An Empirical Functional Central Limit Theorem For Weakly Dependent Sequences

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Abstract. In this paper we obtain a Functional Central Limit Theorem for the empirical process of a stationary sequence under a new weak dependence condition introduced by Doukhan & Louhichi (1999). This result improves on the Empirical Functional Central Limit Theorem in Doukhan & Louhichi (1999). Our proof relies on new moment inequalities and on a Central Limit Theorem. Techniques of proofs come respectively from Louhichi (2000) and Rio (1996). We also deduce a rate of convergence in a Marcinkiewicz-Zygmund Strong Law.

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1 Introduction

In this paper we essentially give a Functional Central Limit Theorem (Theorem 1) for the empirical process of stationary weakly dependent sequences which improves on results in Doukhan & Louhichi (1999). Our dependence frame is described in section 1.1. The main steps to obtain our theorem are a new moment inequality (Lemma 3) and a Central Limit Theorem (CLT) (Theorem 2).

Moment inequalities under independence have already been studied. We recall here Rosenthal inequality:

$$\mathbf{E}|S_n|^r \leq C_r \left\{ (\text{Var } S_n)^{r/2} + n\mathbf{E}|X_1|^r \right\}, \tag{1}$$

where $X = (X_1, \ldots, X_n)$ is a centered vector of independent and identically distributed real valued random variables with finite variance, $S_n = X_1 + \cdots + X_n$, $S_0 = 0$, and $r \in]2, +\infty[$. Doukhan & Louhichi (1999) obtain Rosenthal inequalities under weak dependence, but exponents are restricted to be even integers. Louhichi (2000) get moment inequalities of order $r \in]2, +\infty[$ for a class of sequences satisfying the property "*the non correlation yields the independence.*" This property is called the $(AG)-$property and is satisfied by associated and Gaussian processes. In this paper, we extend Louhichi's moment inequalities to weakly dependent sequences (Lemma 3 stated in section 3). Weak dependence used here is precised by the definition 1 below. Examples of such sequences are described in section 1.2. The tightness of the empirical process is deduced from these moment inequalities. Another application of our moment inequalities is a Marcinkiewicz-Zygmund Strong Law (MZSL) for partial sums of bounded dependent random variables (Corollary 1).

We prove then a Central Limit Theorem from which we deduce the fi-di convergence. Its proof relies on a variation on the Lindeberg-Rio method (Rio, 1996) and not on Bernstein's blocks (used e.g. by Doukhan & Louhichi, 1999, to prove the fi-di convergence).

We relax assumptions of previous authors for both tightness and fi-di convergence.

The paper is organized as follows. Our main result is stated in Section 2. In Section 3 we state our new moment inequalities. We also write a corollary concerning rate of convergence for a Marcinkiewicz-Zygmund Strong Law (MZSL) for partial sums in our dependence frame. Finally sections 4, 5 and 6 are respectively devoted to the proofs of the main result, of the corollary concerning a MZSL and of our moment inequalities. We defer the proof of the Central Limit Theorem and of some technical lemmas respectively to Appendix A and Appendix B.

1.1 Weak dependence

Our dependence frame is a variation on the one in Doukhan & Louhichi (1999). We work here under a causality assumption which is fundamental in the proof of our moment inequalities (see section 6). More precisely, E being some Euclidean space $\mathbb{R}^d$ endowed with its Euclidean norm $\|.\|$, we shall consider a sequence of E-valued random variables $(X_n)_{n\in\mathbb{N}}$. We define $\mathbb{L}^\infty$ as the set of measurable and bounded numerical functions on some space $\mathbb{R}^m$, $m \in \mathbb{N}^*$ and its norm is classically written $\|.\|_\infty$. Moreover, let $u \in \mathbb{N}^*$ be a positive integer. We endow the set $F = E^u$ with the norm $\|(x_1, \ldots, x_u)\|_F = \|x_1\| + \cdots + \|x_u\|$. Let now $h : F = E^u \to \mathbb{R}$ be a numerical function on F, we set

$$\mathrm{Lip}(h) = \sup_{x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_F}$$

the Lipschitz modulus of h. Define $\mathcal{L} = \bigcup_{u=1}^{\infty} \{h \in \mathbb{L}^\infty(E^u, \mathbb{R});\ \mathrm{Lip}(h) < \infty\}$.

Definition 1
The sequence $(X_n)_{n\in\mathbb{N}}$ is $s-$weakly dependent (SWD) if for some sequence $\theta = (\theta_r)_{r\in\mathbb{N}}$ decreasing to zero at infinity, for any u-tuple $(i_1, \ldots, i_u)$, $u \in \mathbb{N}^*$, and for any v-tuple $(j_1, \ldots, j_v)$, $v \in \mathbb{N}^*$, with $i_1 \le \cdots \le i_u \le i_u + r \le j_1 \le \cdots \le j_v$, and $h \in \mathbb{L}^\infty$, $k \in \mathcal{L}$,

$$|\mathrm{Cov}\left(h(X_{i_1}, \ldots, X_{i_u}), k(X_{j_1}, \ldots, X_{j_v})\right)| \le v\ \|h\|_\infty\ \mathrm{Lip}(k)\ \theta_r. \tag{2}$$

1.2 Examples

Before stating our results, we give examples of (SWD) sequences in this section.

Definition 2
Let $(\eta_n)_{n\in\mathbb{Z}}$ be a stationary sequence of real valued random variables and F be a measurable function defined on $\mathbb{R}^{\mathbb{N}}$. The stationary sequence $(X_n)_{n\in\mathbb{Z}}$ defined by $X_n = F(\eta_n, \eta_{n-1}, \eta_{n-2}, \ldots)$ is called a causal Bernoulli shift.

We denote $(\theta_r)_{r\in\mathbb{N}}$ any nonnegative and non increasing sequence such that

$$\mathbf{E}|F(\eta_0, \eta_{-1}, \eta_{-2}, \ldots) - F(\eta_0, \ldots, \eta_{-r}, 0, 0, \ldots)| \le \theta_r.$$

Causal shifts with independent identically distributed (i.i.d.) innovations $(\eta_k)_{k\in\mathbb{Z}}$ satisfy (2) with θ_r (see Doukhan & Louhichi, 1999).

Examples of such situations follow:

- The real-valued functional autoregressive model:
  If $T:\mathbb{R}\to\mathbb{R}$ is such that $|T(u)-T(u')|\le c|u-u'|$ for some $0\le c<1$ and for all $u,\ u'\in\mathbb{R}$, and if $(\eta_n)_{n\in\mathbb{Z}}$ is some i.i.d inovation process satisfying $\mathbf{E}|\eta_0|<\infty$, $(X_n)_{n\in\mathbb{N}}$ defined by

  $$X_n = T(X_{n-1})+\eta_n \tag{3}$$

  is $s-$weakly dependent with $\theta_r = Cc^r$ for some constant $C>0$.

- The non mixing stationary Markov chain with i.i.d. Bernoulli innovations ($P(\eta_0=0)=P(\eta_0=1)=1/2$) $X_n=(X_{n-1}+\eta_n)/2$ is $s-$weakly dependent with $\theta_r=\mathcal{O}(2^{-r})$; its marginal distribution is uniform on $[0,1]$.

- Chaotic expansions associated with the discrete chaos generated by the sequence $(\eta_t)_{t\in\mathbb{Z}}$:
  In a condensed formulation we write, $F(x)=\sum_{k=0}^{\infty}F_k(x),\ x\in\mathbb{R}^{\mathbb{N}}$ for

  $$F_k(x)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}a^{(k)}_{j_1,\dots,j_k}x_{j_1}x_{j_2}\cdots x_{j_k},\ \ k\ge 1$$

  where $F_k(x)$ denotes the k-th order chaos contribution and $F_0(x)=a_0^{(0)}$ is only a centering constant. In short we write in the vectorial notation, $F_k(x)=\sum_{j\in\mathbb{N}^k}a_j^{(k)}x_j$. Processes associated with a finite number of chaos (i.e. $F_k\equiv 0$ if $k>k_0$ for some $k_0\in\mathbb{N}$) are also called *Volterra processes*. A simple and general condition for $\mathbb{L}^1$-convergence of this expansion, still written in a condensed notation, is $\sum_{k=0}^{\infty}\left\{\sum_{j\in\mathbb{N}^k}\left|a_j^{(k)}\right|\mathbf{E}|\eta_0|^k\right\}<\infty$. This condition allows to define the distribution of such shift processes. A suitable bound for θ_r is then

  $$\theta_r=\sum_{k=0}^{\infty}\left\{\sum_{\{j\in\mathbb{N}^k;\|j\|_\infty>r\}}\left|a_j^{(k)}\right|\mathbf{E}|\eta_0|^k\right\}<\infty.$$

  - If $\mathbf{E}|\eta_0|=0$ and

    $$F_k(x)=\sum_{0\le j_1<\cdots<j_k}a^{(k)}_{j_1,\dots,j_k}x_{j_1}x_{j_2}\cdots x_{j_k},\ \ k\ge 1,$$

    then $\theta_r=\left(\sum_{0\le j_1<\cdots<j_k,j_k>r}(a^{(k)}_{j_1,\dots,j_k})^2(\mathbf{E}\eta_0^2)^k\right)^{1/2}$.
  - Linear processes $X_n=\sum_{k=0}^{\infty}a_k\eta_{n-k}$ which include ARMA models are those with $F_k(x)\equiv 0$ for all $k>1$. A first choice is $\theta_r=\mathbf{E}|\eta_0|\sum_{k>r}|a_k|$ for the linear process with i.i.d. innovations such that $\mathbf{E}|\eta_0|<\infty$. For centered and $\mathbb{L}^2$ innovations, another choice is thus $\theta_r=\sqrt{\mathbf{E}|\eta_0|^2\sum_{k>r}|a_k|^2}$.
  - The simple bilinear process with the recurrence equation $X_t=aX_{t-1}+bX_{t-1}\eta_{t-1}+\eta_t$. Such processes are associated with the chaotic representation in

    $$F(x)=\sum_{j=1}^{\infty}x_j\prod_{s=0}^{j-1}(a+bx_s),\ \ x\in\mathbb{R}^{\mathbb{Z}}.$$

    If $c=\mathbf{E}|a+b\xi_0|<1$ then $\theta_r=\frac{c^r(r+1)}{c-1}$ has a geometric decay rate.
  - ARCH(∞): The recurrence equation $X_t=(a+\sum_{j=1}^{\infty}b_jX_{t-j})\eta_t$ has the stationary solution:

    $$X_t=a\sum_{l=0}^{\infty}\sum_{j_1\cdots j_l=1}^{\infty}b_{j_1}\cdots b_{j_l}\eta_t\eta_{t-j_1}\cdots\eta_{t-j_1-\cdots-j_l},$$

    with $a\ge 0$, $b_j\ge 0$ for all j, $\sum_{j=1}^{\infty}b_j<\infty$ and $(\eta_k)_{k\in\mathbb{Z}}$ a sequence of i.i.d. non-negative random variables (see e.g. Giraitis, Kokoszka and Leipus, 1998). Then we can prove that the process $(X_t)_{t\in\mathbb{Z}}$ is $s-$weakly dependent (see Doukhan, 2001).

2 The Empirical Functional Central Limit Theorem

Let $(X_n)_{n\in\mathbb{N}}$ denote a stationary sequence of real valued random variables. In this section we investigate some properties of the empirical process constructed from the stationary sequence $(X_n)_{n\in\mathbb{N}}$.

We get a Fuctional Limit Theorem for the empirical process under the (SWD)-weak dependence condition. We consider a stationary sequence $(X_n)_{n\in\mathbb{N}}$ of random variables with common distribution function F. For $t\in\mathbb{R}$, we consider the following processes:

$F_n(t) := \frac{1}{n}\sum_{k=0}^{n-1} 1_{X_k\le t}$ and $U_n(t) := \sqrt{n}\left(F_n(t) - F(t)\right)$.

We have the following result:

Theorem 1 *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary sequence of real valued random variables with common repartition function F supposed to be Lipschitz. Suppose that (X_n) satisfies the $(SWD)-$dependence condition, with $\theta_r = \mathcal{O}\left((r+1)^{-2-2\sqrt{2}-\nu}\right)$, for some $\nu > 0$. Then the sequence of processes*

$\{U_n(t);\ t\in\mathbb{R}\}_{n>0}$ converges in distribution in the Skohorod space $\mathcal{D}(\mathbb{R})$ to the centered Gaussian process indexed by $\mathbb{R}$ with covariance defined by $\Gamma(s,t) = \sum_{k=-\infty}^{+\infty} Cov(1_{X_0\le s}, 1_{X_{|k|}\le t})$.

In fact, Theorem 1 above can be decomposed in two parts: the tightness and the fi-di convergence.

Lemma 1 (Tightness) *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary sequence of real valued random variables with common distribution function F supposed to be Lipschitz. We assume that $(X_n)_{n\in\mathbb{N}}$ fulfills the $(SWD)-$dependence condition with $\theta_r = \mathcal{O}\left((r+1)^{-2-2\sqrt{2}-\nu}\right)$, for some $\nu > 0$.*

Then the sequence of processes $\{U_n(t);\ t\in\mathbb{R}\}_{n>0}$ is tight in the Skohorod space $\mathcal{D}(\mathbb{R})$.

Lemma 1 above is proved in section 4. It clearly improves on Doukhan & Louhichi (1999) who assume $\theta_r = \mathcal{O}(r^{-(5+\nu)})$, $\nu > 0$. Indeed, in order to obtain tightness, those authors calculate the moment of order 4 of the partial sums. Here we just have to calculate some moment of order $2+\sqrt{2}$ as is shown in the proof of Lemma 1. Lemma 3 in the next section allows us indeed to calculate moment with an order r which is not necessarily an integer.

The fi-di convergence is deduced from the following Central Limit Theorem:

Theorem 2 (Central Limit Theorem) *Let $(X_n)_{n\in\mathbb{N}}$ de a stationary sequence of centered*

$(SWD)-$weakly dependent random variables with $\theta_r = \mathcal{O}\left((r+1)^{-a}\right)$, for some $a > 3/2$. We assume that $(X_n)_{n\in\mathbb{N}}$ is uniformly bounded. If $S_n = X_0 + \cdots X_{n-1}$, we assume that

$\frac{\mathrm{Var}\,(S_n)}{n} \xrightarrow[n\infty]{} \sigma^2 > 0$. *Then*

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1} X_i \xrightarrow[n\infty]{\mathcal{L}} \mathcal{N}(0,\sigma^2).$$

The method to prove Theorem 2 is a variation on Lindeberg-Rio method (Rio, 1996). We deduce from this CLT the following lemma:

Lemma 2 (Fi-di convergence) *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary sequence of real valued random variables with common distribution function F supposed to be Lipschitz. Suppose that $(X_n)_{n\in\mathbb{N}}$ satisfies the $(SWD)-$dependence condition, with a sequence $(\theta_r)_{r\in\mathbb{N}} := ((r+1)^{-a})_{r\in\mathbb{N}}$. Assume that $a > 3$. Then the finidimensional marginals of the process $\{U_n(t);\ t\in\mathbb{R}\}_{n>0}$ converge in distribution to the finidimensional marginals of the centered Gaussian process indexed by $\mathbb{R}$ with covariance* *defined by* $\Gamma(s,t) = \sum_{k=-\infty}^{+\infty} \mathrm{Cov}\left(1_{X_0 \le s}, 1_{X_{|k|} \le t}\right)$.

The proofs of Theorem 2 and Lemma 2 are defered to Appendix A. The condition

$\theta_r = \mathcal{O}\left((r+1)^{-3-\delta}\right)$, for some $\delta > 0$ in Lemma 2 improves the condition

$\theta_r = \mathcal{O}\left(r^{-4}\right)$ obtained by Doukhan & Louhichi (1999) for fi-di convergence.

Now both tightness result and fi-di convergence result yield Theorem 1.

3 Moment inequalities

In the statements of the main results in section 2, we consider a (SWD) sequence $(X_n)_{n\in\mathbb{N}}$. To prove the tightness in Lemma 1, we need moment inequalities for the partial sums of a sequence $(Y_n) = (\varphi(X_n))$. We prove in section 4 that $(Y_n)_{n\in\mathbb{N}}$ is also $s-$weakly dependent. Therefore the goal of the following Lemma is to give moment inequalities for (SWD) sequences. Doukhan & Louhichi (1999) prove moment inequalities for weakly dependent sequences. But the order of these inequalities is an integer not less than 2. Recently, Louhichi (2000) proves moment inequalities of order $r \in]2, +\infty[$ but for sequences satisfying the $(AG)-$property. The following variation on Louhichi's Lemma (2000) entails moment bounds with order $r \in]2, +\infty[$ for $(SWD)-$sequences:

Lemma 3 *Let r be a fixed real number > 2. Let (X_n) be a stationary sequence of centered and (SWD) random variables. Suppose moreover that this sequence is bounded by 1. Let $S_n := X_1 + X_2 + \cdots + X_n$, for $n \ge 1$ and $S_0 = X_0 = 0$. Then there exists a positive constant C_r depending only on r, such that*

$$\mathbf{E}|S_n|^r \le C_r \left[s_n^r + M_{r,n}\right], \tag{4}$$

where $M_{r,n} := n\sum_{i=0}^{n-1}(i+1)^{r-2}\theta_i$, and $s_n^2 := M_{2,n} = n\sum_{i=0}^{n-1}\theta_i$.

We prove Lemma 3 in Section 6.

These moment inequalities also allow us to study the rate of convergence for a Marcinkiewicz-Zygmund Strong Law for partial sums of bounded dependent random variables. Such results appear for example in Lai (1977) and in Berbee (1987) for the mixing case and in Louhichi (1998) for the associated case.

Let us first introduce the notion of $r-$quick convergence as in Lai (1977).

Definition 3 *A sequence $(Z_n)_{n\in\mathbb{N}}$ of random variables converges to 0 $r-$quickly $(r > 0)$ if*

$$\mathbf{E}\left(N_\epsilon\right)^r < \infty \ \forall \varepsilon > 0,$$

where $N_\epsilon := \sup\{n \ge 1 : \ |Z_n| \ge \varepsilon\}$.

Note that $Z_n \longrightarrow 0$ $r-$quickly for some $r > 0$ implies $Z_n \longrightarrow 0$ a.s.

The following corollary is a convergence theorem for $s-$weakly dependent variables.

Corollary 1 *Let $(X_n)_{n\ge1}$ be a stationary sequence of centered and (SWD) random variables. Let r be a fixed real number > 2. Suppose moreover that this sequence is bounded by some positive constant M. Assume that the coefficient of (SWD) satisfies $\theta_q = \mathcal{O}\left((q+1)^{-D}\right)$ with $D > r-1$. Then,*

- *for all* $\frac{1}{2} < \alpha \leq 1$, *for all* $k < (\alpha - \frac{1}{2})r - 1$, *and for all* $\varepsilon > 0$, *we have:*

$$\sum_{n \geq 1} n^k P(\max_{j \leq n} |S_j| \geq \varepsilon n^\alpha) < \infty; \tag{5}$$

- *for all* $1 \geq \alpha > \frac{1}{2}$, *for all* $1 < p\alpha < (\alpha - \frac{1}{2})r + 1$, *we have the four following assertions:*

  1. $\sum_{n \geq 1} n^{p\alpha - 2} P(\max_{j \leq n} |S_j| \geq \varepsilon n^\alpha) < \infty \; \forall \varepsilon > 0$,
  2. $\mathbf{E}\{\sup_{n \geq 0}(|S_n| - \varepsilon n^\alpha)\}^{\frac{p\alpha - 1}{\alpha}} < \infty \; \forall \varepsilon > 0$,
  3. $\sum_{n \geq 1} n^{p\alpha - 2} P(\sup_{j \geq n} j^{-\alpha}|S_j| \geq \varepsilon) < \infty \; \forall \varepsilon > 0$,
  4. $n^{-\alpha} S_n \longrightarrow 0$ $(p\alpha - 1)$ – *quickly.*

The proof of Corollary 1 is written in section 5.

4 Proof of Theorem 1

This section is devoted to the proof of the main result (Theorem 1) stated in section 2. We prove the Functional Limit Theorem (Theorem 1) in 2 steps: the tightness and the fi-di convergence.

Proof of Lemma 1

In the following, C will denote some arbitrary constant which may vary from line to line. Let $s < t$ be two real numbers. We want to apply moment inequalities of Lemma 3 to the sequence $\left(\dfrac{1_{s < X_n \leq t} - (F(t) - F(s))}{2}\right)_{n \in \mathbb{N}}$.

So we have to prove that this sequence which we call $(Y_n)_{n \in \mathbb{N}}$ satisfies a (SWD) dependence condition. For this we have to bound $C_{h,u,v} := \mathrm{Cov}\left(h(Y_{i_1}, \ldots, Y_{i_u}), k\left(Y_{j_1}, \ldots, Y_{j_v}\right)\right)$ for all $i_1 \leq \cdots \leq i_u \leq i_u + r \leq j_1 \leq \cdots \leq j_v$ and for all $h \in \mathbb{L}^\infty$, $k \in \mathcal{L}$. Let $\epsilon > 0$ such that $s + \varepsilon < t - \varepsilon$. Let us write $Y_n = \varphi(X_n)$. We want to smooth the function φ which is not Lipschitz. For this we consider the following Lipschitz function φ^ϵ smoothing φ :

- φ_ϵ is equal to φ on $]-\infty, s - \epsilon] \cup]s + \epsilon, t - \varepsilon] \cup]t + \varepsilon, +\infty[$;
- for $s - \varepsilon < x \leq s + \varepsilon$, $\varphi_\varepsilon(x) = \frac{-1}{8\varepsilon^3}\left(x^3 - 3sx^2 + 3(s^2 - \varepsilon^2)x - s^3 + 3s\varepsilon^2 - 2\varepsilon^3\right) - \frac{F(t) - F(s)}{2}$;
- for $t - \varepsilon < x \leq t + \varepsilon$, $\varphi_\varepsilon(x) = \frac{-1}{8\varepsilon^3}\left(x^3 - 3tx^2 + 3(t^2 - \varepsilon^2)x - t^3 + 3t\varepsilon^2 - 2\varepsilon^3\right) - \frac{F(t) - F(s)}{2}$.

We then have $||\varphi^\epsilon||_\infty \leq ||\varphi||_\infty \leq \frac{1}{2}$ and $\mathrm{Lip}(\varphi^\epsilon) \leq \frac{3}{8\,\epsilon}$.

Then,

$$\begin{aligned}
|C_{h,u,v}| &\leq |\mathrm{Cov}\left(h(Y_{i_1}, \cdots, Y_{i_u}), k\left(\varphi(X_{j_1}), \ldots, \varphi(X_{j_v})\right) - k\left(\varphi^\epsilon(X_{j_1}), \ldots, \varphi^\epsilon(X_{j_v})\right)\right)| \\
&\quad + |\mathrm{Cov}\left(h(Y_{i_1}, \ldots, Y_{i_u}), k\left(\varphi^\epsilon(X_{j_1}), \ldots, \varphi^\epsilon(X_{j_v})\right)\right)| \\
&\leq 2\,||h||_\infty \mathrm{Lip}(k) \textstyle\sum_{i=1}^v \mathbf{E}|\varphi(X_{j_i}) - \varphi^\epsilon(X_{j_i})| + ||h||_\infty \, \mathrm{Lip}(k)\, \mathrm{Lip}(\varphi^\varepsilon)\, \theta_r \, v \\
&\leq 2||h||_\infty \mathrm{Lip}(k) v\, 2\, ||\varphi||_\infty P(X_0 \in]s - \varepsilon, s + \varepsilon] \cup]t - \varepsilon, t + \varepsilon]) + ||h||_\infty \mathrm{Lip}(k) \mathrm{Lip}(\varphi^\varepsilon)\, \theta_r \, v \\
&\leq ||h||_\infty \mathrm{Lip}(k) v \, \left(8\,\varepsilon\, \mathrm{Lip}(F) + \tfrac{3\,\theta_r}{8\epsilon}\right).
\end{aligned}$$

In the following, C denotes some positive constant which may vary from line to line.

So we have $|C_{h,u,v}| \leq C||h||_\infty \operatorname{Lip}(k)\ v\ \left(\varepsilon + \frac{\theta_r}{\varepsilon}\right)$. Then if $\sqrt{\theta_r} < \frac{t-s}{2}$, we take $\epsilon = \sqrt{\theta_r}$, and get

$$|C_{h,u,v}| \leq C\ ||h||_\infty \operatorname{Lip}(k)\ v\ \sqrt{\theta_r}. \tag{6}$$

Moreover $|C_{h,u,v}| = |\mathrm{Cov}\left(h(Y_{i_1}, \ldots, Y_{i_u}), k\left(Y_{j_1}, \ldots, Y_{j_v}\right) - k\left({Y'}_{j_1}, \ldots, {Y'}_{j_v}\right)\right)|$, where we consider $Y'_n = \varphi(X'_n)$ with $(X'_n)_{n\in\mathbb{N}}$ independent of $(X_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, ${X'}_n \sim X_n$. Hence,

$|C_{h,u,v}| \leq 2||h||_\infty \operatorname{Lip}(k)\ \mathbf{E}\sum_{i=1}^{v} |\varphi(X_{j_i}) - \varphi(X'_{j_i})| \leq 2\ ||h||_\infty \operatorname{Lip}(k)\ v\ \frac{1}{2}\ 2\mathbf{E}|1_{s<X_0\leq t}|$ and

$$|C_{h,u,v}| \leq C\ ||h||_\infty \operatorname{Lip}(k)\ v\ \left(F(t) - F(s)\right). \tag{7}$$

From (7) and as F is Lipschitz, we deduce that $|C_{h,u,v}| \leq C\ ||h||_\infty \operatorname{Lip}(k)\ v\ |t-s|$. Hence, if $\sqrt{\theta_r} \geq \frac{t-s}{2}$, $|C_{h,u,v}| \leq C\ ||h||_\infty \operatorname{Lip}(k)\ v\ \sqrt{\theta_r}$. Hence, for all real numbers $s < t$, inequality (6) holds. We conclude with inequalities (6) and (7),

$$|C_{h,u,v}| \leq C\ ||h||_\infty \operatorname{Lip}(k)\ v\ ((F(t) - F(s)) \wedge \sqrt{\theta_r}). \tag{8}$$

So $(Y_n)_{n\in\mathbb{N}}$ is (SWD) with $\theta'_r \leq C\ ((F(t) - F(s)) \wedge \sqrt{\theta_r})$. We prove the tightness by applying Lemma 3 to $(Y_n)_{n\in\mathbb{N}}$. In the following C_r denotes some arbitrary constant depending only on r and which may vary from line to line. Notice that $\frac{2}{\sqrt{n}}\sum_{k=0}^{n-1} Y_k = U_n(t) - U_n(s)$.

If we write $\theta_r = r^{-a}$, $a > 0$, we get

$$\begin{aligned} E\left|\frac{2}{\sqrt{n}}\sum_{k=0}^{n-1} Y_k\right|^r &\leq C_r\left\{\left(\sum_{k=1}^{n} k^{\frac{-a}{2}} \wedge (F(t) - F(s))\right)^{\frac{r}{2}}\right\} \\ &\quad + C_r\left\{n^{\frac{2-r}{2}}\sum_{k=1}^{n}(k+1)^{r-2}[k^{\frac{-a}{2}} \wedge (F(t) - F(s))]\right\} \end{aligned}$$

Hence if $a > 2$ and if $a > 2(r-1)$,

$$E\left|\frac{2}{\sqrt{n}}\sum_{k=0}^{n-1} Y_k\right|^r \leq C_r\{(F(t) - F(s))^{\frac{r(a-2)}{2a}} + n^{\frac{2-r}{2}}\ (F(t) - F(s))^{\frac{2+a-2r}{a}}\}.$$

Now, if we chose $r = 2+\sqrt{2}$, as F is continuous, it follows from Theorem 2.1 in Shao & Yu (1996) that the sequence $\{U_n(t), t \in \mathbb{R}\}$ is tight in the Skohorod space $\mathcal{D}(\mathbb{R})$ as soon as $a > 2 + 2\sqrt{2}$. The choice $r = 2+\sqrt{2}$ minimizes the condition on the dependence coefficient a. This concludes the proof of the tightness. □

To conclude the proof of Theorem 1 we now have to prove Lemma 2 of fi-di convergence. To prove it, we use a Central Limit Theorem (Theorem 2) whose proof is written in Appendix A.

Now Lemmas 1 and 2 together yield both the tightness and the fi-di convergence as soon as $a > 2 + 2\sqrt{2}$. This concludes the proof of Theorem 1.

5 Proof of Corollary 1

In this section we prove Corollary 1 stated in section 3. C_r still denotes some constant, depending only on r, which may vary from line to line.

We apply the moment inequalities to the partial sums of $(\frac{X_n}{M})$. We get as $D > r-1$ and $r > 2$, for all n large enough:

$$\mathbf{E}\left|\frac{S_n}{M}\right|^r \leq C_r\ n^{\frac{r}{2}}\ \left(\sum_{i=0}^{+\infty}\theta_i\right)^{\frac{r}{2}}.$$

We then apply maximal inequalities in Moricz *et al* (1982) and Bienaymé-Tchebycheff inequality to obtain: for $\frac{1}{2} < \alpha \leq 1$, for $k < (\alpha - \frac{1}{2})r - 1$ and for all $\varepsilon > 0$

$$P(\max_{1\leq k\leq n}|S_k| \geq \varepsilon n^{\alpha}) \leq C_{r,\varepsilon} n^{\frac{r}{2}-\alpha r}, \tag{9}$$

where $C_{r,\varepsilon}$ only depends on r and ε.

From (9) we deduce (5) in Corollary 1. Now, Lemma 2 in Chow & Lai (1975) together with (5) yield the four assertions *1.*, *2.*, *3.* and *4.* of Corollary 1.

Next section is devoted to the proof of moment inequalities stated in Lemma 3 of section 3.

6 Proof of Lemma 3

The proof is a variation on Louhichi's method (2000) under our dependence frame. Let $p \geq 2$ be a fixed integer. We define the function $g_p : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ as in Louhichi (2000):

$$g_p(t,x) := \frac{1}{(p+1)!}\left[x^{p+1}1_{0\leq x\leq t} + (x^{p+1} - (x-t)^{p+1})1_{t<x}\right], \tag{10}$$

for any $x \geq 0$ and $g_p(t,x) = g_p(t,-x)$. Then we decompose the proof in several tool steps.

6.1 Step 1: Main terms

Let $p \geq 2$ be a fixed integer and $\mathcal{C}_p$ be the class of real-valued, p times continuously differentiable functions f such that $f(0) = \cdots = f^{(p)}(0) = 0$.
Let $\mathcal{F}_p(b_p, b_{p+1})$ be the subclass of $\mathcal{C}_{p+1}$ such that $\|f^{(p)}\|_\infty \leq b_p$ and $\|f^{(p+1)}\|_\infty \leq b_{p+1}$, where $\|f^{(i)}\|_\infty = \sup_{x\in\mathbb{R}}|f^{(i)}(x)|$ and $f^{(i)}$ denotes the differential of order i of the function f.
We recall a result in Louhichi (2000) which is a generalization of equation (4.3) in Rio (1995).

Lemma 4 *Let p be a fixed integer, $p \geq 2$. Let $\phi_p \in \Phi_p$, where*

$$\Phi_p := \{\phi : \mathbb{R}^+ \to \mathbb{R}^+; \phi \textit{ convex}, \phi(0) = \phi'(0) = \cdots = \phi^{(p)}(0) = 0,\ \phi^{(p)} \textit{ nondecreasing, concave}\}.$$

Suppose that $\lim_{x\to\infty}\phi_p^{(p+1)}(x) = 0$. *Then*

$$\phi_p(x) = \int_0^{+\infty} g_p(t,x)\nu_p(dt),$$

where ν_p is the Stieltjes measure of $-\phi_p^{(p+1)}$ defined by $\nu_p(dt) = -d\phi_p^{(p+1)}(t)$.

Lemma 4 above together with Fubini's Theorem yields

$$\mathbf{E}\phi_p(|S_n|) = \int_0^{+\infty}\mathbf{E}g_p(t,S_n)\nu_p(dt). \tag{11}$$

So we deduce the estimation of $\mathbf{E}\phi_p(|S_n|)$ from that of $\mathbf{E}g_p(t,S_n)$. Hence the goal of this step is to bound $\mathbf{E}f(S_n)$ for a "good" set of real valued functions f containing the functions $x \longrightarrow g_p(t,x)$, $t \geq 0$.

We notice that the function $x \longrightarrow g_p(t,x)$ as defined by (10) belongs to the set $\mathcal{F}_p(b_p, b_{p+1})$, with $b_p = t$ and $b_{p+1} = 1$. Hence we give in this step an estimation of $\mathbf{E}f(S_n)$, for $f \in \mathcal{F}_p(b_p, b_{p+1})$.
Let us first exhibit the main terms which appear in the proof.

$$\mathbf{E}_{p-1,k} = \sum_{0=:i_0 \leq i_1 \leq \ldots \leq i_{p-1} \leq k-1} \left| \mathbf{E} X_{k-i_0} X_{k-i_1} \cdots X_{k-i_{p-1}} \right|;$$

$$\mathbf{E}_{p-2,k}(\Delta f) = \sup_{0 \leq u \leq 1} \sum_{0=:i_0 \leq i_1 \leq \ldots \leq i_{p-2} \leq k-1} \left| \mathbf{E} X_{k-i_0} X_{k-i_1} \cdots X_{k-i_{p-2}} \Delta_{p-2,k}(f) \right|,$$

where

$$\begin{aligned} \Delta_{p-2,k}(f) = \Delta_{p-2,k}(f,u) &= \left[f(S_{k-i_{p-2}-1} + uX_{k-i_{p-2}}) - f(S_{k-i_{p-2}-1}) \right] \\ &= uX_{k-i_{p-2}} \int_0^1 f'\left(S_{k-i_{p-2}-1} + uvX_{k-i_{p-2}}\right) dv; \end{aligned}$$

and

$$\mathbf{E}_{p-2,k}(f) = \sum_{0=:i_0 \leq i_1 \leq \ldots \leq i_{p-2} \leq k-1} |\mathbf{E} X_{k-i_0} X_{k-i_1} \cdots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})|.$$

Then we denote by $A_{p,n}$(respectively by $A_{p,n}(f)$, $A_{p,n}(\Delta f)$) the sum $\sum_{k=1}^n \mathbf{E}_{p,k}$ (respectively $\sum_{k=1}^n \mathbf{E}_{p,k}(f)$, $\sum_{k=1}^n \mathbf{E}_{p,k}(\Delta f)$).

For a real valued function f that belongs to the set $\mathcal{F}_p(b_p, b_{p+1})$, the quantity $|\mathbf{E}(f(S_n))|$ is evaluated by means of the main terms $\mathbf{E}_{p-2,k}(f^{(p-1)})$ and $\mathbf{E}_{p-2,k}(\Delta f^{(p-1)})$ as shows the following lemma.

Lemma 5 *Let p be a fixed integer, $p \geq 2$. Let (X_n) be a sequence of (SWD) random variables, centered and bounded by 1. There exists a positive constant C_p depending only on p, such that for any $f \in \mathcal{F}_p(b_p, b_{p+1})$*

$$\begin{aligned} |\mathbf{E}(f(S_n))| &\leq C_p \Bigg\{ s_n^p (b_p \wedge b_{p+1} s_n) + (b_p \wedge b_{p+1}) \sum_{i=0}^{n-1} (n-i)\theta_i \\ &\quad + \sum_{k=1}^n \mathbf{E}_{p-2,k}(f^{(p-1)}) + \sum_{k=1}^n \mathbf{E}_{p-2,k}(\Delta f^{(p-1)}) \Bigg\}. \end{aligned}$$

The covariance terms in Louhichi (2000) are replaced here by bounds depending on $(\theta_r)_{r \geq 0}$.

Lemma 5 is proved in Appendix B.

6.2 Step 2: Evaluation of the main terms $\mathbf{E}_{p-2,k}(f)$ and $\mathbf{E}_{p-2,k}(\Delta f)$

The purpose of the second step is to evaluate the main terms $\mathbf{E}_{p-2,k}(f)$ and $\mathbf{E}_{p-2,k}(\Delta f)$ of Lemma 5. We need first the following preparatory lemmas.

6.2.1 Preparatory lemmas:

Let us recall the following notations: $M_{r,n} := n\sum_{i=0}^{n-1}(i+1)^{r-2}\theta_i$, $s_n^2 := M_{2,n} = n\sum_{i=0}^{n-1}\theta_i$.

Lemma 6 *[Doukhan and Louhichi (1999)]*
Let (X_n) be a centered sequence of (SWD) random variables. Suppose that (X_n) is uniformly bounded by 1. Then, for any integer $p \geq 2$, there exists a positive constant C_p for which

$$\sum_{k=1}^{n} \mathbf{E}_{p-1,k} \leq C_p \left\{ s_n^p + M_{p,n} \right\}.$$

Lemma 6 is established by Doukhan and Louhichi (1999), in order to give moment inequalities with integer order p.
The following lemma will often be used in the sequel and is proved in Appendix B.

Lemma 7 *Hölder's inequalities*
For all $p \geq 4$ and $m \in \{3, \ldots, p-1\}$, we have

$$M_{m,n}M_{p-m,n} \leq s_n^{2p/(p-2)} M_{p,n}^{(p-4)/(p-2)} \leq \left\{ s_n^p + M_{p,n} \right\}. \tag{12}$$

Now define

$$M_{m,n}(b_1, b_2) := n\sum_{i=0}^{n-1}(b_1 \wedge b_2(i+1))(i+1)^{m-2}\theta_i,$$

then

$$s_n^{p-m}(b_1 \wedge b_2 s_n)M_{m,n} \leq s_n^p(b_1 \wedge b_2 s_n) + M_{p,n}(b_1, b_2). \tag{13}$$

Also

$$s_n^{p-m}M_{m,n}(b_1, b_2) \leq s_n^p(b_1 \wedge b_2 s_n) + M_{p,n}(b_1, b_2). \tag{14}$$

6.2.2 The basic lemma:

The following lemma is the basic technical lemma of this section.

Lemma 8 *Let f be a real valued function of the set $\mathcal{F}_1(b_1, b_2)$. Let (X_n) be a centered sequence of (SWD) random variables. Suppose that (X_n) is uniformly bounded by 1. Then, for any integer $p \geq 2$, there exists a positive constant C_p depending only on p, for which*

$$\begin{aligned} &\sum_{k=1}^{n} \mathbf{E}_{p-2,k}(\Delta f) + \sum_{k=1}^{n} \mathbf{E}_{p-2,k}(f) \\ &\quad \leq C_p \left\{ s_n^p(b_1 \wedge b_2 s_n) + \sum_{m=2}^{p-2} M_{m,n}M_{p-m,n}(b_1, b_2) + M_{p,n}(b_1, b_2) \right\} \end{aligned} \tag{15}$$

The right hand side term of (15) is similar to the one obtained in Lemma 5 of Louhichi (2000). However, details of the proof are different due to the kind of dependence assumed.

Proof of Lemma 8
Using induction on $p \geq 2$, we will prove that each of the terms $\sum_{k=1}^{n} \mathbf{E}_{p-2,k}(\Delta f)$ and $\sum_{k=1}^{n} \mathbf{E}_{p-2,k}(f)$

is bounded by the right hand side (RHS) of (15).
For $p=2$, we refer to Louhichi (1998). We can also deduce the calculations for the case $p=2$ from the general case that we state just below.
Suppose now that (15) holds for the order $(p-1)$, we will prove it for p.
Our purpose is then to evaluate the following sums:

$$\sum_{k=1}^{n} \sum_{0=:i_0\le i_1\le \ldots \le i_{p-2}\le k-1} \left|\mathbf{E}X_{k-i_0}X_{k-i_1}\cdots X_{k-i_{p-2}}f(S_{k-i_{p-2}-1})\right|, \tag{16}$$

and

$$\sum_{k=1}^{n} \sum_{0=:i_0\le i_1\le \ldots \le i_{p-2}\le k-1} \left|\mathbf{E}X_{k-i_0}X_{k-i_1}\cdots X_{k-i_{p-2}}\Delta_{p-2,k}f\right|, \tag{17}$$

We argue as Doukhan & Portal (1983) : let $0=: i_0\le i_1\le \ldots \le i_{p-2}\le k-1$ be a fixed sequence of increasing integers, let m be the smallest integer for which $r_m := i_{m+1}-i_m = \max_{1\le q\le p-2}(i_q-i_{q-1})$. Finally let $\sum^{(m)}_{0=:i_0\le i_1\le\ldots\le i_{p-2}}$ denote the sums over the subdivisions $i_1\le\ldots\le i_{p-2}\le k-1$ for which the big lag $\max_{1\le q\le p-2}(i_q-i_{q-1})$ is reached at the index m.
We also set $\sum^{(m),r_m}_{0=:i_0\le i_1\le\ldots\le i_{p-2}\le k-1}$ for sums over the subdivisions $i_1\le\ldots\le i_{p-2}$ such that the big lag $i_{m+1}-i_m=\max_{1\le q\le p-2}(i_q-i_{q-1})$ equals r_m.
Hence, if B denotes a subset of $\mathbb{N}^{p-2}$ and g is a positive function on $\mathbb{N}^{p-2}$, then

$$\sum_{(0=:i_0\le i_1\le\ldots\le i_{p-2}\le k-1)\in B} g(i_1,\ldots,i_{p-2}) \le \sum_{m=0}^{p-3}\sum_{r_m=0}^{k-1}\sum^{(m),r_m}_{(0=:i_0\le i_1\le\ldots\le i_{p-2}\le k-1)\in B} g(i_1,\cdots,i_{p-2}). \tag{18}$$

We mean by the notation $(0=: i_0\le i_1\le\ldots\le i_{p-2}\le k-1)\in B$ that $(i_1,\ldots,i_{p-2})\in B$ and that $0=: i_0\le i_1\le\ldots\le i_{p-2}\le k-1$.

Study of the bound of (16):

We note the following decomposition: for g continuous and differentiable, if $g(0)=0$,

(19)
$$g(S_n)=\sum_{k=1}^{n}[g(S_k)-g(S_{k-1})]=\sum_{k=1}^{n}X_k\, g^{'}(S_{k-1}+u_kX_k),\ \text{where for all } 1\le k\le n,\ 0<u_k<1.$$

Now we use calculations on the function f' instead of f. f' appears thanks to Taylor's formula (cf. equality (19)). Let us go into further details.
First, we write

$$f(S_{k-i_{p-2}-1})=[f(S_{k-i_{p-2}-1})-f(S_{k-i_{p-2}-r_m-1})]+f(S_{k-i_{p-2}-r_m-1}).$$

The last decomposition yields :

$$\begin{aligned} &|\mathbf{E}(X_kX_{k-i_1}\cdots X_{k-i_{p-2}}f(S_{k-i_{p-2}-1}))| \\ \le\ & |\mathbf{E}(X_kX_{k-i_1}\cdots X_{k-i_{p-2}}[f(S_{k-i_{p-2}-1})-f(S_{k-i_{p-2}-r_m-1})])| \\ &+|\mathbf{E}(X_kX_{k-i_1}\cdots X_{k-i_{p-2}}f(S_{k-i_{p-2}-r_m-1}))| \\ =:\ & J_1(i_1,\ldots,i_{p-2})+J_2(i_1,\ldots,i_{p-2}). \end{aligned} \tag{20}$$

Evaluation of $\sum_{i_1\le\ldots\le i_{p-2}} J_2(i_1,\ldots,i_{p-2})$:

The relation $\mathbf{E}(XY)=\mathrm{Cov}(X,Y)+\mathbf{E}(X)\mathbf{E}(Y)$ writes

(21)
$$\begin{aligned}\mathbf{E}(X_kX_{k-i_1}\cdots X_{k-i_{p-2}}f(S_{k-i_{p-2}-r_m-1})) &= \mathrm{Cov}(X_kX_{k-i_1}\cdots X_{k-i_{p-2}},f(S_{k-i_{p-2}-r_m-1}))\\ &\quad+\mathbf{E}(X_kX_{k-i_1}\cdots X_{k-i_{p-2}})\mathbf{E}(f(S_{k-i_{p-2}-r_m-1})).\end{aligned}$$

The function f belongs to the set $\mathcal{F}_1(b_1,b_2)$, hence using Taylor's formula (recall that $f(0)=f'(0)=0$), we obtain

$$|f(x)|\le b_1|x|,\quad |f(x)|\le\frac{b_2x^2}{2},$$

this yields

$$|\mathbf{E}(f(S_{k-i_{p-2}-r_m-1}))|\le s_n(b_1\wedge b_2s_n). \tag{22}$$

We deduce from the last bound, Lemmas 6 and 7, that

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{0\le i_1\le\ldots\le i_{p-2}\le k-1}|\mathbf{E}(X_kX_{k-i_1}\cdots X_{k-i_{p-2}})\mathbf{E}(f(S_{k-i_{p-2}-r_m-1}))|\\ &\le C_{p-1}\left(M_{p-1,n}+s_n^{p-1}\right)s_n(b_1\wedge b_2s_n)\le C_p\left(M_{p,n}(b_1,b_2)+s_n^p(b_1\wedge b_2s_n)\right).\end{aligned} \tag{23}$$

In order to evaluate the first term of the RHS of (21), we use the following lemma whose proof is defered to Appendix B.

Lemma 9 *Let (X_n) be a centered sequence of (SWD) random variables. Suppose that (X_n) is uniformly bounded by 1. Then, for any integer $p\ge 2$, there exists a positive constant C_p such that for any $f\in\mathcal{F}_1(b_1,b_2)$:*

$$\sum_{k=1}^{n}\sum_{m=0}^{p-2}\sum_{r_m=0}^{k-1}\sum_{0=:i_0\le i_1\le\ldots\le i_{p-1}}^{(m),r_m}\sup_{u\in[0,1]}|\mathrm{Cov}(X_k\cdots X_{k-i_m},X_{k-i_{m+1}}\cdots X_{k-i_{p-1}}f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}}))|$$

$$\le C_p\{M_{p,n}(b_1,b_2)+\sum_{m=2}^{p-2}M_{m,n}M_{p-m,n}(b_1,b_2)+s_n^p(b_1\wedge b_2s_n)\}.$$

Now, the first equality in (19) and Taylor's formula yield :

(24)
$$\begin{aligned}&\sum_{0\le i_1\le\ldots\le i_{p-2}\le k-1}^{(m),r_m}|\mathrm{Cov}(X_kX_{k-i_1}\cdots X_{k-i_{p-2}},f(S_{k-i_{p-2}-r_m-1}))|\\ &\qquad\le\sum_{(i)}^{(m),r_m}\sum_{i_{p-1}=i_{p-2}+r_m+1}^{k-1}\sup_{u\in[0,1]}|\mathrm{Cov}(X_k\cdots X_{k-i_{p-2}},X_{k-i_{p-1}}f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}}))|,\end{aligned}$$

where (i) means $0\le i_1\le\ldots\le i_{p-2}$.

Remark 4 The two sums $\sum_{0\le i_1\le\ldots\le i_{p-2}}^{(m),r_m}\sum_{i_{p-1}=i_{p-2}+r_m+1}^{k-1}$ are taken over the subdivisions $0\le i_1\le\cdots\le i_{p-1}\le k-1$ such that $r_m:=i_{m+1}-i_m=\max_{1\le q\le p-2}(i_q-i_{q-1})$ and that $i_{p-1}-i_{p-2}\ge r_m+1$, hence $i_{p-1}-i_{p-2}$ is the big lag of the subdivision $0 \le i_1 \le \cdots \le i_{p-1} \le k-1$. So if we sum some positive quantities, $\sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{0\le i_1 \le \ldots \le i_{p-2}}^{(m),r_m} \sum_{i_{p-1}=i_{p-2}+r_m+1}^{k-1} \le \sum_{m=0}^{p-2} \sum_{r_m=0}^{k-1} \sum_{0\le i_1\le\ldots\le i_{p-2}\le i_{p-1}}^{(m),r_m}$. $\diamond$

We take the sums over $m:\ 0 \le m \le (p-3)$, $r_m:\ 0 \le r_m \le k-1$ and over $k:\ 1 \le k \le n$ in (24), and we use Remark 4 and Lemma 9 to obtain :

$$\sum_{k=1}^{n} \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{0\le i_1 \le \ldots i_{p-2} \le k-1}^{(m),r_m} |\mathrm{Cov}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1}))|$$

$$\le C_p\{M_{p,n}(b_1,b_2) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1,b_2) + s_n^p(b_1 \wedge b_2 s_n)\} \tag{25}$$

We deduce from (21), (23) and (25) that

$$\sum_{k=1}^{n} \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{0\le i_1 \le \ldots i_{p-2} \le k-1}^{(m),r_m} |J_2(i_1,\ldots,i_{p-2})|$$

$$\le C_p\{M_{p,n}(b_1,b_2) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1,b_2) + s_n^p(b_1 \wedge b_2 s_n)\}. \tag{26}$$

Evaluation of $\sum_{i_1\le\ldots\le i_{p-2}} J_1(i_1,\ldots,i_{p-2})$:

Let us note that

$$|\mathbf{E}XY| \le |\mathrm{Cov}(X,Y)| + |\mathbf{E}X||\mathbf{E}Y|.$$

Using decomposition (19) and Remark 4, we then deduce

$$\sum_{0\le i_1\le\ldots\le i_{p-2}\le k-1}^{(m),r_m} |\mathbf{E}(X_k X_{k-i_1} \cdots X_{k-i_{p-2}}[f(S_{k-i_{p-2}-1}) - f(S_{k-i_{p-2}-r_m-1})])|$$

$$\begin{aligned}
\le\ & \sum_{0\le i_1\le\ldots\le i_{p-2}\le k-1}^{(m),r_m} |\mathrm{Cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-2}}[f(S_{k-i_{p-2}-1}) - f(S_{k-i_{p-2}-r_m-1})])| \\
& + \sum_{i_1\le\ldots\le i_{p-2}}^{(m),r_m} |\mathbf{E}(X_k \cdots X_{k-i_m})\mathbf{E}(X_{k-i_{m+1}} \cdots X_{k-i_{p-2}}[f(S_{k-i_{p-2}-1}) - f(S_{k-i_{p-2}-r_m-1})])| \\
\le\ & \sum_{i_1\le\ldots\le i_{p-1}}^{(m),r_m} \sup_{u\in[0,1]} |\mathrm{Cov}(X_k \ldots X_{k-i_m}, X_{k-i_{m+1}} \ldots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}})| \\
& + \sum_{i_1\le\ldots\le i_{p-2}}^{(m),r_m} [|\mathbf{E}(X_k X_{k-i_1} \ldots X_{k-i_m})\mathbf{E}(X_{k-i_{m+1}} \ldots X_{k-i_{p-2}} f(S_{k-i_{p-2}-1})| \\
& + |\mathbf{E}(X_k \ldots X_{k-i_m})\mathrm{Cov}(X_{k-i_{m+1}} \ldots X_{k-i_{p-2}}, f(S_{k-i_{p-2}-r_m-1}))| \\
& + |\mathbf{E}(X_k \ldots X_{k-i_m})\mathbf{E}(X_{k-i_{m+1}} \ldots X_{k-i_{p-2}})\mathbf{E}(f(S_{k-i_{p-2}-r_m-1}))|].
\end{aligned} \tag{27}$$

To give a bound for the sums over k, m and r_m of the RHS of (27), we use Lemma 9 in order to evaluate the first term, and for the other terms we use lemmas 6 and 7 and, respectively the inductive assumption, inequality (25) and inequality (22). We thus obtain :

$$\sum_{k=1}^{n} \sum_{m=0}^{p-3} \sum_{r_m=0}^{k-1} \sum_{0\le i_1 \le\ldots\le i_{p-2} \le k-1}^{(m),r_m} |J_1(i_1,\ldots,i_{p-2})|$$

$$\leq C_p\{\sum_{m=2}^{p-2} M_{m,n}\ M_{p-m,n}(b_1,b_2)+M_{p,n}(b_1,b_2)+s_n^p(b_1\wedge b_2 s_n)\}. \tag{28}$$

Inequalities (20), (26), (28) entail

$$\sum_{k=1}^{n}\sum_{0\leq i_1\leq\ldots\leq i_{p-2}\leq k-1}\left|\mathbf{E}X_kX_{k-i_1}\cdots X_{k-i_{p-2}}f(S_{k-i_{p-2}-1})\right|$$
$$\leq C_p\left\{s_n^p(b_1\wedge b_2 s_n)+\sum_{m=2}^{p-2}M_{m,n}M_{p-m,n}(b_1,b_2)+M_{p,n}(b_1,b_2)\right\}.$$

The last inequality proves that the quantity in (16) is bounded by the RHS of (15).
Let us now give a bound for (17).

Study of the bound of (17):

We obtain, using again the fact $|\mathbf{E}(XY)|\leq|\mathrm{Cov}(X,Y)|+|\mathbf{E}X||\mathbf{E}Y|$, a Taylor expansion, the inductive assumption and Lemma 9 :

$$\sum_{k=1}^{n}\sup_{0\leq u\leq 1}\sum_{i_1\leq\ldots\leq i_{p-2}\leq k-1}\left|\mathbf{E}X_kX_{k-i_1}\cdots X_{k-i_{p-2}}\Delta_{p-2,k}f\right|$$
$$\leq C_p\left\{s_n^p(b_1\wedge b_2 s_n)+\sum_{m=2}^{p-2}M_{m,n}M_{p-m,n}(b_1,b_2)+M_{p,n}(b_1,b_2)\right\}.$$

This last bound proves that the quantity in (17) is bounded by the RHS of (15), for all integer $p\geq 2$. □

6.3 Step 3 : End of the proof of Lemma 3

We are now in position to conclude. Let us first recall that the function $x\to g_p(t,x)$ belongs to the set $\mathcal{F}_p(b_p,b_{p+1})$, with $b_p=t$ and $b_{p+1}=1$.
We note also that if f belongs to the set $\mathcal{F}_p(b_p,b_{p+1})$, then $f^{(p-1)}\in\mathcal{F}_1(b_1,b_2)$ with $b_1=b_p$ and $b_2=b_{p+1}$.

As $p\geq 2$, $(t\wedge 1)\sum_{i=0}^{n-1}(n-i)\theta_i\leq n\sum_{i=0}^{n-1}(t\wedge(i+1))(i+1)^{p-2}\theta_i=M_{p,n}(t,1)$.

Hence we deduce combining Lemmas 5 and 8

$$\mathbf{E}g_p(t,S_n)\leq C_p\left\{\sum_{m=2}^{p-2}M_{m,n}M_{p-m,n}(t,1)+M_{p,n}(t,1)+s_n^p(t\wedge s_n)\right\}. \tag{29}$$

Let $\Phi_p\in\mathbf{\Phi_p}$. Taking into account Lemma 4 and the fact $g^{(p)}(x)=x\wedge t$, we deduce that

$$\phi_p^{(p)}(x)=\int_0^{+\infty}(t\wedge x)\nu_p(dt). \tag{30}$$

Inequalities (11), (29) and (30) yield :

$$\begin{aligned}\mathbf{E}\phi_p(|S_n|)\quad&\leq\quad C_p\left\{\sum_{m=2}^{p-2}M_{m,n}\left(n\sum_{i=0}^{n-1}\phi_p^{(p)}(i+1)(i+1)^{p-m-2}\theta_i\right)\right.\\ &\left.\quad+\quad n\sum_{i=0}^{n-1}\phi_p^{(p)}(i+1)(i+1)^{p-2}\theta_i+s_n^p\phi_p^{(p)}(s_n)\right\}. \end{aligned}\tag{31}$$

Now, using the concavity of $\phi_p^{(p)}$ yields :

$$\phi_p(x) = \frac{x^p}{(p-1)!}\int_0^1 (1-t)^{p-1}\phi_p^{(p)}(tx)dt \geq x^p\phi_p^{(p)}(x)\int_0^1 \frac{t(1-t)^{p-1}}{(p-1)!}dt.$$

Hence we deduce that

$$x^p\phi_p^{(p)}(x) \leq C_p\phi_p(x). \tag{32}$$

We conclude, combining inequalities (31) and (32) :

$$\begin{aligned}\mathbf{E}\phi_p(|S_n|) \quad &\leq \quad C_p\Big\{\textstyle\sum_{m=2}^{p-2} M_{m,n}\left(n\sum_{i=0}^{n-1}\phi_p(i+1)(i+1)^{-m-2}\theta_i\right) \\ &\quad + n\textstyle\sum_{i=0}^{n-1}\phi_p(i+1)(i+1)^{-2}\theta_i + \phi_p(s_n)\Big\}\end{aligned}$$

Finally, for the suitable choice of the function $\phi_p : \phi_p(x) = x^r$, for $r \in]p, p+1]$, and using inequality (12) in Lemma 7, we finish the proof of Lemma 3.

7 Appendix A

This appendix is devoted to the proofs of Theorem 2 and Lemma 2.

Theorem 6 in Doukhan & Louhichi (1999) proves the fi-di convergence in another weak dependence frame. Following their approach but replacing their dependence conditions by (SWD) sequences, we get the fi-di convergence as soon as (X_i) is (SWD) with $\theta_r = \mathcal{O}\left(r^{-a}\right)$, $a \geq 4$. This is enough to yield the Empirical Functional Central Limit Theorem (Theorem 1). However, we prove here a general Central Limit Theorem (Theorem 2 stated in section 2) under weak dependence and we apply it to prove the fi-di convergence. It allows us to weaken assumption required for the fi-di convergence to $\theta_r = \mathcal{O}\left(r^{-a}\right)$ with $a > 3$ (Lemma 2). To prove Theorem 2 we do not use Bernstein's blocks as in Doukhan & Louhichi (1999) but a variation on the Lindeberg-Rio method (Rio, 1996).

Proof of Theorem 2: In the following, if $x \in \mathbb{R}_+$, $[x]$ denotes the integer part of x.

We define $S_n = X_0 + \cdots + X_{n-1}$, and $S_k = X_0 + \cdots + X_{k-1}$. Let $M := \sup_{n\in\mathbb{N}} ||X_n||_\infty$.

Let $v_k = \mathrm{Var}\,(S_k) - \mathrm{Var}\,(S_{k-1})$. $(S_0 = 0)$.

By assumption, $\sigma_n^2 = \frac{\mathrm{Var}\,(S_n)}{n} \underset{n\infty}{\longrightarrow} \sigma^2 > 0$. Moreover $v_k = \sum_{i=-(k-1)}^{+(k-1)} \mathrm{Cov}\left(X_0, X_{|i|}\right)$.

Hence, as $(X_n)_{n\in\mathbb{N}}$ is (SWD) and uniformly bounded by M, $\left|\mathrm{Cov}\left(X_0, X_{|i|}\right)\right| \leq M\ \theta_{|i|}$ and $v_k \underset{k\infty}{\longrightarrow} \sum_{i=-\infty}^{+\infty}\mathrm{Cov}\left(X_0, X_{|i|}\right) < \infty$. Then Césaro's Theorem yields $\sigma^2 := \lim_{n\infty}\frac{\mathrm{Var}\,(S_n)}{n} = \lim_{k\infty} v_k$. Hence there exists some positive integer k_0 such that

$$k \geq k_0 \Rightarrow v_k \geq \frac{\sigma^2}{2} > 0. \tag{33}$$

Set now $Y_k \sim \mathcal{N}(0, v_{k+1})$, $k \geq k_0 - 1$. The sequence $(Y_k)_{k\geq k_0-1}$ is assumed to be independent and independent of the sequence $(X_k)_{k\in\mathbb{N}}$. We also set, for $k \geq k_0 - 2$, $T_{k,n} = \sum_{j=k+1}^{n-1} Y_j$; empty sums are, as usual, set equal to 0.

Let $\eta \sim \mathcal{N}\left(0, 1\right)$, and $h_{k,n}(x) = \mathbf{E}h(x + T_{k,n}) := \mathbf{E}1_{y\geq x+T_{k,n}}$, where $y \in \mathbb{R}$. In the following, C will denote some arbitrary constant which may vary from line to line but which is independent from y, k, n. We are in position to use Rio's decomposition. We define

$$\Delta_n(h_{k,n}) = \sum_{k=k_0-1}^{\left[\frac{n}{3}\right]} \Delta_{k,n}(h_{k,n}),\ \text{where}\ \Delta_{k,n}(h_{k,n}) = \mathbf{E}\left(h_{k,n}(S_k + X_k) - h_{k,n}(S_k + Y_k)\right). \tag{34}$$

Hence

$$\Delta_n(h_{k,n}) = \mathbf{E}\left(h(S_{[\frac{n}{3}]+1} + T_{[\frac{n}{3}],n}) - h(S_{k_0-1} + T_{k_0-2,n})\right), \text{ where } h(x) := 1_{y\geq x}. \tag{35}$$

Assume that there exists $\alpha_n \xrightarrow[n\to\infty]{} 0$ such that for all y,

$$|\Delta_n(h_{k,n})| \leq \alpha_n. \tag{36}$$

Setting $y = \sqrt{n}\, z$, we get for all z,

$$\left| P\left(\frac{S_{[\frac{n}{3}]+1} + T_{[\frac{n}{3}],n}}{\sqrt{n}} \leq z\right) - P\left(\frac{S_{k_0-1} + T_{k_0-2,n}}{\sqrt{n}} \leq z\right)\right| \leq \alpha_n. \tag{37}$$

Then, as $\frac{S_{k_0-1}+T_{k_0-2,n}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0,\sigma^2\right)$,

$$\frac{S_{[\frac{n}{3}]+1}}{\sqrt{n}} + \frac{Y_{[\frac{n}{3}]+1} + \cdots + Y_{n-1}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0,\sigma^2\right). \tag{38}$$

Moreover

$$\frac{Y_{[\frac{n}{3}]+1} + \cdots + Y_{n-1}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0,\frac{2}{3}\sigma^2\right). \tag{39}$$

(38) and (39) yield

$$\frac{S_{[\frac{n}{3}]+1}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0,\frac{\sigma^2}{3}\right). \tag{40}$$

Finally (40) yields

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0,\sigma^2\right). \tag{41}$$

Let us now prove (36). To bound the terms $\Delta_{k,n}(h_{k,n})$ one writes

$\Delta_{k,n}(h_{k,n}) = \Delta^{(1)}_{k,n}(h_{k,n}) - \Delta^{(2)}_{k,n}(h_{k,n})$, with

$$\Delta^{(1)}_{k,n}(h) = \mathbf{E}h(S_{k+1} + T_{k,n}) - \mathbf{E}h(S_k + T_{k,n}) - \frac{v_{k+1}}{2}\mathbf{E}h''(S_k + T_{k,n}), \tag{42}$$

$$\Delta^{(2)}_{k,n}(h) = \mathbf{E}h(S_k + Y_k + T_{k,n}) - \mathbf{E}h(S_k + T_{k,n}) - \frac{v_{k+1}}{2}\mathbf{E}h''(S_k + T_{k,n}). \tag{43}$$

<u>Study of $\Delta^{(2)}_{k,n}(h)$:</u> Using a Taylor's decomposition we get

$$\begin{aligned}\Delta^{(2)}_{k,n}(h_{k,n}) \quad = \quad & \mathbf{E}\left\{h'_{k,n}\left(S_k\right)Y_k\right\} \\ & +\mathbf{E}\left\{\tfrac{1}{2}h''_{k,n}\left(S_k\right)\left(Y_k^2 - v_{k+1}\right) + \tfrac{1}{6}h^{(3)}_{k,n}\left(S_k + \rho_{k,n}Y_k\right)Y_k^3\right\},\end{aligned}$$

where $0 < \rho_{k,n} < 1$. Using the independence of the process $(Y_n)_{n\in\mathbb{N}}$ and the process $(X_n)_{n\in\mathbb{N}}$, one deduces that,

$$\left|\Delta^{(2)}_{k,n}(h_{k,n})\right| = \left|\frac{1}{6}\mathbf{E}\left\{h^{(3)}_{k,n}\left(S_k + \rho_{k,n}Y_k\right)Y_k^3\right\}\right| \leq ||h^{(3)}_{k,n}||_\infty\, \frac{2v_{k+1}^{3/2}}{3\sqrt{2\pi}} \leq ||h^{(3)}_{k,n}||_\infty\, C\, \left(\sum_{i=0}^{k}\theta_i\right)^{\frac{3}{2}}. \tag{44}$$

Study of $\Delta^{(1)}_{k,n}(h)$: Set $\Delta^{(1)}_{k,n}(h_{k,n}) = \mathbf{E}\delta^{(1)}_{k,n}(h_{k,n})$, then, from Taylor's formula again (with some random $\tau_{k,n} \in]0,1[$), we write
$\delta^{(1)}_{k,n}(h_{k,n}) = X_k h^{'}_{k,n}(S_k) + \frac{1}{2} h^{"}_{k,n}(S_k)(X_k^2 - v_{k+1}) + \frac{1}{6}\left(h^{(3)}_{k,n}(S_k + \tau_{k,n}X_k)X_k^3\right)$.
We analyze separatedely the terms in the previous expression

$$\frac{1}{6}|\mathbf{E}h^{(3)}_{k,n}(S_k+\tau_{k,n}X_k)X_k^3| \le M^3\,\frac{||h^{(3)}_{k,n}||_\infty}{6} \le C\,||h^{(3)}_{k,n}||_\infty. \tag{45}$$

We then write

$$\mathbf{E}\left\{X_k h^{'}_{k,n}(S_k) + \frac{1}{2}h^{"}_{k,n}(S_k)(X_k^2 - v_{k+1})\right\}$$

$$= \mathbf{E}\left\{X_k h'_{k,n}(S_k)\right\} + \frac{1}{2}\mathrm{Cov}\left(h''_{k,n}(S_k), X_k^2\right) - \left(\mathbf{E}h''_{k,n}(S_k)\right)\sum_{i=1}^{k}\mathbf{E}\left(X_0X_i\right). \tag{46}$$

We set $X_p = 0$ for all $p < 0$ and $S_0 = 0$. Using $g(S_i) - g(0) = \sum_{j=1}^{i}\left(g(S_j) - g(S_{j-1})\right)$, a Taylor's decomposition and the independence properties of the sequence $(Y_k)_{k\ge k_0-1}$ we get

$$\left|\mathrm{Cov}\left(h''_{k,n}\left(S_k\right), X_k^2\right)\right| \le C\,||h^{(3)}_{k,n}||_\infty \sum_{i=1}^{k}\theta_i \tag{47}$$

and

$$X_k h'_{k,n}(S_k) = X_k \sum_{i=1}^{k}\left\{X_{i-1}h''_{k,n}(S_{i-1}) + \frac{X_{i-1}^2}{2}h^{(3)}_{k,n}(S_{i-1}+\nu_{k,n,i}X_{i-1})\right\}, \tag{48}$$

where $0 < \nu_{k,n,i} < 1$.

Moreover using the weak dependence of $(X_n)_{n\in\mathbb{N}}$,

$$\left|\mathbf{E}X_k\frac{X_{i-1}^2}{2}h^{(3)}_{k,n}\left(S_{i-1}+\nu_{k,n}X_{i-1}\right)\right| \le C\,||h^{(3)}_{k,n}||_\infty\theta_{k-i+1}. \tag{49}$$

Let now $j := \sup(0, 2i-k)$. We write

$$\begin{aligned}\mathbf{E}X_kX_{i-1}h''_{k,n}(S_{i-1}) &= \mathbf{E}\left\{h''_{k,n}(S_{j-1})X_kX_{i-1}\right\}\\ &\quad+\mathbf{E}\left\{\left(h''_{k,n}(S_{i-1}) - h''_{k,n}(S_{j-1})\right)X_kX_{i-1}\right\}.\end{aligned}$$

We also have, using a Taylor's decomposition

$$|h''_{k,n}(S_{i-1}) - h''_{k,n}(S_{j-1})| \le C\,||h^{(3)}_{k,n}||_\infty(k-i).$$

Hence,

$$\begin{aligned}\left|\mathbf{E}\left\{\left(h''_{k,n}(S_{i-1}) - h''_{k,n}(S_{j-1})\right)X_kX_{i-1}\right\}\right| &\le C\;\;||h^{(3)}_{k,n}||_\infty\;\;(\,k-i\,)\,\theta_{k-i+1}\\ &\le C\,||h^{(3)}_{k,n}||_\infty(k-i+1)\theta_{k-i+1}.\end{aligned} \tag{50}$$

Then, still using a Taylor's decomposition

$$\left|\mathrm{Cov}\left(h''_{k,n}(S_{j-1}), X_kX_{i-1}\right)\right| \le C||h^{(3)}_{k,n}||_\infty\sum_{l=1}^{j-1}\theta_{i-l}. \tag{51}$$

Then, using inequalities (48), (49), (50), (51) and $\sum_{i=1}^{k}\sum_{l=1}^{j-1}\theta_{i-l} \leq \sum_{i=1}^{k-1} i\theta_i$,

$$\left|\mathbf{E}\left(X_k h'_{k,n}(S_k)\right) - \sum_{i=1}^{k}\mathbf{E}\left(h''_{k,n}(S_{j-1}\right)\mathbf{E}\left(X_k X_{i-1}\right)\right| \leq C||h^{(3)}_{k,n}||_\infty \sum_{p=1}^{k} p\,\theta_p. \tag{52}$$

It remains to bound

$$\left\{\sum_{i=1}^{k}\mathbf{E}h''_{k,n}(S_{j-1})\mathbf{E}\left(X_k X_{i-1}\right)\right\} - \left\{\mathbf{E}h''_{k,n}(S_k)\sum_{i=1}^{k}\mathbf{E}\left(X_0 X_i\right)\right\}.$$

We can rewrite it

$$\sum_{i=1}^{k}\mathbf{E}\left(h''_{k,n}(S_{j-1}) - h''_{k,n}(S_k)\right)\mathbf{E}\left(X_k X_{i-1}\right).$$

Using a Taylor's decomposition and as $(k-j+1) \leq 2\,(k-i+1)$,

$$\left|\sum_{i=1}^{k}\mathbf{E}\left(h''_{k,n}(S_{j-1}) - h''_{k,n}(S_k)\right)\mathbf{E}\left(X_k X_{i-1}\right)\right| \leq C||h^{(3)}_{k,n}||_\infty \sum_{i=1}^{k} i\,\theta_i. \tag{53}$$

Then, summing inequalities (45), (47), (52) and (53),

$$\left|\Delta^{(1)}_{k,n}(h_{k,n})\right| \leq C\ ||h^{(3)}_{k,n}||_\infty \left(1+\sum_{i=1}^{k}\theta_i + \sum_{i=1}^{k} i\,\theta_i\right) \leq C\ ||h^{(3)}_{k,n}||_\infty \left(1+\sum_{i=1}^{k} i\,\theta_i\right). \tag{54}$$

Hence using inequalities (44) and (54),

$$\left|\sum_{k=k_0-1}^{[\frac{n}{3}]}\Delta_{k,n}(h_{k,n})\right| \leq C\ \left(\left[\frac{n}{3}\right] - k_0 + 2\right) \sup_{k_0-1\leq k\leq[\frac{n}{3}]} ||h^{(3)}_{k,n}||_\infty \left(1+\sum_{i=1}^{[\frac{n}{3}]} i\,\theta_i\right). \tag{55}$$

It remains to bound $\sup_{k_0-1\leq k\leq[\frac{n}{3}]}||h^{(3)}_{k,n}||_\infty$.

If $\Phi_1(z) := \frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}$, we can prove with (33) that $\sup_{k_0-1\leq k\leq[\frac{n}{3}]}||h^{(3)}_{k,n}||_\infty \leq C\ n^{\frac{-3}{2}}\sigma^{-3}||\Phi''_1||_\infty$, where C is independent of y (see also Rio, 1996).

(56) Hence $$\left|\sum_{k=k_0-1}^{[\frac{n}{3}]}\Delta_{k,n}(h_{k,n})\right| \leq C\ n^{\frac{-1}{2}}\sigma^{-3}||\Phi''_1||_\infty \left(1+\sum_{i=1}^{n} i\,\theta_i\right).$$

Let denote α_n the right hand side of (56). α_n does not depend on y and it tends to 0 as soon as $a > 3/2$. Hence, by (41), this concludes the proof of Theorem 2. □

We are now in position to prove the fi-di convergence. We prove Lemma 2 for one-dimensional marginals. If one wants to prove it for any finidimensional marginals, it is sufficient to use the following proof with $Z_k(t_1,\ldots,t_r) = \sum_{j=1}^{r}\alpha_j\left(1_{X_k\leq t_j} - F(t_j)\right)$, where $(t_1,\ldots,t_r) \in \mathbb{R}^r$ and for arbitrary numbers $\alpha_1,\ldots,\alpha_r$. To prove the convergence of $(Z_n(t))_{n\in\mathbb{N}} := \left(1_{X_n\leq t} - F(t)\right)_{n\in\mathbb{N}}$ for some $t\in\mathbb{R}$, we apply Theorem 2 to the sequence $(Z_n(t))_{n\in\mathbb{N}}$. We first need the following proposition:

Proposition 1 *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary sequence with common distribution function F supposed to be Lipschitz. Suppose that $(X_n)_{n\in\mathbb{N}}$ satisfies the $(SWD)-$weak dependence condition, with a sequence $(\theta_r)_{r\in\mathbb{N}}$. Then the sequence $(Z_n(t))_{n\in\mathbb{N}} := \left(1_{X_n\le t} - F(t)\right)_{n\in\mathbb{N}}$, which is uniformly bounded, satisfies the $(SWD)-$weak dependence condition, with a sequence $(\theta'_r)_{r\in\mathbb{N}}$ such that there exists some positive constant D satisfying $\theta'_r \le D\sqrt{\theta_r}$.*

Proof of Proposition 1: In the following, C will denote some arbitrary constant which may vary from line to line but which is independent from u, v, r. Let $t \in \mathbb{R}$. We want to prove that $(Z_n(t))_{n\in\mathbb{N}}$ satisfies a $(SWD)-$dependence condition. For this we have to bound $C_{h,u,v} := \mathrm{Cov}\left(h(Z_{i_1},\dots,Z_{i_u}), k\left(Z_{j_1},\dots,Z_{j_v}\right)\right)$ for all $i_1 \le \dots \le i_u \le i_u + r \le j_1 \le \dots \le j_v$ and for all $h \in \mathbb{L}^\infty$, $k \in \mathcal{L}$. We write Z_i for $Z_i(t)$. Let us write $Z_n = \varphi(X_n)$. We want to smooth the function φ which is not Lipschitz. Let $\varepsilon > 0$. We consider the following Lipschitz function φ^ϵ smoothing φ :

$$\begin{array}{ll} x \in [0, t-\varepsilon]\cup]t+\varepsilon, 1] & \varphi_\varepsilon(x) = \varphi(x), \\ t-\varepsilon < x \le t+\varepsilon & \varphi_\varepsilon(x) = \frac{1}{4\varepsilon^3}\left(x^3 - 3tx^2 + 3(t^2-\varepsilon^2)x - t^3 + 3t\varepsilon^2 + 2\varepsilon^3\right) - F(t). \end{array}$$

We have $||\varphi^\varepsilon||_\infty \le ||\varphi||_\infty \le 1$ and $\mathrm{Lip}(\varphi^\varepsilon) \le \frac{3}{4\,\varepsilon}$. Hence, arguing as for Lemma 3,

$$\begin{aligned} |C_{h,u,v}| &\le \left|\mathrm{Cov}\left(h(Z_{i_1},\dots,Z_{i_u}), k\left(\varphi(X_{j_1}),\dots,\varphi(X_{j_v})\right) - k\left(\varphi^\varepsilon(X_{j_1}),\dots,\varphi^\varepsilon(X_{j_v})\right)\right)\right| \\ &\quad + \left|\mathrm{Cov}\left(h(Z_{i_1},\dots,Z_{i_u}), k\left(\varphi^\varepsilon(X_{j_1}),\dots,\varphi^\varepsilon(X_{j_v})\right)\right)\right| \\ &\le C\,||h||_\infty \mathrm{Lip}(k) v\left(\varepsilon + \tfrac{\theta_r}{\varepsilon}\right). \end{aligned}$$

Then if we take $\varepsilon = \sqrt{\theta_r}$, $|C_{h,u,v}| \le C\;||h||_\infty\mathrm{Lip}(k)\;v\;\sqrt{\theta_r}$. So $(Y_n)_{n\in\mathbb{N}}$ is (SWD) with $\theta'_r \le C\;\sqrt{\theta_r}$. This concludes the proof of Proposition 1. □

Now, let $S_k(t) = \sum_{i=0}^{k-1} Z_i(t)$ for $1 \le i \le n$, $S_0(t) = 0$. Let us prove that $\frac{\mathrm{Var}\,(S_n(t))}{n} \underset{n\infty}{\longrightarrow} \sigma^2 \ge 0$.

Let $v_k(t) = \mathrm{Var}\;(S_k(t)) - \mathrm{Var}\;(S_{k-1}(t))$. As soon as $\theta_r = \mathcal{O}\left((r+1)^{-2-\delta}\right)$, $\delta > 0$, we have $v_k(t) \underset{k\infty}{\longrightarrow} \sigma^2(t) = \sum_{i=-\infty}^{+\infty} \mathrm{Cov}\left(Z_0(t), Z_{|i|}(t)\right) < \infty$. Césaro's Theorem yields then $\frac{\mathrm{Var}\,(S_n(t))}{n} = \frac{\sum_{k=1}^n v_k(t)}{n} \underset{n\infty}{\longrightarrow} \sigma^2(t)$.

If $\sigma^2(t) = 0$, the convergence of $\frac{S_n(t)}{n}$ is a simple fact. If $\sigma^2(t) > 0$, thanks to Proposition 1, we are in position to apply Theorem 2 to $(Z_n(t))_{n\in\mathbb{N}}$ to obtain the fi-di convergence as soon as $\theta_r = \mathcal{O}\left((r+1)^{-a}\right)$ with $a > 3$. This concludes the proof of Lemma 2.

8 Appendix B

Proof of Lemma 5 By induction on $p \ge 2$, and using decomposition (19), we get for any $f \in \mathcal{C}_{p-1}$

$$|\mathbf{E}(f(S_n))| \le C_p \sum_{k=1}^n \left[\mathbf{E}_{p-2,k}(f^{(p-1)}) + \mathbf{E}_{p-2,k}(\Delta f^{(p-1)})\right] + \sum_{k=1}^n \mathbb{R}_{p,k}(f), \tag{57}$$

where $\mathbb{R}_{p,k}(f) := \left|\mathbf{E}\left[f(S_k) - f(S_{k-1}) - \dots - \frac{X_k{}^{p-1}}{(p-1)!} f^{(p-1)}(S_{k-1})\right]\right|$.

Let now f be a fixed function of the set $\mathcal{F}_p(b_p, b_{p+1})$.
In view of (57), Lemma 5 is proved if we suitably evaluate $\mathbb{R}_{p,k}(f)$. Clearly

$$\mathbb{R}_{p,k}(f) \le \mathbb{R}_{p+1,k}(f) + |\mathrm{Cov}(X_k^p, f^{(p)}(S_{k-1}))| + \mathbf{E}(|X_k^p|)\mathbf{E}|f^{(p)}|(S_{k-1}) =: I_{1,k} + I_{2,k} + I_{3,k}.$$

Taylor's formula and the (SWD) property yield :

$$\begin{aligned} I_{1,k} &\le C_p \mathbf{E}\left(|X_k|^p (b_p \wedge b_{p+1}|X_k|)\right), \\ I_{2,k} &\le C_p \left[b_p \mathbf{E}|X_k|^2 \wedge b_{p+1} \textstyle\sum_{i=0}^{k-1} \theta_i \right] \le C_p (b_p \wedge b_{p+1}) \textstyle\sum_{i=0}^{k-1} \theta_i. \end{aligned}$$

$$\begin{aligned} \text{Finally } I_{3,k} &\le C_p\ \mathbf{E}|X_k^p|(b_p \wedge b_{p+1}s_n) 1_{|X_k|\le s_n} + C_p\ \mathbf{E}|X_k^p|(b_p \wedge b_{p+1}s_n)1_{|X_k|>s_n} \\ &\le C_p\ s_n^{p-2}(b_p \wedge b_{p+1}s_n)\mathbf{E}(X_k^2) + C_p\ \mathbf{E}\left(|X_k|^p(b_p \wedge b_{p+1}|X_k|)\right). \end{aligned} \tag{58}$$

The proof of Lemma 5 is complete by noting that, as $(X_n)_{n\in\mathbb{N}}$ is bounded by 1, $\sum_{k=1}^n \mathbf{E}\left(|X_k|^p(b_p \wedge b_{p+1}|X_k|)\right) \le (b_p \wedge b_{p+1}) n\theta_0$, and $\sum_{k=1}^n \mathbf{E}(X_k^2) \le s_n^2$. □

Proof of Lemma 7 For all positive integer m, and all positive real numbers α and β that are conjugate (i.e $1/\alpha + 1/\beta = 1$) there holds, thanks to Hölder inequality

$$\sum_{i=0}^{n-1} (i+1)^{m-2}\theta_i \le \left(\sum_{i=0}^{n-1} \theta_i\right)^{1/\alpha} \left(\sum_{i=0}^{n-1} (i+1)^{(m-2)\beta}\theta_i\right)^{1/\beta}.$$

Thus $M_{m,n} \le s_n^{2/\alpha} \left(M_{p,n}\right)^{1/\beta}$. Hence, for $\alpha = \frac{p-2}{p-m}$ and $\beta = \frac{p-2}{m-2}$, $M_{m,n} \le s_n^{2(p-m)/(p-2)} M_{p,n}^{(m-2)/(p-2)}$.

As we also have $M_{p-m,n} \le s_n^{2m/(p-2)} M_{p,n}^{(p-m-2)/(p-2)}$, we get the first inequality in (12). Then, bounding $M_{p,n}$ and s_n^p by the sum $M_{p,n} + s_n^p$, we get the second inequality of (12).
Now, we only prove (13) ((14) will hold with the same method). Clearly

$$\begin{aligned} s_n^{p-m}(b_1 \wedge b_2 s_n)\left(n \textstyle\sum_{(i+1)\le(s_n\wedge n)} (i+1)^{m-2}\theta_i\right) &\le s_n^{p-m}(b_1 \wedge b_2 s_n) s_n^{m-2}\left(n \textstyle\sum_{i=0}^{n-1} \theta_i\right) \\ &\le s_n^p (b_1 \wedge b_2 s_n) \end{aligned} \tag{59}$$

$$n \sum_{i:\ (i+1)\ge s_n}^{n-1} \left(s_n^{p-m}(b_1 \wedge b_2 s_n)(i+1)^{m-2}\theta_i\right) \le M_{p,n}. \tag{60}$$

Collecting inequalities (59) and (60) we obtain (13). □

Proof of Lemma 9 We write

$$f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}}) = [f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}}) - f'(S_{k-i_{p-1}-1})] + f'(S_{k-i_{p-1}-1})$$

so the covariance quantity of Lemma 9 is decomposed into two terms :

$$|\mathrm{Cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1}))| \tag{61}$$

$$|\mathrm{Cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} [f'(S_{k-i_{p-1}-1} + uX_{k-i_{p-1}}) - f'(S_{k-i_{p-1}-1})])|. \tag{62}$$

Step 1 : Study of the terms of type (61)

We use the following decomposition:

$$\begin{aligned} &|\mathrm{Cov}(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-1}))| \le \\ &\left|\mathrm{Cov}\left(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} [f'(S_{k-i_{p-1}-1}) - f'(S_{k-i_{p-1}-r_m-1})]\right)\right| \\ &+ \left|\mathrm{Cov}\left(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-r_m-1})\right)\right|. \end{aligned}$$

a) The (SWD) property yields

$$|J(k, r_m, i_1, \ldots, i_{p-1})| :=$$

$$\left|\mathrm{Cov}\left(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}}[f'(S_{k-i_{p-1}-1}) - f'(S_{k-i_{p-1}-r_m-1})]\right)\right|$$

$$\leq C_p\ (b_1 \wedge b_2 r_m)\ \theta_{r_m},\ \text{where } r_m = i_{m+1} - i_m = \max_{1\leq q\leq p-2}(i_q - i_{q-1}).$$

$$\text{Thus} \quad \sum_{r_m=0}^{k-1} \sum_{0=:i_0\leq i_1\leq\ldots\leq i_{p-1}}^{(m),r_m} |J(k, r_m, i_1, \ldots, i_{p-1})| \leq \sum_{l=0}^{k-1} C_p\ (b_1 \wedge b_2 l)\ l^{p-2}\ \theta_l.$$

Summing up over $m \in \{0, \ldots, p-2\}$, $k \in \{0, \ldots, n\}$ in the last inequality yields

$$\sum_{k=1}^{n} \sum_{m=0}^{p-2} \sum_{r_m=0}^{k-1} \sum_{0\leq(i)\leq k-1}^{(m),r_m} |J(k, r_m, i_1, \ldots, i_{p-1})|$$

$$\leq \sum_{k=1}^{n} \sum_{l=0}^{k-1} C_p\ l^{p-2}\ (b_1 \wedge b_2(l+1))\ \theta_l \leq C_p\ M_{p,n}(b_1, b_2), \tag{63}$$

where (i) denotes $0 =: i_0 \leq i_1 \leq \ldots \leq i_{p-1} \leq k-1$.

b) We now use the fact

$\mathrm{Cov}(X, YZ) = \mathrm{Cov}(XY, Z) + \mathbf{E}(XY)\mathbf{E}(Z) - \mathbf{E}(X)\mathrm{Cov}(Y, Z) - \mathbf{E}(X)\mathbf{E}(Y)\mathbf{E}(Z)$, to get

$$\left|\mathrm{Cov}\left(X_k \cdots X_{k-i_m}, X_{k-i_{m+1}} \cdots X_{k-i_{p-1}} f'(S_{k-i_{p-1}-r_m-1})\right)\right|$$

$$\leq \left|\mathrm{Cov}\left(X_k \cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)\right|$$

$$+ \left|\mathbf{E}(X_k \cdots X_{k-i_{p-1}})\mathbf{E}(f'(S_{k-i_{p-1}-r_m-1}))\right|$$

$$+ \left|\mathbf{E}(X_k \cdots X_{k-i_m})\mathrm{Cov}(X_{k-i_{m+1}} \cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1}))\right|$$

$$+ \left|\mathbf{E}(X_k \cdots X_{k-i_m})\mathbf{E}(X_{k-i_{m+1}} \cdots X_{k-i_{p-1}})\mathbf{E}(f'(S_{k-i_{p-1}-r_m-1}))\right|$$

$$=: I_1(i_1, \ldots, i_{p-1}) + I_2(i_1, \ldots, i_{p-1}) + I_3(i_1, \ldots, i_{p-1}) + I_4(i_1, \ldots, i_{p-1}).$$

Using Lemmas 6, 7 and arguing as for inequality (22) yield

$$\sum_{k=1}^{n} \sum_{0=:i_0\leq i_1\leq\ldots\leq i_{p-1}\leq k-1} I_2(i_1, \ldots, i_{p-1}) + \sum_{k=1}^{n} \sum_{0=:i_0\leq i_1\leq\ldots\leq i_{p-1}\leq k-1} I_4(i_1, \ldots, i_{p-1})$$

$$\leq C_p\{s_n^p(b_1 \wedge b_2 s_n) + M_{p,n}(b_1, b_2) + \sum_{m=2}^{p-2} M_{m,n} M_{p-m,n}(b_1, b_2)\}. \tag{64}$$

Let us now study $I_1(i_1, \ldots, i_{p-1})$.

On one hand we have, using the SWD-property:

$$|\mathrm{Cov}\left(X_k \cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)| \leq C_p\ b_1\ \theta_{r_m}. \tag{65}$$

On the other hand, using Taylor's decomposition, we get:

$$|\mathrm{Cov}\left(X_k\cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)| \tag{66}$$

$$= |\sum_{i_p=i_{p-1}+r_m+1}^{k-1}\mathrm{Cov}(X_k\cdots X_{k-i_{p-1}}, f'(S_{k-i_p})-f'(S_{k-i_p-1}))| \leq \sum_{l=r_m+1}^{k-1} C_p\,\theta_l\, b_2.$$

Moreover we write:

$$\begin{aligned}|\mathrm{Cov}\left(X_k\cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)| &= |\mathrm{Cov}\left(X_k\cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)|1_{b_1\leq r_m b_2}\\ &\quad +|\mathrm{Cov}\left(X_k\cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)|1_{b_1> r_m b_2}.\end{aligned}$$

Inequalities (65) and (66) yield

$$\sum_{k=1}^{n}\sum_{m=0}^{p-2}\sum_{r_m=0}^{k-1}\sum_{(i)}^{(m),r_m} |\mathrm{Cov}\left(X_k\cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)|1_{b_1\leq r_m b_2}\leq C_p\, M_{p,n}(b_1,b_2).$$

If now we note that: $\sum_{r_m=0,r_m<\frac{b_1}{b_2}}^{k-1}\sum_{l=r_m+1}^{k-1} l^{p-2}\ \theta_l \leq \sum_{l=1}^{k-1}\ \sum_{r_m=0}^{l\wedge(b_1/b_2)} l^{p-2}\ \theta_l$, using once more inequality (66) we also get

$$\sum_{k=1}^{n}\sum_{m=0}^{p-2}\sum_{r_m=0}^{k-1}\sum_{(i)}^{(m),r_m} |\mathrm{Cov}\left(X_k\cdots X_{k-i_{p-1}}, f'(S_{k-i_{p-1}-r_m-1})\right)|1_{b_1> r_m b_2}\leq C_p\, M_{p,n}(b_1,b_2).$$

So,

$$\sum_{k=1}^{n}\sum_{i_1\leq\ldots\leq i_{p-1}\leq k-1} I_1(i_1,\ldots,i_{p-1})\leq C_p\, M_{p,n}(b_1,b_2). \tag{67}$$

To bound $I_3(i_1,\ldots,i_{p-1})$, we use the (SWD) property, as for the bound of $I_1(i_1,\ldots,i_{p-1})$, and Lemmas 6 and 7. We obtain

$$\sum_{k=1}^{n}\sum_{(i)} I_3(i_1,\ldots,i_{p-1})\leq C_p\{M_{p,n}(b_1,b_2)+s_n^p(b_1\wedge b_2 s_n)+\sum_{m=2}^{p-2}M_{m,n}M_{p-m,n}(b_1,b_2)\},$$

which provides a bound for the sum of the terms of type (61).

Step 2 : Study of the terms of type (62)

With Taylor,

$$|\mathrm{Cov}\left(X_k\cdots X_{k-i_m}, X_{k-i_{m+1}}\cdots X_{k-i_{p-1}}[f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}})-f'(S_{k-i_{p-1}-1})]\right)|$$

$$\leq C_p(b_1\wedge b_2)\theta_{r_m}.$$

So, if (k) denotes $1\leq k\leq n$ and if (i) denotes $i_1\leq\ldots\leq i_{p-1}\leq k-1$,

$$\begin{aligned}&\sum_{(k),(i)}|\mathrm{Cov}(X_k\cdots X_{k-i_m}, X_{k-i_{m+1}}\cdots X_{k-i_{p-1}}[f'(S_{k-i_{p-1}-1}+uX_{k-i_{p-1}})-f'(S_{k-i_{p-1}-1})])|\\ &\qquad\leq n\, C_p\sum_{i=0}^{n-1} i^{p-2}(b_1\wedge b_2(i+1))\theta_i\leq C_p\, M_{p,n}(b_1\wedge b_2).\end{aligned} \tag{68}$$

Inequalities (63), (64), (67) and (68) yield the proof of Lemma 9. □

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