NEWTON'S AERODYNAMIC FOR NON CONVEX BODIES

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Abstract. We characterize the solution to the Newton minimal resistance problem in a specific class of hollow profiles, satisfying a q-concavity condition. We treat two-dimensional bodies and radially symmetric three-dimensional bodies.

1. Introduction

The simplest model for computing the aerodynamic resistance of a body moving through a rare, homogeneous incompressible fluid was proposed by Newton in 1685. In particular Newton assumed the fluid as constituted by independent small particles hitting against the surface of the body at most once and then being reflected according to the elastic impact laws. This model is not realistic in the ordinary air and at a relatively low altitude, but it becomes approximately valid when considering the motion of flying vehicles in a rarefied atmosphere (e.g. missiles, artificial satellites etc...).

If a body covers a prescribed cross section Ω at its rear end and moves with constant velocity (orthogonally with respect to Ω), and if its shape can be described by the graph of a function $u : \Omega \to \mathbb{R}$, Newton aerodynamic resistance reads

$$D_\Omega(u) = \frac{\rho v_0^2}{2} \int_\Omega \frac{dx}{1 + |\nabla u|^2}, \tag{1.1}$$

where ρ is the density of the fluid and v_0 is the velocity of the body. The arising problem consists in finding the shape of the body which undergoes the least resistance among those satisfying Newton's assumptions and having same length and caliber. In this perspective a natural class of profiles for studying the Newton problem in the format of the Calculus of Variations can be informally defined by

$$\mathcal{S}^M(\Omega) = \{u : \Omega \to [0, M] \colon \text{almost every fluid particle hits the body at most once}\},$$

where $M > 0$. The condition can be rigorously stated as follows: for Ω an open bounded convex subset of $\mathbb{R}^n$, we say that $u \colon \Omega \to \mathbb{R}$ is a *single shock function on* Ω if u is a.e. differentiable in Ω and

$$u\left(x - \tau \nabla u(x)\right) \leq u(x) + \frac{\tau}{2}\left(1 - |\nabla u(x)|^2\right)$$

holds for a.e. $x \in \Omega$ and for every $\tau > 0$ such that $x - \tau \nabla u(x) \in \Omega$. $S^M(\Omega)$ is then defined as the class of single shock functions on Ω that take values in $[0, M]$. The specified maximal cross section Ω and the restriction on the body length (not exceeding $M > 0$) represent given design constraints.

Unfortunately, as shown in [B], [BFK2], $\mathcal{S}^M(\Omega)$ lacks of the necessary compactness properties in order to gain the existence of a global minimizer. Notice also that the form of D_Ω favors rapid oscillating shapes, so the choice of the class of competing functions is a delicate issue. Nevertheless, in [CL1, CL2] existence of global minimizers is shown among radial profiles in the $W^{1,\infty}_{loc}(\Omega) \cap C^0(\bar{\Omega})$-closure of polyhedral functions $u : \Omega \to [0, M]$ (Ω begin a ball) satisfying the single shock condition. On the other hand, it has been recently shown in [P1, P2] that a minimizer

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in the class of functions $S^M(\Omega)$ does not exist and that the infimum of the integral in (1.1) in this class is

$$\int_\Omega \frac{1}{2}\left(1-\frac{M}{\sqrt{M^2+d^2(x)}}\right)dx,$$

where $d(x)=dist(x,\partial\Omega)$.

Among the different choices in the literature, the most classical problem is set in the class

$$\mathcal{C}^M(\Omega):=\{u:\Omega\to[0,M]:\quad \text{u is concave}\}, \tag{1.2}$$

whose elements automatically satisfy the single shock property. Moreover, such class is compact with respect to the strong $W^{1,1}$ topology on compact subsets of Ω, a property ensuring existence of global minimizers. The aim of the paper is to show that the class of competing functions can be enlarged without giving up the mentioned compactness property and mostly that a complete characterization of 1D and radial 2D minimizers is possible without assuming concavity of the profile. As we are interested in minimizing the resistance functional in a class of possibly hollow profiles, we choose the class of q-concave functions u (i.e., $x\mapsto u(x)-\frac{q}{2}|x|^2$ is concave), with height not exceeding M. That is, for given $M>0$ and $q\geq 0$, we let

$$\mathcal{C}_q^M(\Omega):=\{\,u\colon\Omega\to[0,M]|\ u \text{ is } q\text{-concave}\},$$

and we wish to find the minimal resistance among profiles in $\mathcal{C}_q^M(\Omega)$. If $q=0$, we are reduced to the standard Newton problem among concave functions, while if $q>0$ it is possible to show that $\mathcal{C}_q^M(\Omega)\subset\mathcal{S}^M(\Omega)$ whenever $q\,\mathrm{diam}(\Omega)\leq 2$ and that this condition is sharp: indeed if $\Omega=(-1,1)$ and $u(x)=\frac{a}{2}x^2$, if $a>1$ then any particle moving vertically downwards to hit the body close to the points ±1 will meet the graph of u a second time.

We shall now state two main results. The first regards the one-dimensional case, i.e., Ω is a line segment. The second is concerned with the radial two-dimensional case, Ω being a ball in $\mathbb{R}^2$. Detailed proofs are contained in the forthcoming paper [MMOP]. Here we will limit ourselves to describe the key steps towards the proof of the second result.

2. Main results

One-dimensional case. For every $w\in W^{1,1}_{loc}(a,b)$, the one-dimensional resistance functional is given by

$$D_{(a,b)}(w)=\frac{\rho v_0^2}{2}\int_a^b\frac{dy}{1+w'(y)^2}.$$

By setting $y(x)=\frac{1}{2}(b-a)(x+1)+a$, $x\in(-1,1)$ and $u(x):=\frac{2}{b-a}\,w(y(x))$, we get

$$D_{(a,b)}(w)=\frac{\rho v_0^2(b-a)}{4}\int_{-1}^1\frac{dx}{1+u'(x)^2}. \tag{2.1}$$

It is worth noticing that the integral functional appearing in formula (2.1) represents an adimensional number that can be regarded as a *shape coefficient*. Then the problem we shall solve is

$$\min_{u\in\mathcal{C}_q^M}\int_{-1}^1\frac{dx}{1+u'(x)^2} \tag{2.2}$$

for $M>0$ and $q\in[0,1]$, where

$$\mathcal{C}_q^M:=\{\,u\colon[-1,1]\to[0,M]|\ u \text{ is } q\text{-concave}\}.$$

Admissible functions are here defined on the close interval $[-1,1]$, and notice that it is not restrictive to assume they are continuous up to the boundary. Existence of minimizers is very easy to prove: indeed if $(u_n)\subset\mathcal{C}_q^M$ is a minimizing sequence then $u_n(x)-\frac{q}{2}x^2$ is a sequence of concave and equibounded functions, hence there exists a concave w such that $u_n'(x)\to w'(x)+qx$ a.e. (possibly on a subsequence) and existence follows by the Lebesgue convergence theorem.

Our first main result is the following.

Theorem 2.1. *Let $M > 0$ and $q \in [0,1]$ be such that $2M \geq q$. Then problem (2.2) has a unique solution given by*

$$u_{M;q}(x) := \begin{cases} \begin{cases} \frac{q}{2}(x^2 - \gamma_{M;q}^2) + M & \text{if} \quad |x| \leq \gamma_{M;q} \\ \frac{M}{1-\gamma_{M;q}}(1-|x|) & \text{if} \quad \gamma_{M;q} \leq |x| \leq 1 \end{cases} & \text{if} \quad M \in (0,1) \\ M(1-|x|) & \text{if} \quad M \in [1,+\infty), \end{cases}$$

where $\gamma_{M;q} \in (0,1)$ is the unique minimizer of the function $R_{M;q} : [0,1] \to \mathbb{R}$ defined by

$$R_{M;q}(\gamma) =: \begin{cases} \frac{2}{q}\arctan(q\gamma) + \frac{2(1-\gamma)^3}{M^2 + (1-\gamma)^2} & \text{if } q > 0 \\ 2\gamma + \frac{2(1-\gamma)^3}{M^2 + (1-\gamma)^2} & \text{if } q = 0. \end{cases}$$

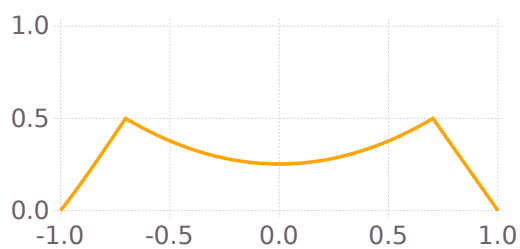

FIGURE 1. Numerical illustration for $M = 0.5$ and $q = 1$ (1d theorem case).

Theorem 2.1 is stated under the specific *high profile* assumption $2M \geq q$, which ensures that the constructed solution fits the maximal height interval $[0, M]$. On the other hand, the restriction $q \leq 1$ corresponds to the single shock condition in this case, as mentioned in the introduction.

Radial two-dimensional case. In this case we let $\Omega = B_R(0)$ be the open ball in $\mathbb{R}^2$, with center 0 and radius R. For every radially symmetric function w on $B_R(0)$ whose radial profile (still denoted by w) is in $W^{1,1}_{loc}(0,R)$, the resistance functional is

$$D_{B_R(0)}(w) = \pi\rho v_0^2 \int_0^R \frac{s\,ds}{1 + w'(s)^2}.$$

As before it can be adimensionalized by setting $u(r) := R^{-1}v(Rr)$, $r \in (0,1)$, thus obtaining

$$D_{B_R(0)}(w) = \pi\rho v_0^2 R^2 \int_0^1 \frac{r\,dr}{1 + u'(r)^2}.$$

Then, given $M > 0$ and $q \geq 0$, by setting

$$\mathcal{R}_q^M := \left\{ u\colon [0,1] \to [0,M] \,\middle|\, r \mapsto u(r) - \tfrac{q}{2}r^2 \text{ is nonincreasing and concave} \right\},$$

we shall solve the problem

$$\min_{u \in \mathcal{R}_q^M} \int_0^1 \frac{r\,dr}{1 + u'(r)^2}. \tag{2.3}$$

We will still work with the high profile assumption $2M \geq q$ and the single shock assumption $0 \leq q \leq 1$. Existence of minimizers is again standard and our second main result is the characterization

of the solution to problem (2.3). It is given by a parabolic profile in $[0, a]$, and a strictly decreasing profile satisfying the radial two-dimensional Euler-Lagrange equation

$$\frac{-ru'(r)}{(1+u'(r)^2)^2} = \text{const}$$

in $(a, 1]$. The optimal value of a is uniquely determined in $(0, 1)$. In order to write down the solution, which is a little less explicit, we need to introduce some further notation. We let $(-\infty, -1] \ni t \mapsto h(t) := -t(1+t^2)^{-2}$. For $a \in (0, 1)$, let

$$\varphi(a) := -\int_a^1 h^{-1}\left(\frac{a}{4r}\right) dr \tag{2.4}$$

and

$$\gamma_q(a) := \sqrt{\frac{1}{2}\big(3a^2q^2 + 1 + \sqrt{9a^4q^4 + 10a^2q^2 + 1}\big)}, \quad \zeta_q(a) := -\int_a^1 h^{-1}\left(\frac{ah(-\gamma_q(a))}{r}\right) dr.$$

We are now in a position to state our 2D result.

Theorem 2.2. *Let $M > 0$ and assume that $0 \le q \le 1$ and $2M \ge q$. Then there exists a unique $a_M \in (0, 1)$ such that $\varphi(a_M) = M$, and there exists a unique $a_* \in [a_M, 1)$ such that $\zeta_q(a_*) = M$. Moreover, there exists a unique solution to problem (2.3), given by*

$$u_{M;q}(r) := \begin{cases} \dfrac{q}{2}(r^2 - a_*^2) + M & \text{if} \quad r \in [0, a_*] \\[2ex] -\displaystyle\int_r^1 h^{-1}\left(\frac{a_* h(-\gamma_q(a_*))}{s}\right) ds & \text{if} \quad r \in (a_*, 1]. \end{cases} \tag{2.5}$$

It is worth noticing that $\gamma_0(a) \equiv 1$, hence when $q = 0$ we get $a_* = a_M$, and we recover the classical concave radial minimizer.

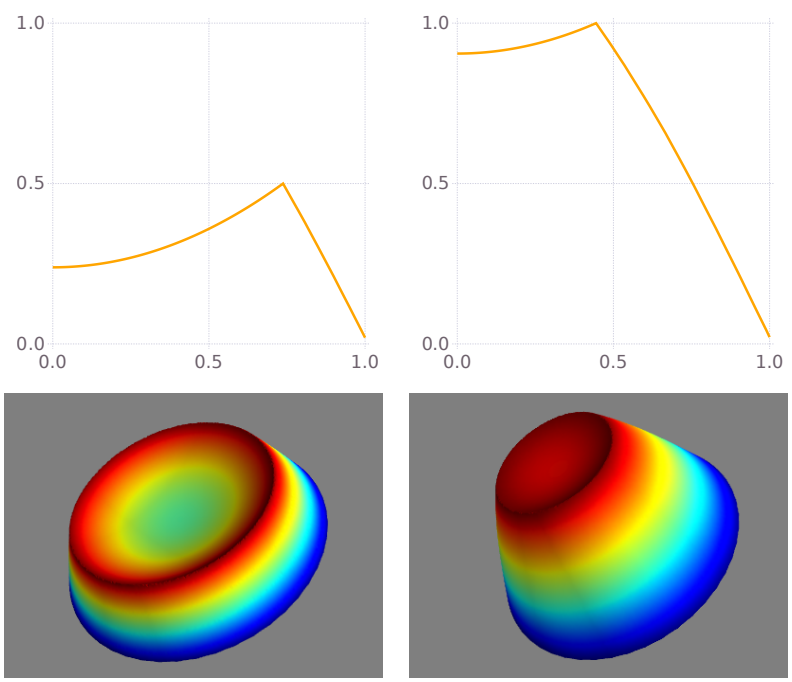

FIGURE 2. Numerical illustrations for $M = 0.5$ and $M = 1$ (2d theorem case), both for $q = 1$.

Figure 1 and Figure 2 show the graphs of optimal profiles obtained by numerical simulations, which correspond to the results given by Theorem 2.1 and Theorem 2.2 respectively.

3. Outline of the proofs

We shall discuss the proof of Theorem 2.2, referring to [MMOP] for the full arguments. For $0 \le a \le b$ and locally absolutely continuous functions u on (a,b), we let

$$\mathcal{D}_{(a,b)}(u) := \int_a^b \frac{r\,dr}{1+(u'(r))^2}.$$

We shall also use the notations $\mathfrak{p}_q(r) := \frac{q}{2}r^2$ and $\mathfrak{p}_{q,s}(r) := \frac{q}{2}(r-s)^2$. The proof is based on the combination of the following results, each giving solution to a partial problem.

Proposition 3.1. *Let $q \ge 0$. Let $0 \le a \le b$. Then $\mathcal{D}_{(a,b)}(\mathfrak{p}_{q,b}) \ge \mathcal{D}_{(a,b)}(\mathfrak{p}_{q,a})$ and equality holds if and only if $q = 0$ or $a = b$.*

Proof. Let $q > 0$. Let $\varphi(t) := t\arctan t - \log(1+t^2)$, $t \in [0,+\infty)$. Since $\varphi(0) = 0 = \varphi'(0)$ and $\varphi''(t) = 2t^2(t^2+1)^{-2} > 0$ for every $t \in (0,+\infty)$ then $\varphi(t) > 0$ for every $t \in (0,+\infty)$. Since $\mathcal{D}_{(a,b)}(\mathfrak{p}_{q,b}) - \mathcal{D}_{(a,b)}(\mathfrak{p}_{q,a}) = \frac{1}{q^2}\varphi(q(b-a))$ the result follows. If $q = 0$ the result is trivial. □

The following is the key lemma.

Lemma 3.2. *Let $q \ge 0$, $a > 0$, $H \in \mathbb{R}$. The minimization problem*

$$\min\left\{\mathcal{D}_{(0,a)}(u) : r \mapsto u(r) - \tfrac{q}{2}r^2 \text{ is concave nonincreasing on } [0,a],\ u(r) \le u(a) = H \text{ on } [0,a]\right\}$$

admits the unique solution $u_*(r) := \frac{q}{2}(r^2 - a^2) + H$.

Proof. If $q = 0$ the result is trivial. Let $q > 0$. Since $r \mapsto u(r) - \frac{q}{2}r^2$ is concave nonincreasing we get $u'(r) \le qr$ a.e in $(0,a)$. If $u' \ge 0$ a.e. in $(0,a)$, then either $u'(r) = qr$ a.e. in $(0,a)$ or by pointwise estimating the integrand we get $\mathcal{D}_{(0,a)}(u) > \mathcal{D}_{(0,a)}(\mathfrak{p}_q) = \mathcal{D}_{(0,a)}(u_*)$.

Suppose instead that that there are negativity points of the left derivative u'_- on $(0,a)$. Since u is q-concave, u'_- is upper semicontinuous on $(0,a)$, therefore the set $I := \{r \in (0,a) : u'_-(r) < 0\}$ is open, thus a (at most) countable union of (nonempty) disjoint open intervals (α_j, β_j). Moreover, if $\beta_j < a$ there holds $u'_-(\beta_j) = 0$ (left continuity of u'_-). A direct consequence of q-concavity and of the restriction $u(r) \le u(a)$ on $[0,a]$ is that $u'_-(r) \ge \frac{q}{2}(r-a)$ on $(0,a)$, therefore if instead $\beta_j = a$ we still have $\lim_{r\to a^-} u'_-(r) = 0$. On the other hand, q-concavity yields $0 \ge u'_-(r) \ge q(r-\beta_j)$ on any interval (α_j, β_j). Since $u'_- < 0$ at some point in $(0,a)$, there is at least one of these intervals (α_j, β_j). If there exists and index j such that $\alpha_j > 0$, Proposition 3.1 entails

$$\begin{aligned}\int_I \frac{rdr}{1+u'(r)^2} &= \sum_j \int_{\alpha_j}^{\beta_j} \frac{rdr}{1+u'(r)^2} \ge \sum_j \int_{\alpha_j}^{\beta_j} \frac{rdr}{1+q^2(r-\beta_j)^2} \\ &\ge \sum_j \int_{\alpha_j}^{\beta_j} \frac{rdr}{1+q^2(r-\alpha_j)^2} > \sum_j \int_I \frac{rdr}{1+q^2r^2} = \int_I \frac{rdr}{1+q^2r^2}.\end{aligned}$$

By taking into account that $\displaystyle\int_{[0,a]\setminus I} \frac{rdr}{1+u'(r)^2} \ge \int_{[0,a]\setminus I} \frac{rdr}{1+q^2r^2}$ we get $\mathcal{D}_{(0,a)}(u) > \mathcal{D}_{(0,a)}(u_*)$. The remaining case is $I = (0,\beta)$ for some $\beta \in (0,a]$. If $\beta < a$, q-concavity and Proposition 3.1 yield $\mathcal{D}_{(0,a)}(u) \ge \mathcal{D}_{(0,\beta)}(\mathfrak{p}_q) + \mathcal{D}_{(\beta,a)}(\mathfrak{p}_{q,\beta}) > \mathcal{D}_{(0,a)}(\mathfrak{p}_q) = \mathcal{D}_{(0,a)}(u_*)$. If $\beta = a$, we use $u'(r) \le \frac{q}{2}(r-a)$ a.e. on $(0,a)$ and we get $\mathcal{D}_{(0,a)}(u) \ge \mathcal{D}_{(0,a)}(\mathfrak{p}_{q/2}) > \mathcal{D}_{(0,a)}(\mathfrak{p}_q) = \mathcal{D}_{(0,a)}(u_*)$. □

Another important result is the following

Lemma 3.3. *Let $q \ge 0$. Let $0 \le a \le \gamma \le b$ and $q(b-\gamma) \le 2$. Let moreover $u\colon [a,b] \to \mathbb{R}$ be an absolutely continuous function such that*

(i) *$u(\gamma) = u(b) \ge u(r)$ for any $r \in [\gamma, b]$ and the restriction of u on $[\gamma, b]$ is q-concave;*
(ii) *$u'(r) \le -1$ a.e. on (a,γ).*

Then $\mathcal{D}_{(a,b)}(u) \ge \mathcal{D}_{(a,b)}(w_u)$, where $w_u(r) := \begin{cases} u(r+\gamma-a) + u(a) - u(b) & \text{if } r \in [a, a+b-\gamma) \\ u(r-b+\gamma) & \text{if } r \in [a+b-\gamma, b]. \end{cases}$

Equality holds if and only if $\gamma = a$ or $\gamma = b$.

Proof. We show the proof of this result in the extremal case $u(r) = \frac{q}{2}(r-\gamma)(r-b) + m$ on $[\gamma, b]$, for some $m \in \mathbb{R}$. The full proof would require in fact a generalized version of Lemma 3.2, that we omit. Let $q > 0$. It is easily seen, by taking (ii) into account, that

$$\int_a^b \frac{r\,dr}{1+{w_u}'(r)^2} = \int_\gamma^b \frac{(r+a-b)\,dr}{1+u'(r)^2} + \int_a^\gamma \frac{(r+b-\gamma)\,dr}{1+u'(r)^2}$$

$$\leq (a-\gamma)\int_\gamma^b \frac{dr}{1+u'(r)^2} + \int_a^b \frac{r\,dr}{1+u'(r)^2} + \frac{1}{2}(b-\gamma)(\gamma-a).$$

Since $u'(r) = qr - \frac{q}{2}(\gamma+b)$ on (γ, b), a direct computation shows that

$$\mathcal{D}_{(a,b)}(w_u) - \mathcal{D}_{(a,b)}(u) \leq (a-\gamma)\int_\gamma^b \frac{dr}{1+u'(r)^2} + \frac{1}{2}(b-\gamma)(\gamma-a) = \frac{\alpha-\gamma}{q}\psi\left(\tfrac{q}{2}(\beta-\gamma)\right)$$

where $\psi(z) := 2\arctan z - z$. Since $\psi(z) > 0$ for every $z \in (0,1]$ and $\frac{q}{2}(\beta-\gamma) \in [0,1]$, the result follows. If $q = 0$ the term $\frac{2}{q}\arctan(\frac{q}{2}(\beta-\gamma))$ becomes $\beta-\gamma$ and the result follows as well. □

There is a close relation between minimization of the resistance among concave and monotonic profiles. The following lemma is reminiscent of the results by Marcellini [M] (we refer to Section 5 therein). It is also related to the property $|u'| \notin (0,1)$ for concave minimizers of the resistance functional, which is established by Buttazzo, Ferone and Kawohl in [BFK2, Theorem 2.3]. We omit the proof, which follows the same line therein.

Lemma 3.4. *Let $0 \leq a < 1$, $m_1 > m_2$ and*

$$\mathcal{W} := \left\{u \in W^{1,1}_{loc}(a,1) \colon u' \leq 0 \text{ a.e. in } (a,1),\ u(a) = m_1,\ u(1) = m_2\right\},$$

where the boundary values are understood as limits. Then $\mathcal{D}_{(a,1)}$ admits a minimizer on $\mathcal{W}$ which is concave in $(a,1)$. If $u_* \in \arg\min_{\mathcal{W}} \mathcal{D}_{(a,1)}$, then $|u_*'(r)| \notin (0,1)$ for a.e. $r \in (a,1)$.

All the necessary elements for the proof of Theorem 2.2 are now settled.

Proof of Theorem 2.2. Let $M > 0$, $q \in [0,1]$, $2M \geq q$. Let u be solution to (2.3) It is not restrictive to assume that u is continuous up to the boundary of $[0,1]$. Let $m := \max\{u(r) : r \in [0,1]\}$ and $a := \max\{r \in [0,1] : u(r) = m\}$. We immediately see that $a < 1$. Indeed, if $a = 1$, Lemma 3.2 and a direct comparison with $w(r)$ given by $w(r) = \frac{q}{2}(r^2 - (1-\delta)^2) + M$ if $r \in [0, 1-\delta]$ and $w(r) = \frac{m}{\delta}(1-r) + M - m$ if $r \in (1-\delta, 1]$ yield contradiction for small enough δ (notice that w is admissible since $2M \geq q$).

Step 1 - u decreases on the side. We claim that u is strictly decreasing in $[a,1]$. For simplicity, we show the proof in a particular case, which however features the key point: we shall reach a contradiction by supposing that u admits a unique local maximum point a'' on $(a,1]$. We let a' the unique point in (a, a'') such that $u(a') = u(a'')$ and notice that u is nonincreasing on (a, a'). We can apply first Lemma 3.4 and then Lemma 3.3 (since $q \leq 1$). In this way, we get $\mathcal{D}_{(0,1)}(u) > \mathcal{D}_{(0,1)}(u_*)$, where u_* is constant on (a,c) for some $c \in [a, a')$, $u_*(c + a'' - a') = u_*(c) = u_*(a)$, u_* has slope less than or equal to -1 a.e. on $(c + a'' - a', a')$ and $u_* = u$ outside (a, a'). By construction, u_* is q-concave in $(0, c + a'' - a')$ and nonincreasing on $(c + a'' - a', 1)$, with $u_*(1) = u(1)$. If we apply Lemma 3.4 again on $(c + a'' - a', 1)$, we find a q-concave function on $[0,1]$ with not greater resistance and a contradiction. The general case requires approximation of u with functions having finitely many local maxima on $(a,1]$, to which one recursively applies the above argument.

Step 2 - $u' \leq -1$ a.e. on the side. We give other structure properties of the minimizer u. Its restriction to $[a,1]$ minimizes the resistance functional in $(a,1)$ among all nonincreasing v in $[a,1]$, such that $v(a) = m$, $v(1) = u(1)$, $v \in W^{1,1}_{loc}(a,1)$. Indeed, if this was not the case, the concave minimizer of the resistance in such class, given by Lemma 3.4, would give contradiction. Lemma 3.4 yields $|u'| \notin (0,1)$ a.e. in $(a,1)$ and by taking into account that u is q-concave (thus the right derivative u_+' is right continuous) we get that if $u_+'(a) = 0$ then there exists $\delta > 0$ such that $u' = 0$

in $(a, a+\delta)$, contradicting the definition of a. Then $u' \le -1$ a.e. in $(a,1)$ still by q-concavity since u' can only jump downwards. Simple comparison arguments also show that $m = M$ and $u(1) = 0$.

Step 3 - Euler-Lagrange equation. Let $h : (-\infty - 1] \to \mathbb{R}$ be defined by $h(t) = -t(1+t^2)^{-2}$. Notice that the inverse function h^{-1} is defined on $(0, \frac{1}{4}]$, it is smooth, increasing and there hold $\lim_{r\to 0} h^{-1}(r) = -\infty$ and $h^{-1}(\frac{1}{4}) = -1$. Let $\varphi(a)$, $a \in (0,1)$, be defined by (2.4). It is readily seen, from the definition of h, that $\lim_{a\to 1}\varphi(a) = 0$, $\lim_{a\to 0}\varphi(a) = +\infty$ and $\varphi' < 0$ on $(0,1)$. Then there exists a unique a_M in $(0,1)$ such that $\varphi(a_M) = M$ and $[a_M, 1) = \{a \in (0,1) : \varphi(a) \le M\}$. For every $a \in [a_M, 1)$ let $\psi_a : (0, \frac{a}{4}] \to [0, +\infty)$ be defined by

$$\psi_a(\eta) := -\int_a^1 h^{-1}\left(\frac{\eta}{r}\right) dr.$$

Similarly as above we may check that for any $a \in [a_M, 1)$ there is $\psi_a'(\eta) < 0$ on $(0, \frac{a}{4})$, and moreover $\lim_{\eta\to 0}\psi_a(\eta) = +\infty$, $\lim_{\eta\to a/4} = \varphi(a) \le M$. Hence for every $a \in [a_M, 1)$ there exists a unique number $\eta \in (0, a/4]$ such that $\psi_a(\eta) = M$ is satisfied, and we denote it by $\eta(a)$. Notice that $\psi_a(\eta)$ strictly decreases with a for each $\eta \in (0, \frac{a}{4}]$ so that the function $[a_M, 1) \ni a \mapsto \eta(a)$ is strictly decreasing, and it satisfies $\eta(a_M) = \frac{a}{4}$, $\lim_{a\to 1}\eta(a) = 0$, and $\psi_a(\eta(a)) = M$ on $[a_M, 1)$. We concentrate on $(a,1)$, where $u' \le -1$ is a.e. thanks to Step 2, and we use the first variation of the resistance functional

$$\int_a^1 \frac{ru'\phi' dr}{(1+u'^2)^2} = 0 \quad \text{for every } \phi \in C_0^1(a,1),$$

that is there exists a constant $c > 0$ such that $-ru' = c(1+u'^2)^2$ a.e. in $(a,1)$. We get therefore $h(u'(r)) = c/r$. Hence, $4c/r \in (0,1]$ for every $r \in (a,1)$, that is $0 < c \le a/4$. Since $u(1) = 0$, $u(a) = M$, then c has to satisfy $\psi_a(c) = M$, that is, $c = \eta(a)$, which also implies $\varphi(a) \le M$, then $a \in [a_M, 1)$.

Step 4 - Global structure of the solution. Summing up if $u \in C^0([0,1])$ solves (2.3), there exist $a \in [a_M, 1)$ and a unique $\eta = \eta(a) \in (0, a/4]$ such that (also using Lemma 3.2) u takes the form

$$u(r) = \begin{cases} \frac{q}{2}(r^2 - a^2) + M & \text{if } r \in [0,a] \\ -\int_r^1 h^{-1}\left(\frac{\eta(a)}{s}\right) ds & \text{if } r \in (a,1], \end{cases}$$

and then

$$\mathcal{D}_{(0,1)}(u) = \int_0^a \frac{r\,dr}{1+q^2r^2} + \int_a^R \frac{r\,dr}{1+|h^{-1}(\eta(a)/r)|^2}. \tag{3.1}$$

We are now left to minimize over $a \in [a_M, 1)$. That is, we have to solve $\min_{a\in[a_M,1)} \mathcal{E}(a)$, where $\mathcal{E}(a)$ is defined by the right hand side of (3.1). After some computations using the implicit expression for $\eta(a)$, that is, $\psi_a(\eta(a)) = M$, it is possible to show that this one variable minimization problem admits a unique solution $a_* \in [a_M, 1)$, which is characterized by means of functions γ_q and ζ_q that appear in Theorem 2.2. We omit the details of this computation. The consequence is that problem 2.3 admits a unique solution that can be expressed by 2.5. □

Conclusive remarks. The proof of Theorem 2.1 is in the same spirit of that of Theorem 2.2 and follows the same line. Some extra difficulties arise since in Theorem 2.1 we do not assume a priori that the solution is symmetric on $[-1,1]$. On the other hand, it is not possible to drop the radial symmetry assumption in the two-dimensional case, since it is well known that the minimization problem for D_Ω on the class $\mathcal{C}^M$ from (1.2) exhibits symmetry break in case Ω is a ball in $\mathbb{R}^2$, see [BFK1]. A second difference between one- and two-dimensional case is that in the latter we always have $a_M > 0$, for any value of M. On the other hand, we see in Theorem 2.1 that there is no parabolic profile in the center if $M \ge 1$, so that in such case the concave and the q-concave solutions coincide.

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