Algebraic structures from shapes

Tom Hirschowitz

CNRS, Université de Savoie

CCS '09

Intro: what is algebra?

In "Algebraic structures from shapes",

What is algebra?

- Universal algebra $<$ Lawvere theories $<$ Sketches $<$ ...
- Here: (finite limit) sketches.
- Beyond algebra.
- Everything works as algebra.

Dictionary

Algebra	Sketches
Theory	Sketch
Model	Finite limit preserving functor to Set
Model morphism	Natural transformation
Term model	Initial model

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "U"] \end{tikzcd} \qquad \begin{tikzcd} {[S, \mathrm{Set}]} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "U"] \end{tikzcd}$$

So what?

The category of algebras has a model structure, hence:

- limits and colimits,
- local closure,
- a factorisation system.

Canonical construction of the free algebra over a theory.

All you need to know about sketches for this talk

- There is a notion of a local right adjoint (lra) monad (in a minute).
- For such a lra monad T on a presheaf category:

Theorem (Leinster,Weber)

The category of T-algebras is locally presentable, i.e., sketchable.

Moreover, the construction provides a minimal sketch, in a certain sense.

Goal

- Here: technique for constructing lra monads.
- Project: make this into a theory.

Motivation for such a theory

Algebraic approach to CCS:

- design new algebraic structures
- modelling computation (e.g., prog. languages).

Such a theory would make attempts cheaper.

Shapes

Consider the category $\mathcal{S}$ looking like:

$$
\begin{tikzcd}
\cdots & n & \cdots & p & \cdots \\
 & & \star \arrow[ull, bend left, "\sigma_0"] \arrow[ull, bend left=60, "\sigma_{n-1}"] \arrow[ull, bend left=80, "\tau"] \arrow[urr, bend left, "\sigma_0"] \arrow[urr, bend left=60, "\sigma_{p-1}"] \arrow[urr, bend left=80, "\tau"] & &
\end{tikzcd}
$$

Presheaves over shapes

The category $\hat{\mathcal{S}} = [\mathcal{S}^{op}, \mathrm{Set}]$ has multigraphs as objects.

Exercise

Draw the multigraph for:

- $F(\star) = \{x, y, z\}$,
- $F(0) = \{e_0\}$,
- $F(2) = \{e_2\}$,
- $F(3) = \{e_3\}$,
- $F(\tau)(e_0) = y$,
- $F(\tau)(e_2) = x$,
- $F(\sigma_0)(e_2) = x$,
- $F(\sigma_1)(e_2) = y$,
- $\ldots$

A first construction

We now:

- define a recipient presheaf $\mathcal{R}$ on $\mathcal{S}$;
- and a sequence of subpresheaves $\mathcal{T}_0, \mathcal{T}_1, \ldots \subseteq \mathcal{R}$;
- consider the union $\mathcal{T}_\omega = \bigcup_{i \in \omega} \mathcal{T}_i \subseteq \mathcal{R}$.

We obtain:

The elements of $\mathcal{T}_\omega(n)$ are the morphisms $n \to 1$ in the free multicategory over 1.

Furthermore, multicategorical composition is computed by pushout.

The recipient presheaf

Consider $\mathcal{R}\colon \hat{\mathcal{S}} \to \mathrm{Set}$ defined by:

- $\mathcal{R}(\star) = y\star$, seen as a functor $\mathcal{S} \cong 1^{op} \to \hat{\mathcal{S}}$,
- $\mathcal{R}(n)$ is the set of diagrams

$$\begin{tikzcd}
 & \star \arrow[d, "t"'] & \\
 & f & \\
\star \arrow[ur, "s_0"] & \cdots & \star \arrow[ul, "s_{n-1}"']
\end{tikzcd}$$

  with f finite,
- modulo isomorphism of diagrams $\mathcal{S}/n \to \hat{\mathcal{S}}$.

Intuition

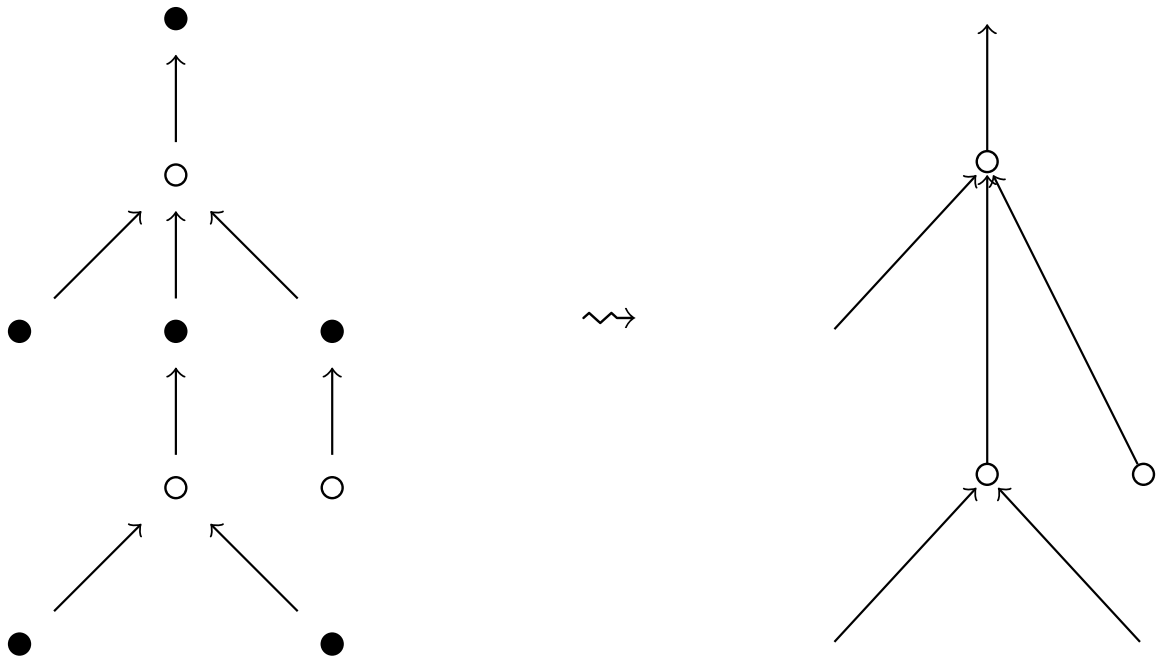
Multigraphs with arity and handles.

An invariant and pictures

The diagrams we'll construct will satisfy a:

Linearity constraint

Each vertex is the source of at most one multi-edge, and is the target of at most one multi-edge.



Boot

Consider the following presheaf $\mathcal{T}_0 \subseteq \mathcal{R}$:

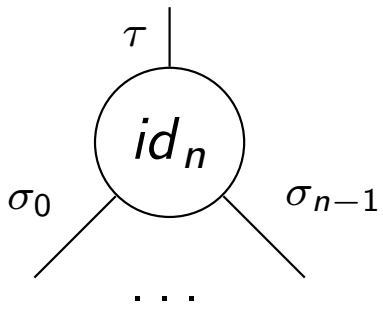
- $\mathcal{T}_0(\star) = \mathcal{R}(\star) = y\star$,
- $\mathcal{T}_0(n)$ is the singleton

$$\begin{tikzcd}
 & \star \arrow[d, "\tau"'] & \\
 & n & \\
\star \arrow[ur, "\sigma_0"] & \dots & \star, \arrow[ul, "\sigma_{n-1}"']
\end{tikzcd}
\qquad \text{plus} \qquad
\begin{tikzcd}
\star \arrow[d, equal] \\
\star \arrow[d, equal] \\
\star
\end{tikzcd}$$

(when $n = 1$).

Boot

Pictorially, $\mathcal{T}_0$ has, for each n, one element



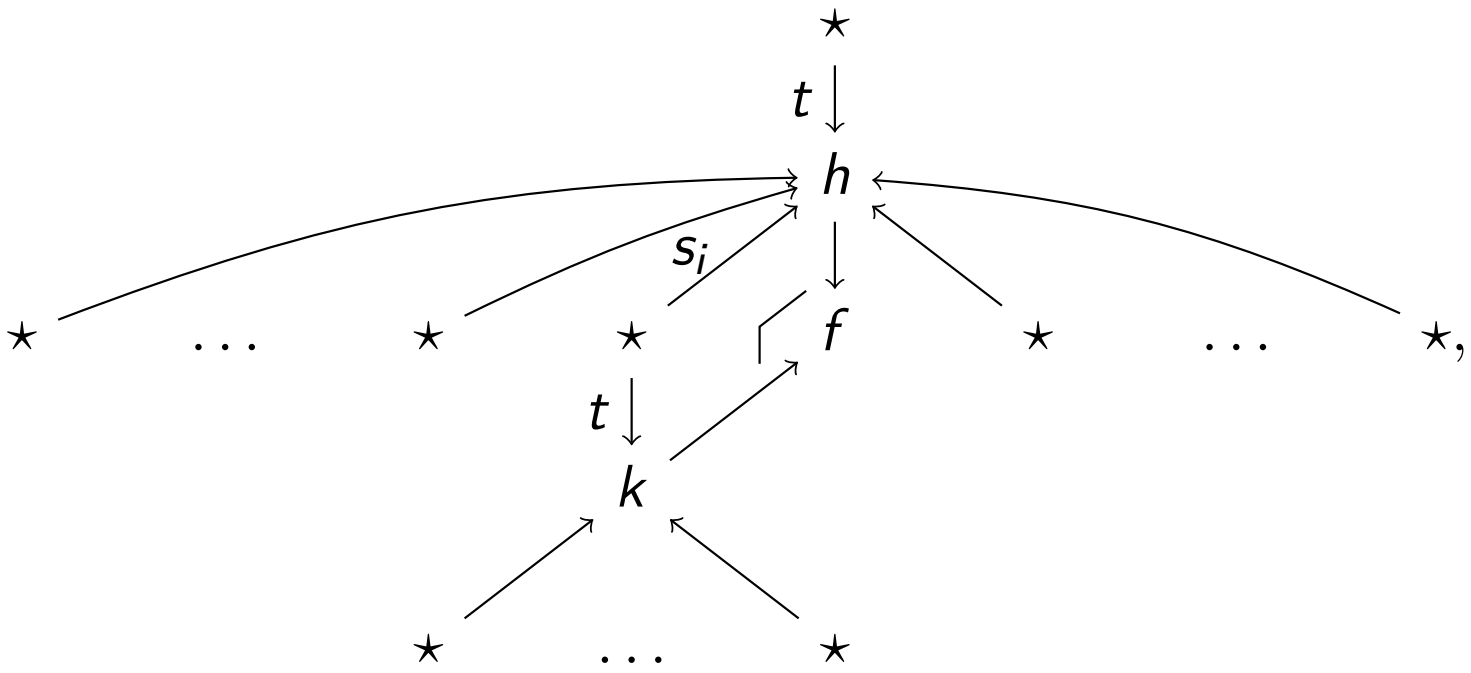
plus

(when $n = 1$).
Role of the diagram: distinguish the dangling wires.

Step

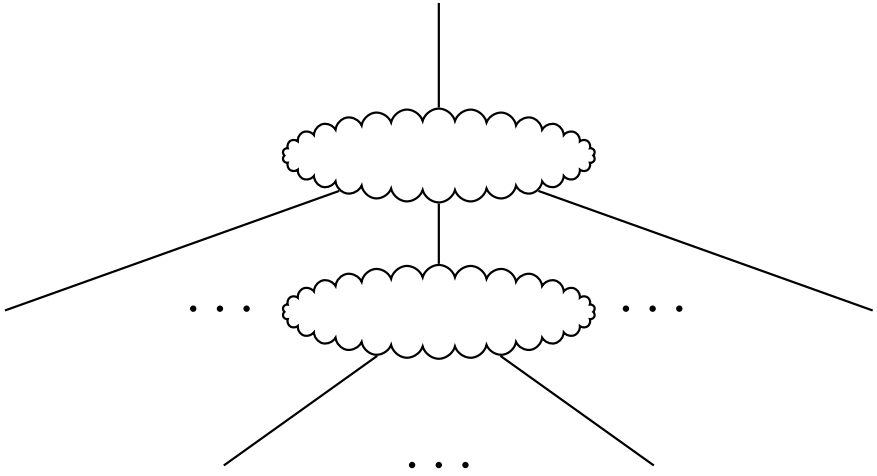
Now define $\mathcal{T}_{n+1}$ to be the union of $\mathcal{T}_n$ and T'_n, which has:

- $T'_n(\star) = \emptyset$
- $T'_n(m)$ is the set of diagrams $(f, \overline{s}, t)$ as above such that:
  - there exist $p + q - 1 = m$, $i \in p$, and
  - diagrams $h \in \mathcal{T}_n(p)$ and $k \in \mathcal{T}_n(q)$, such that f is:



Picture

This just glues two multigraphs together along the chosen edge:



Wrap up

Definition

Let $\mathcal{T}_\omega$ be the union of all the $\mathcal{T}_n$.

Theorem (Coherence at 1)

For all n, $\mathcal{T}_\omega(n)$ is isomorphic to the set $M(n)$ of morphisms $n \to 1$ in the free multicategory on 1.
Furthermore, composition and identities are given by the operations on the $\mathcal{T}_n$'s, e.g,

$$\begin{tikzcd}
\mathcal{T}_\omega(p) \times \mathcal{T}_\omega(q) \arrow[r, phantom, "\cong"] \arrow[d, "\text{glueing at } i"'] & M(p) \times M(q) \arrow[d, "\circ_i"] \\
\mathcal{T}_\omega(p+q-1) \arrow[r, phantom, "\cong"] & M(p+q-1)
\end{tikzcd}$$

commutes.

The monad

Consider the functor:

$$\begin{array}{ccc} \hat{\mathcal{S}} & \longrightarrow & \hat{\mathcal{S}} \\ F & \mapsto & \mathcal{T}F, \end{array}$$

where $\mathcal{T}(F)(s) = \coprod_{x \in \mathcal{T}_\omega(s)} \hat{\mathcal{S}}(x(id_s), F)$.

WTF?

Sorry, I have to show you this key formula.

Understanding the key formula

$$\mathcal{T}(F)(s) = \coprod_{x \in \mathcal{T}_\omega(s)} \hat{\mathcal{S}}(x(id_s), F)$$

- Recall that $\mathcal{T}_\omega(s)$ is a set of diagrams $\hat{\mathcal{S}}/s^{op} \to \mathrm{Set}$,

e.g.,

$$\begin{tikzcd}
 & \star \arrow[d, "t"'] & \\
 & f & \\
\star \arrow[ur, "s_0"] & \cdots & \star, \arrow[ul, "s_{n-1}"']
\end{tikzcd}$$

- So that for $x \in \mathcal{T}_\omega(s)$,
  - $x(id_s)$ is a presheaf on $\mathcal{S}$,
  - here f.
  - And a natural transformation $f \to F$ is a labelling of f in F.

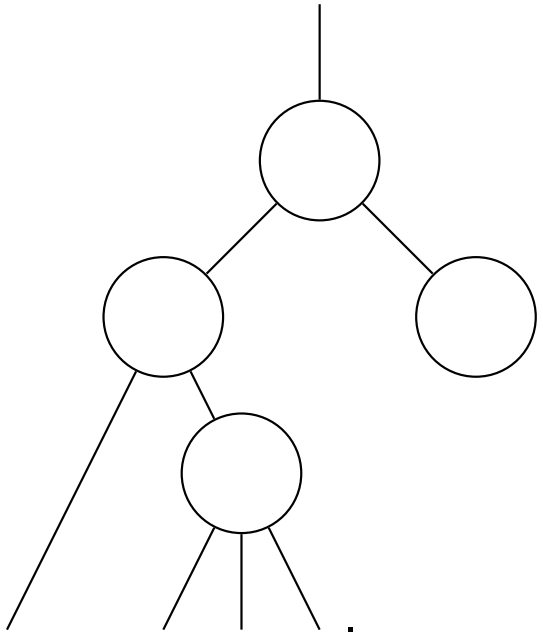
Example

Recall

- $F(\star) = \{x, y, z\}$,
- $F(0) = \{e_0\}$,
- $F(2) = \{e_2\}$,
- $F(3) = \{e_3\}$,
- $F(\tau)(e_0) = y$,
- $F(\tau)(e_2) = x$,
- $F(\sigma_0)(e_2) = x$,
- $F(\sigma_1)(e_2) = y$,
- $\ldots$

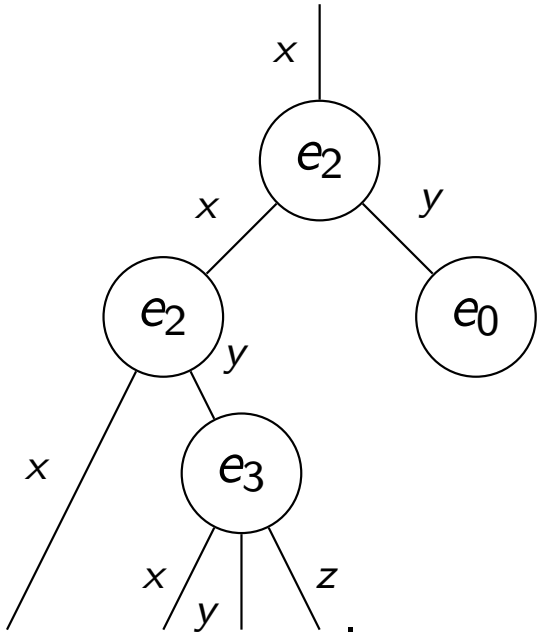
Example

- Consider $f =$



Example

- Consider $f =$
- A possible natural transformation $f \to F$ yields



- Actually, here, there is only one.

Results

Theorem (Not me)

- *$\mathcal{T}$ is a lra monad, and a club [Kelly]:*
  - *$\mathcal{T}$ preserves pullbacks.*
  - *Naturality squares for μ and η are pullbacks.*
  - *Generic factorisations.*
  - *$\mathcal{T}$ is sketchable, i.e., algebraic.*
- *$\mathcal{T}$-algebras are multicategories.*

Iterating the process

- We could do the same for categories, with $\mathcal{S} =$

$$\begin{tikzcd} 1 \\ \star \arrow[u, bend left, "\sigma"] \arrow[u, bend right, "\tau"'] \end{tikzcd}$$

- The elements of $\mathcal{T}_\omega(1)$ are cospans

$$\star \longrightarrow f \longleftarrow \star.$$

Extension

- Consider two such cospans f and g.
- They consist of multigraphs looking like

$$— \bullet — \bullet — \ldots — \bullet — \quad .$$

  with, say, n and p occurrences of $\bullet$, respectively.
- Consider the full subcategory $\mathcal{S}[f, g]$ of $\hat{\mathcal{S}}$ containing:
  - the representables, plus
  - the colimit (f, g) of the diagram

$$\begin{tikzcd} & f & \\ \star \arrow[ur, bend left] \arrow[dr, bend right] & & \star. \arrow[ul, bend right] \arrow[dl, bend left] \\ & g & \end{tikzcd}$$

Extension

This subcategory $\mathcal{S}[f,g]$ looks like $\mathcal{S}$ plus one object:

$$
\begin{tikzcd}
 & (f,g) \arrow[d, shift left, "\sigma_{n-1}"'] \arrow[d, shift right, "\tau_0"] \arrow[d, phantom, "\cdots"] \\
 & 1 \arrow[u, bend left=60, "\sigma_0"] \arrow[u, bend right=60, "\tau_{p-1}"'] \\
 & \star \arrow[u, bend left, "\sigma"] \arrow[u, bend right, "\tau"']
\end{tikzcd}
$$

with $\sigma\sigma_0 = \sigma\tau_0$, $\tau\sigma_{n-1} = \tau\tau_{p-1}$, and

$\tau\sigma_i = \sigma\sigma_{i+1}$ $\tau\tau_j = \sigma\tau_{j+1}$.

Extension

Now, we:

- do this for every equivalence class of pairs of cospans,
- and take the colimit:

$$\begin{tikzcd}
 & & \mathcal{S} \arrow[dl] \arrow[dr] & & \\
\dots & \mathcal{S}[f,g] \arrow[dr] & \dots & \mathcal{S}[h,k] \arrow[dl] & \dots \\
 & & \mathcal{S}_2 \arrow[ul, phantom, "\frown"] & &
\end{tikzcd}$$

Extension

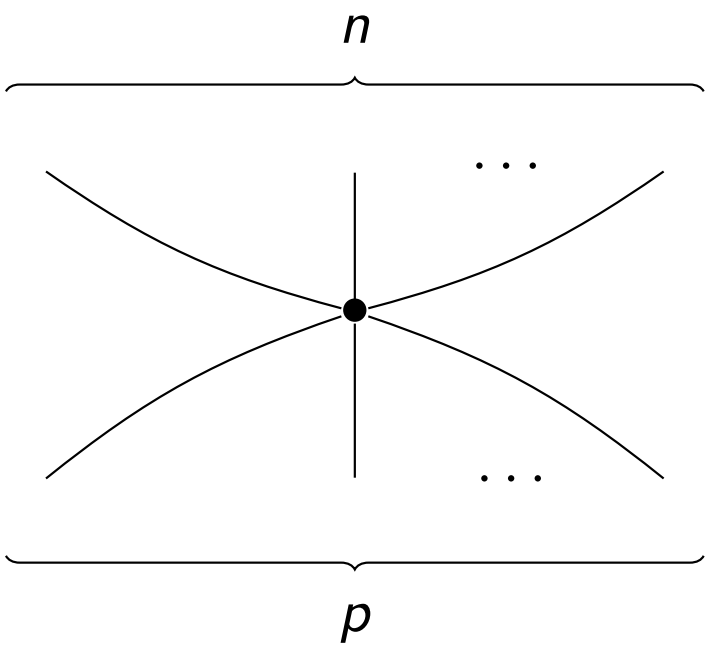
The (category of elements of the) representable presheaf (f, g) on $\mathcal{S}_2$ looks like:

$$
\begin{tikzcd}
 & \overbrace{1 \arrow[dr] & \star \arrow[l] \arrow[r] \arrow[dr] & 1 \arrow[d] & \dots \arrow[l] \arrow[r] \arrow[dl] & 1 \arrow[dl]}^{n} & \\
\star \arrow[ur] \arrow[dr] \arrow[rrr] & & & (f, g) & & & \star \arrow[ul] \arrow[dl] \arrow[lll] \\
 & \underbrace{1 \arrow[ur] & \star \arrow[l] \arrow[r] \arrow[ur] & 1 \arrow[u] & \dots \arrow[l] \arrow[r] \arrow[ul] & 1 \arrow[ul]}_{p} &
\end{tikzcd}
$$

with all triangles commuting.

Picture

We picture it as:



This allows to define:

- Horizontal composition by glueing along backgrounds.
- Vertical composition by glueing along wires.

Not yet proved

- One defines $\mathcal{T}_\omega$ using these operations and identities,
- then $\mathcal{T}$,

and obtains:

The corresponding $\mathcal{T}$-algebras are exactly 2-categories.

Conclusion

- A technique to define algebraic structures
  - of the kind where to seek coherence results,
  - with coherence built-in.
- Now?
  - Make this into a theory (i.e., prove a general result)?
  - Handle weak structures (here, we might have a size problem)?