R}.$$

Both sides are analytic on $\mathscr{O} = \{z \in \mathbb{C}; -\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}\}$. Continuity for $\arg(z) = \pm\frac{\pi}{4} \Longrightarrow$ for all $u \in \mathbb{R}$ $\quad \mathbb{E}\left(\mathrm{e}^{-iuT_b^-}\right) = \frac{\mathrm{e}^{\sqrt{2ui}b}}{\psi(\sqrt{2iuD})}$.

$$\mu(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\sqrt{2ui}b}}{\psi(\sqrt{2iuD})}\,\mathrm{e}^{-iut}\,du.$$

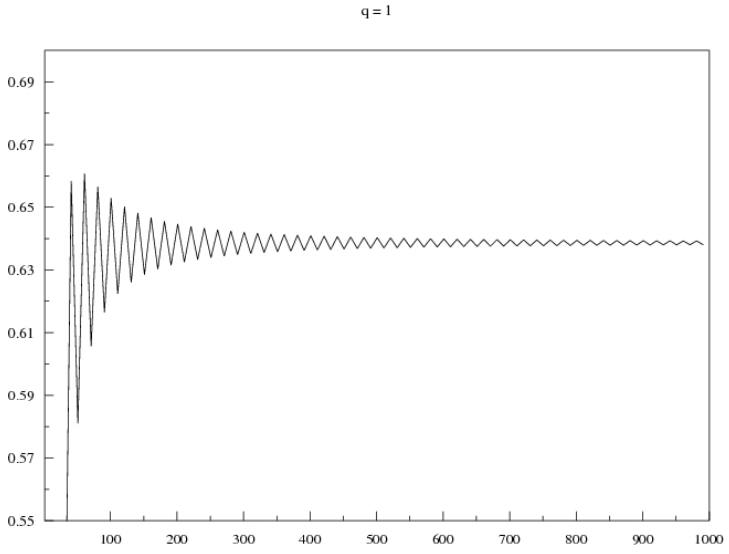
Moreover, $\mathbb{E}\left(\mathrm{e}^{-iuT_b^-}\right) = \mathscr{O}\left(\mathrm{e}^{-|b|\sqrt{|u|}}\right)$ when $|u| \to \infty \Longrightarrow \mu$ is $\mathscr{C}^\infty$. ▲

Practical implementation

- For $2\alpha/t = 18.4$, $p = q = 15$,
  $\left|f(t) - E(q,p,t)\right| \leq S_0 10^{-8} + t\left|f'(0) - \alpha f(0)\right| 10^{-11}$.
- Very few terms are needed to achieve a very good accuracy.
- The computation of $E(q,p,t)$ only requires the computation of $p+q$ terms.

Numerical convergence for a PUOC I

Consider a Parisian Up Out Call with
$S_0 = 110$, $r = 0.1$, $\sigma = 0.2$, $T = 1$, $L = 110$, $D = 0.1$ year.



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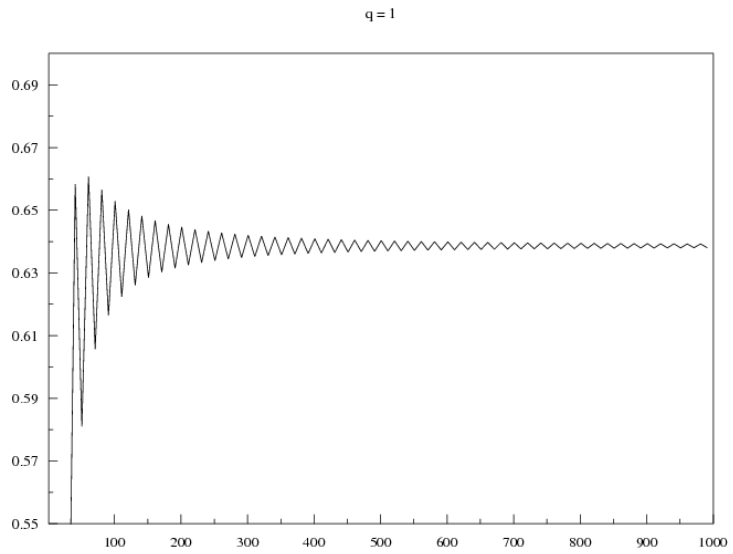


To achieve an acceptable accuracy without any Euler summation, $p \approx 1000$ is required whereas the use of the Euler summation enables to cut down to $p = q \approx 15$ (e.g. only 30 terms to be computed).

Numerical convergence for a PUOC II

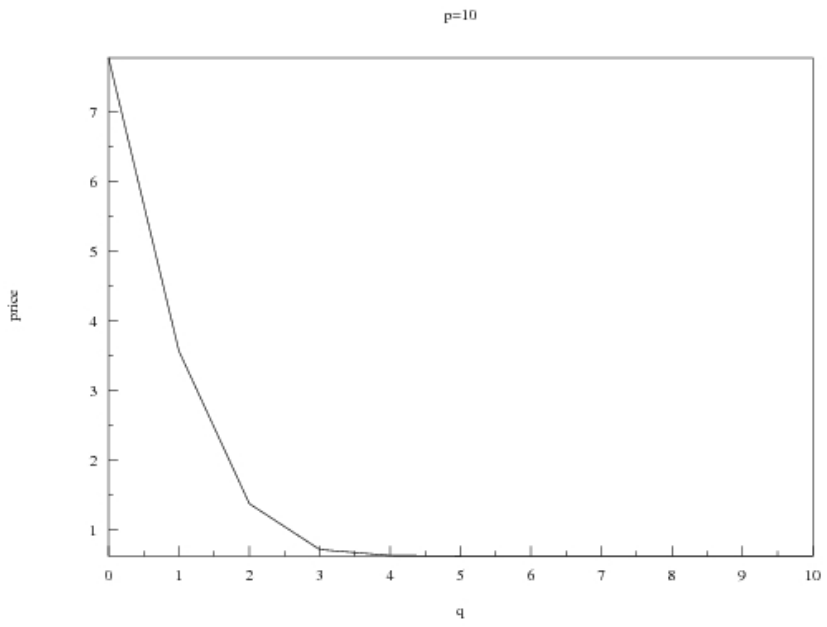


FIG.: Convergence of the Euler summation w.r.t. q for $p = 10$

Improved Monte Carlo method for a PUOC

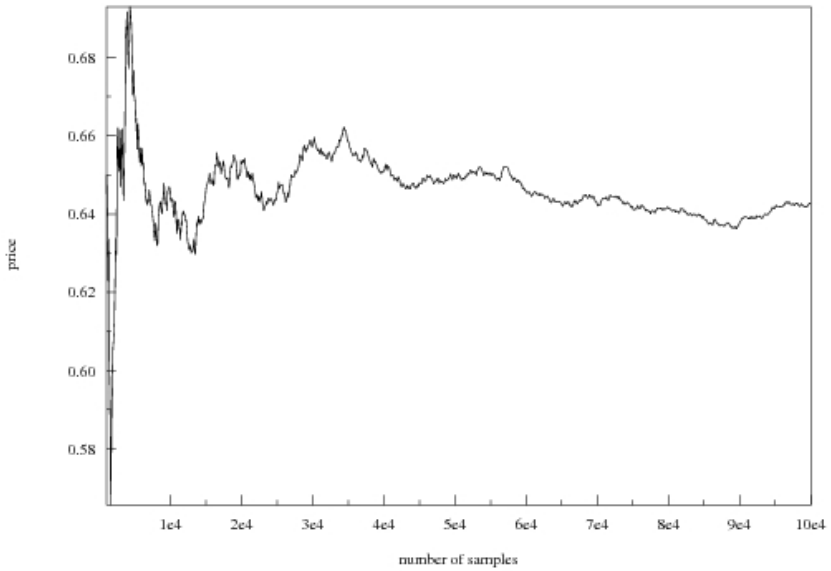
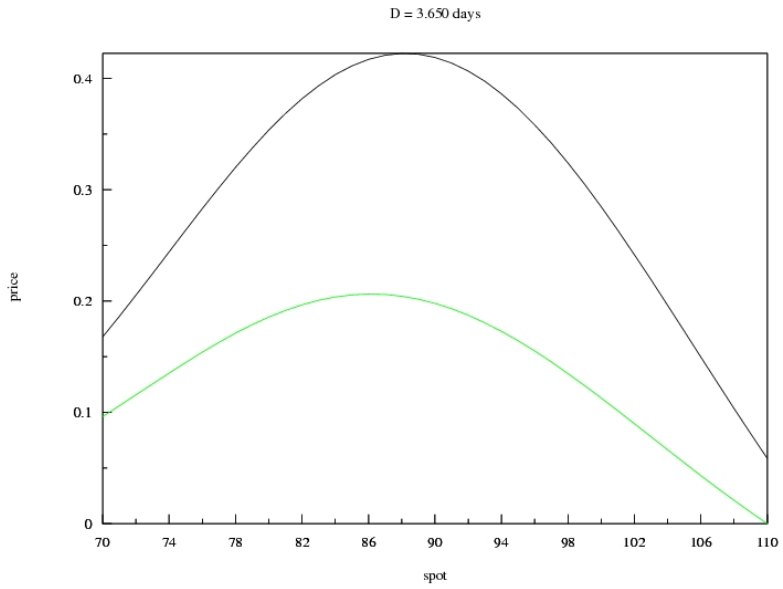


FIG.: Convergence of the Improved Monte Carlo Method with 250 time steps.

Convergence of a PUOC to a standard Up Out Call Barrier option



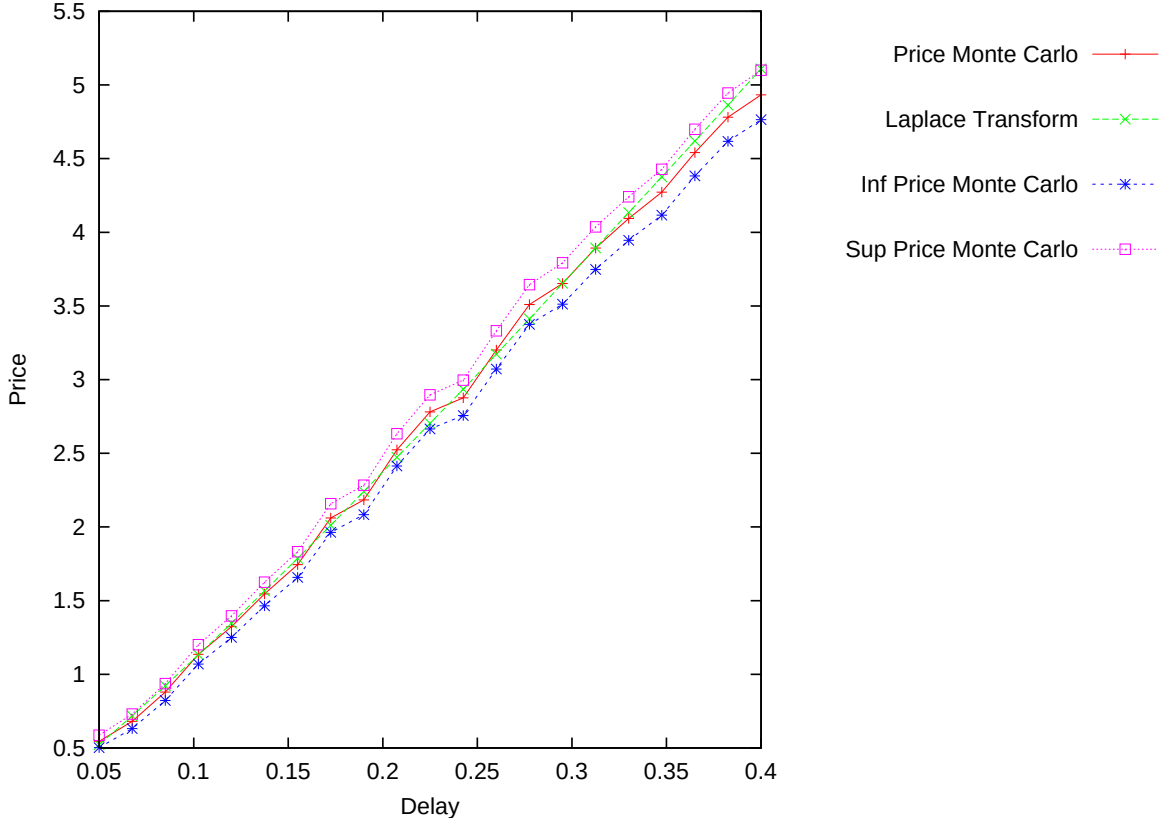


FIG.: Comparison of the improved MC with the Laplace method

Hedging

- Is the replicating portfolio generated by the delta? Probably yes, but not straightforward.
- How to compute the delta
  - Differentiate the Laplace transforms of the prices to obtain the Laplace transforms of the deltas.
  - Use some automatic differentiation tools to compute the Laplace transforms of the deltas.

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