A new approach to topological ligaments in shape optimization

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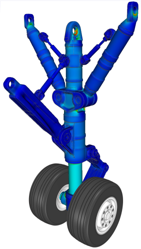
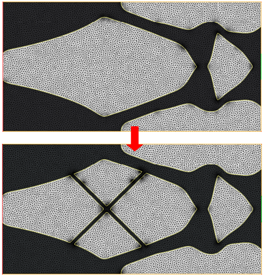
Foreword

- Shape optimization aims at finding the "best" design of a physical device Ω with respect to a measure of performance $J(\Omega)$.
- This discipline has raised a tremendous enthusiasm in the academic and industrial communities.
- Most numerical algorithms rely on a notion of "derivative" for the mapping $\Omega \mapsto J(\Omega)$...
- ... which, in turn, calls for a definition of "small variations" of a given shape Ω.
- We introduce a method to appraise the sensitivity of $J(\Omega)$ with respect to the graft of a thin bar to Ω.
- This task relies on a connection with the mathematical field of small inhomogeneities.



Optimization of a landing gear (courtesy of Ansys).



"Optimized" addition of thin bars to a shape with poor topology.

Contents

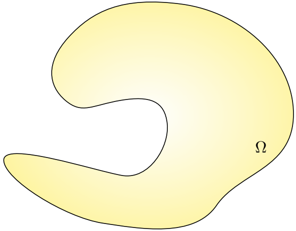
Different sensitivities with respect to the domain (I)

A typical shape and topology optimization problem reads:

$$\min_{\Omega} J(\Omega) \text{ s.t. } C(\Omega) \leq 0,$$

where

- Ω is a shape, e.g. an elastic structure.
- $J(\Omega)$ measures the physical performance of Ω.
- $C(\Omega)$ is a constraint functional.



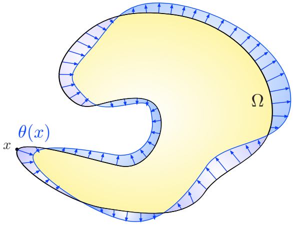
- Most numerical algorithms rely on the "derivatives" of $\Omega \mapsto J(\Omega)$ and $\Omega \mapsto C(\Omega)$.
- Multiple notions of derivative with respect to the design exist, which are based on as many descriptions of "small variations of shapes".

❶ *Hadamard's boundary variation method.*

Variations of a shape are considered under the form

$$\Omega_\theta := (\mathrm{Id} + \theta)(\Omega),$$

where $\theta : \mathbb{R}^d \to \mathbb{R}^d$ is a "small" vector field [HenPi].



This gives rise to the notion of shape derivative $J'(\Omega)(\theta)$ for a function $\Omega \mapsto J(\Omega)$:

$$J(\Omega_\theta) = J(\Omega) + J'(\Omega)(\theta) + \mathrm{o}(\theta).$$

A. Henrot and M. Pierre, *Shape Variation and Optimization*, EMS Tracts in Mathematics Vol. 28, 2018.

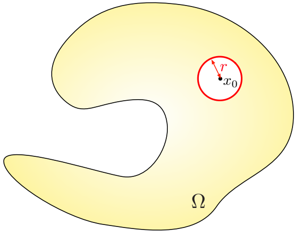
Different sensitivities with respect to the domain (III)

❷ *Nucleation of a tiny hole.*

Variations of Ω are considered under the form

$$\Omega_{x_0,r} := \Omega \setminus \overline{B(x_0, r)},$$

where $x_0 \in \Omega$ and $r \ll 1$ [NoSo].



This yields the notion of topological derivative $\mathrm{d}J_T(\Omega)(x_0)$ for a function $\Omega \mapsto J(\Omega)$:

$$J(\Omega_{x_0,r}) = J(\Omega) + r^d \mathrm{d}J_T(\Omega)(x_0) + \mathrm{o}(r^d).$$

A. A. Novotny and J. Sokołowski, *Topological derivatives in shape optimization*, Springer Science & Business Media, 2012.

Different sensitivities with respect to the domain (III)

❸ *Graft of a thin ligament.*

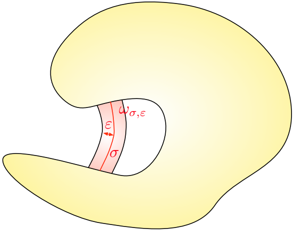
One third means to define "small" variations of Ω is:

$$\Omega_{\sigma,\varepsilon} := \Omega \cup \omega_{\sigma,\varepsilon},$$

where

$$\omega_{\sigma,\varepsilon} := \left\{ x \in \mathbb{R}^d, \;\; d(x,\sigma) < \varepsilon \right\}$$

is a tube with thickness $\varepsilon \ll 1$ around a curve σ [NaSo]



Such variations pave the way to a notion of topological ligament derivative:

$$J(\Omega_{\sigma,\varepsilon}) = J(\Omega) + \underbrace{\varepsilon^{d-1}}_{\approx |\omega_{\sigma,\varepsilon}|} \mathrm{d}J_L(\Omega)(\sigma) + \mathrm{o}(\varepsilon^{d-1}).$$

This topic has been seldom investigated in the literature. Unfortunately,

- The mathematical derivation of such asymptotic formulas is very difficult.
- The resulting formulas are difficult to use in practice.

S. Nazarov and J. Sokolowski, *The topological derivative of the dirichlet integral due to formation of a thin ligament*, Siberian Mathematical Journal, 45 (2004), pp. 341–355.

Guidelines of this work

- We approximate the considered "one-phase and void" shape optimization problems by two-phase problems, featuring an "ersatz", nearly degenerate phase.

- This allows to approximate rigorous topological ligament asymptotic expansions by formulas pertaining to the field of small inhomogeneities.

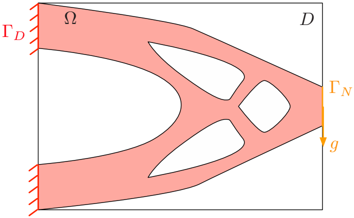
- We present a formal energy method to obtain such expansions with a minimum amount of technicality.

- We use the derived formulas to add bars to a shape in an optimal way, in several practical contexts.

Shape optimization of elastic structures

Shapes are bounded Lipschitz domains $\Omega \subset D$ in $\mathbb{R}^d$.

- They are clamped on a fixed subset $\Gamma_D \subset \partial D$.
- Traction loads $g : \Gamma_N \to \mathbb{R}^d$ are applied on $\Gamma_N \subset \partial D$.
- The remaining part $\Gamma = \partial\Omega \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N})$ is traction-free.



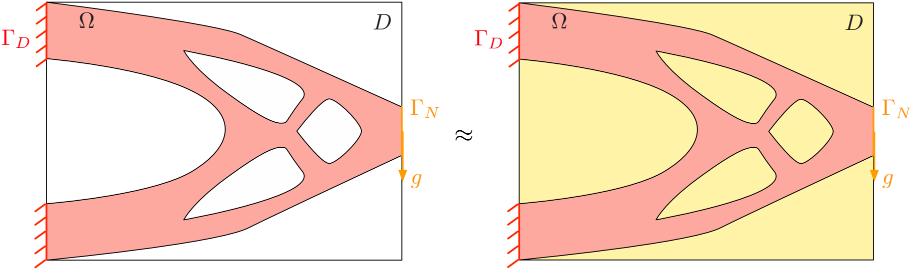
The elastic displacement $u_\Omega \in H^1(\Omega)^d$ is the unique solution to

$$\begin{cases} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma, \end{cases}$$

where A is the (homogeneous) Hooke's law of the material.

The ersatz material approximation (I)

We approximate this setting by "filling the void" $D \setminus \overline{\Omega}$ with a soft material ηA, $\eta \ll 1$.

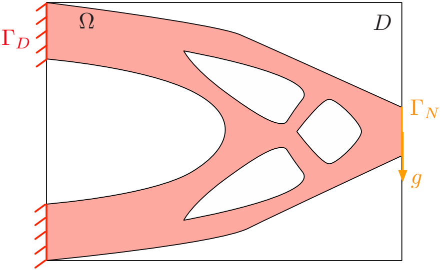


$$\begin{cases} -\text{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma. \end{cases}$$

$$\begin{cases} -\text{div}(A_\eta e(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ A_\eta e(u_\Omega)n = g & \text{on } \Gamma_N, \\ A_\eta e(u_\Omega)n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}), \end{cases}$$

$$A_\eta = \begin{cases} A & \text{if } x \in \Omega, \\ \eta A & \text{otherwise.} \end{cases}$$

We may as well use a smoothed version $\widetilde{A_\eta}$ of A_η.



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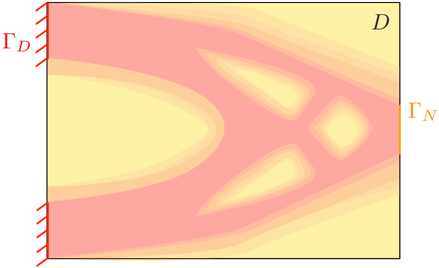


$$\begin{cases} -\mathrm{div}(Ae(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(u_\Omega)n = g & \text{on } \Gamma_N, \\ Ae(u_\Omega)n = 0 & \text{on } \Gamma. \end{cases}$$

$$\begin{cases} -\mathrm{div}(\widetilde{A_\eta} e(u_\Omega)) = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_D, \\ \widetilde{A_\eta} e(u_\Omega)n = g & \text{on } \Gamma_N, \\ \widetilde{A_\eta} e(u_\Omega)n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}), \end{cases}$$

$$\widetilde{A_\eta} = (\text{smoothed}) \begin{cases} A & \text{if } x \in \Omega, \\ \eta A & \text{otherwise.} \end{cases}$$

The ersatz material approximation (III)

- A quantity of interest $J(\Omega)$, depending on Ω via u_Ω can be given an approximate counterpart by the same token.

  Example: The shape functional

  $$J(\Omega) = \int_\Omega j(x, u_\Omega(x))\,\mathrm{d}x \text{ where } j : \mathbb{R}^d_x \times \mathbb{R}^d_u \to \mathbb{R} \text{ is smooth}$$

  can be approximated as

  $$J(\Omega) \approx \int_D j(x, u_\eta(x))\,\mathrm{d}x, \quad \text{up to modifying } j.$$

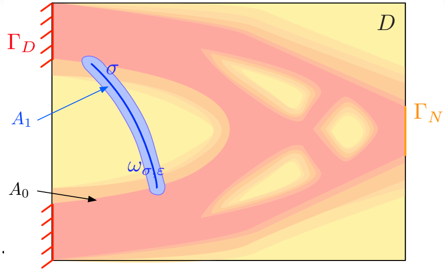
- In the same spirit, we aim to construct an ersatz material approximation

  $$J_\sigma(\varepsilon) \approx J(\Omega_{\sigma,\varepsilon}), \text{ where } \Omega_{\sigma,\varepsilon} = \Omega \cup \omega_{\sigma,\varepsilon}.$$

The ersatz material approximation: perturbed setting (I)

- Let $A_0(x)$ be a smooth Hooke's law in D.
- The "background displacement u_0 is the $H^1(D)^d$ solution to:

$$\begin{cases} -\text{div}(A_0 e(u_0)) = 0 & \text{in } D, \\ u_0 = 0 & \text{on } \Gamma_D, \\ A_0 e(u_0) n = g & \text{on } \Gamma_N, \\ A_0 e(u_0) n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}). \end{cases}$$



- In a perturbed situation, the properties $A_0(x)$ are traded for $A_1(x)$ in a tube

$$\omega_{\sigma,\varepsilon} := \{x \in D, \ \ d(x, \sigma) < \varepsilon\}$$

with "small" thickness $\varepsilon \ll 1$ around a curve σ.

- The perturbed elastic displacement u_ε is the solution to:

$$\begin{cases} -\text{div}(A_\varepsilon e(u_\varepsilon)) = 0 & \text{in } D, \\ u_0 = 0 & \text{on } \Gamma_D, \\ A_\varepsilon e(u_0) n = g & \text{on } \Gamma_N, \\ A_\varepsilon e(u_0) n = 0 & \text{on } \partial D \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N}), \end{cases} \quad \text{where } A_\varepsilon(x) = \begin{cases} A_1(x) & \text{if } x \in \omega_{\sigma,\varepsilon}, \\ A_0(x) & \text{otherwise.} \end{cases}$$

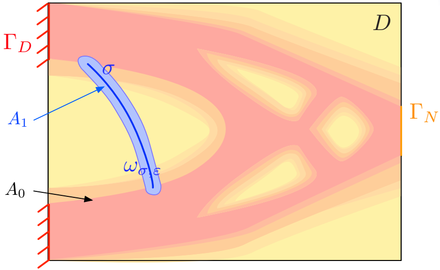
The ersatz material approximation: perturbed setting (II)

The perturbed version of a quantity

$$J_\sigma(0) := \int_D j(u_0)\,\mathrm{d}x$$

reads

$$J_\sigma(\varepsilon) = \int_D j(u_\varepsilon)\,\mathrm{d}x.$$



- Intuitively, the asymptotic expansion of $J'_\sigma(0)$,

$$J_\sigma(\varepsilon) = J_\sigma(0) + \varepsilon^{d-1} J'_\sigma(0) + \mathrm{o}(\varepsilon^{d-1})$$

  measures the sensitivity of J_σ with respect to changing material properties from A_0 to A_1 in the thin tube $\omega_{\sigma,\varepsilon}$.

- When A_0 is obtained from $\Omega \subset D$ by the ersatz material approximation, i.e.

$$A_0(x) = \begin{cases} A & \text{if } x \in \Omega, \\ \eta A & \text{if } x \in D \setminus \overline{\Omega}, \end{cases} \qquad \text{where } \eta \ll 1,$$

  $J'_\sigma(0)$ is an approximate sensitivity of $J(\Omega)$ with respect to the addition of $\omega_{\sigma,\varepsilon}$.

A glimpse of "small" inhomogeneities (I)

To set ideas, let us consider a model problem in the conductivity setting.

- $D \subset \mathbb{R}^d$ is a smooth bounded domain, filled by a material with smooth conductivity $\gamma_0 \in \mathcal{C}^\infty(\overline{D})$.
- A smooth current g is applied on ∂D such that $\int_{\partial D} g \, \mathrm{d}s = 0$.
- The "background" potential u_0 is the unique $H^1(D)$ solution such that $\int_D u_0 \, \mathrm{d}x = 0$ to the boundary-value problem

$$\begin{cases} -\mathrm{div}(\gamma_0 \nabla u_0) = 0 & \text{in } D, \\ \gamma_0 \frac{\partial u_0}{\partial n} = g & \text{on } \partial D. \end{cases}$$

- In a perturbed situation, D contains inhomogeneities with conductivity $\gamma_1 \in \mathcal{C}^\infty(\mathbb{R}^d)$, occupying a "small" subset $\omega_\varepsilon \Subset D$.
- The perturbed potential $u_\varepsilon \in H^1(D)$ satisfies $\int_D u_\varepsilon \, \mathrm{d}x = 0$ and

$$\begin{cases} -\mathrm{div}(\gamma_\varepsilon \nabla u_\varepsilon) = 0 & \text{in } D, \\ \gamma_0 \frac{\partial u_\varepsilon}{\partial n} = g & \text{on } \partial D, \end{cases} \quad \text{where } \gamma_\varepsilon(x) := \begin{cases} \gamma_1(x) & \text{if } x \in \omega_\varepsilon, \\ \gamma_0(x) & \text{otherwise.} \end{cases}$$

A glimpse of “small” inhomogeneities (II)

- A general representation formula for u_ε in the low-volume limit $|\omega_\varepsilon| \to 0$ was derived in [CapVo]: for $x \in \partial D$ and a subsequence of the ε,

$$u_\varepsilon(x) = u_0(x) + |\omega_\varepsilon| \int_D (\gamma_1 - \gamma_0)(y)\mathcal{M}(y)\nabla u_0(y) \cdot \nabla_y N(x,y)\,\mathrm{d}\mu(y) + \mathrm{o}(|\omega_\varepsilon|),$$

  where

  - The probability measure μ describes the “limiting position” of the subsets ω_ε.
  - The polarization tensor $\mathcal{M}(y)$ accounts for the “limiting behavior” of a rescaled version of the field u_ε inside ω_ε.
  - $N(x,y)$ is the Neumann function of the background problem.

- The relevant quantity to measure the “smallness” of ω_ε is the volume $|\omega_\varepsilon|$.

- This formula can be refined when particular geometries are assumed for ω_ε.

Y. Capdeboscq and M. S. Vogelius, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, ESAIM: M2AN, 37 (2003), pp. 159–173.

"Small" inhomogeneities: examples (I)

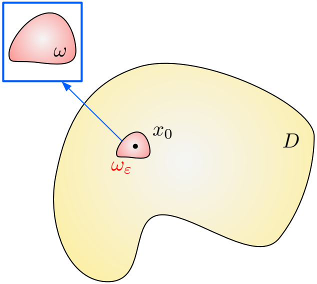
❶ Diametrically small inhomogeneities read

$$\omega_\varepsilon = x_0 + \varepsilon\omega,$$

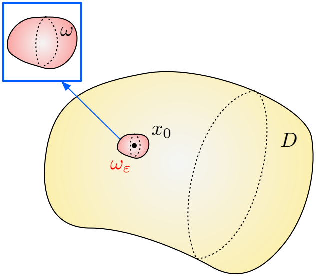
where $\omega \Subset \mathbb{R}^d$ is a given bounded subset.

Then,

- μ is a multiple of δ_{x_0},
- $\mathcal{M}$ involves the solution to an exterior problem, posed on ω and $\mathbb{R}^d \setminus \overline{\omega}$.
- <u>*References:*</u> [ASe, CeMoVo]



A 2d diametrically small inhomogeneity



A 3d diametrically small inhomogeneity

❷ Thin inhomogeneities have small thickness about a codimension 1 entity:

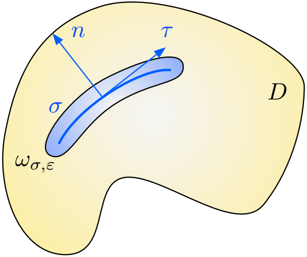
$$\omega_{\sigma,\varepsilon} = \left\{x \in \mathbb{R}^d,\ \ d(x,\sigma) < \varepsilon\right\} \text{ if } d = 2,$$

and

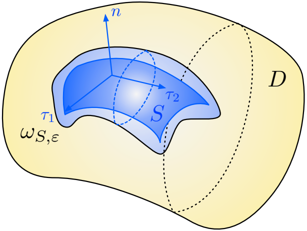
$$\omega_{S,\varepsilon} = \left\{x \in \mathbb{R}^d,\ \ d(x,S) < \varepsilon\right\} \text{ if } d = 3,$$

where $\sigma \Subset D$ and $S \Subset D$ are (open or closed) curve and hypersurface in $\mathbb{R}^2$, $\mathbb{R}^3$, respectively.

- μ is an integration measure on σ or S,
- $\mathcal{M}$ is diagonal in a local basis $(\tau_1, \ldots, \tau_{d-1}, n)$ attached to σ or S.
- *References:* [BeFranVo, KheZri]



A 2d thin inhomogeneity



A 3d thin inhomogeneity

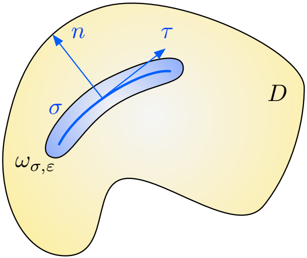
"Small" inhomogeneities: examples (III)

❸ Tubular inhomogeneities are of the form

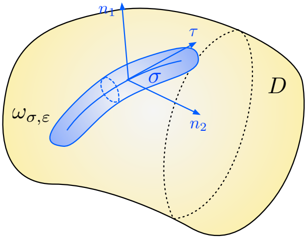
$$\omega_{\sigma,\varepsilon} = \left\{ x \in \mathbb{R}^d, \;\; d(x,\sigma) < \varepsilon \right\},$$

where $\sigma \Subset D$ is an (open or closed) curve in $\mathbb{R}^d$.

- μ is an integration measure on σ,
- $\mathcal{M}$ is diagonal in a local basis $(\tau, n_1, \ldots, n_{d-1})$ attached to σ.
- *References:* [BeCapGoFran, CapGrieKno]



In 2d, tubular inhomogeneities coincide with thin inhomogeneities



A 3d tubular inhomogeneity

"Small" inhomogeneities: extensions and applications

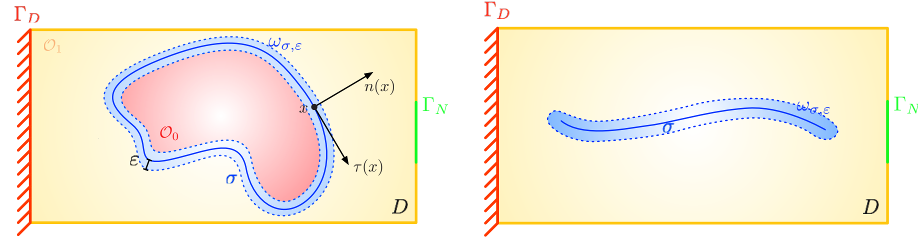
- These questions have been considered in various more challenging physical settings, such as
  - that of the linearized elasticity equations [BeFran, BeBoFranMa]
  - that of the Maxwell system [AmVoVo, Grie].
- These asymptotic formulas pave the way to multiple numerical methods for the detection or the reconstruction of small inhomogeneities [AmKa].
- They also allow for the optimization of the placement and shape of inhomogeneities:
  - Topological derivatives in shape optimization [NoSo].
  - Optimization of the placement of tubular inhomogeneities [present work].

The model context of the conductivity equation

- For simplicity, we consider the model setting of the conductivity equation.
- The functions u_0 and $u_\varepsilon \in H^1(D)$ are the solutions to the respective equations:

$$\left\{\begin{array}{ll} -\text{div}(\gamma_0 \nabla u_0) = f & \text{in } D, \\ u_0 = 0 & \text{on } \Gamma_D, \\ \gamma_0 \frac{\partial u_0}{\partial n} = g & \text{on } \Gamma_N, \\ \gamma_0 \frac{\partial u_0}{\partial n} = 0 & \text{on } \partial D \setminus (\Gamma_D \cup \Gamma_N), \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll} -\text{div}(\gamma_\varepsilon \nabla u_\varepsilon) = f & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \Gamma_D, \\ \gamma_0 \frac{\partial u_\varepsilon}{\partial n} = g & \text{on } \Gamma_N, \\ \gamma_0 \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial D \setminus (\Gamma_D \cup \Gamma_N), \end{array}\right.$$

$$\text{where } \gamma_\varepsilon(x) = \left\{\begin{array}{ll} \gamma_1(x) & \text{if } x \in \omega_{\sigma,\varepsilon} \\ \gamma_0(x) & \text{otherwise.} \end{array}\right.$$



The base curve σ may be open (left) or closed (right).

Theorem 1.

The following expansion holds at any point $x \in D \setminus \sigma$:

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \mathrm{o}(\varepsilon), \text{ where } u_1(x) := \int_\sigma \mathcal{M}(y)\nabla u_0(y) \cdot \nabla_y N(x,y)\, \mathrm{d}\ell(y).$$

Here,

- *$N(x,y)$ is the Green's function of the background operator;*
- *For any point $y \in \sigma$, the polarization tensor $\mathcal{M}(y)$ is a symmetric 2×2 matrix, whose expression reads, in the local orthonormal frame $(\tau(y), n(y))$:*

$$\mathcal{M}(y) = \begin{pmatrix} 2(\gamma_1(y) - \gamma_0(y)) & 0 \\ 0 & 2\gamma_0(y)\left(1 - \frac{\gamma_0(y)}{\gamma_1(y)}\right) \end{pmatrix}.$$

The main result (II)

- This result is proved in [BeFranVo, KheZri] by using different techniques.

- The conclusion holds regardless of σ being closed or open.
  - When σ is closed, u_1 can be characterized by a variational equation.
  - When σ is open, the interpretation of u_1 as the solution to a "classical PDE" is more difficult.

- This indicates that "the endpoints" of σ contribute only at higher order to the expansion of u_ε. This phenomenon is observed in all known investigations about thin or tubular inhomogeneities.

- In the following, we present a formal energy argument, which allows to "easily" derive the correct formula (in the case of closed σ).

The main result: sketch of proof (I)

Sketch of the proof:

We consider the error

$$r_\varepsilon := \frac{1}{\varepsilon}(u_\varepsilon - u_0)$$

which is the unique solution in the space

$$H^1_{\Gamma_D}(D) := \left\{ u \in H^1(D), \;\; u = 0 \text{ on } \;\Gamma_D \right\}.$$

to the following variational problem:

$$\forall v \in H^1_{\Gamma_D}(D), \;\; \int_D \gamma_\varepsilon \nabla r_\varepsilon \cdot \nabla v \, \mathrm{d}x = -\frac{1}{\varepsilon} \int_{\omega_{\sigma,\varepsilon}} (\gamma_1 - \gamma_0) \nabla u_0 \cdot \nabla v \, \mathrm{d}x.$$

Equivalently, r_ε is the unique solution to the minimization problem

$$\min_{u \in H^1_{\Gamma_D}(D)} E_\varepsilon(u), \text{ where } E_\varepsilon(u) := \frac{1}{2} \int_D \gamma_\varepsilon |\nabla u|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \int_{\omega_{\sigma,\varepsilon}} (\gamma_1 - \gamma_0) \nabla u_0 \cdot \nabla u \, \mathrm{d}x.$$

The main result: sketch of proof (II)

Step 1: We derive a representation formula for the values $r_\varepsilon(x)$ at $x \in D \setminus \sigma$ in terms of the values of $r_\varepsilon(x)$ inside $\omega_{\sigma,\varepsilon}$.

This task relies on the Green's function $N(x,y)$ of the background operator:

For all $x \in \Omega$, $y \mapsto N(x,y)$ satisfies

$$\begin{cases} \mathrm{div}_y(\gamma_0(y)\nabla_y N(x,y)) = \delta_{y=x} & \text{in } D, \\ \gamma_0(y)\frac{\partial N}{\partial n_y}(x,y) = 0 & \text{for } y \in \partial D \setminus \overline{\Gamma_D}, \\ N(x,y) = 0 & \text{for } y \in \Gamma_D, \end{cases}$$

N(x,y) can be constructed from (and behaves like) the modified fundamental solution to Laplace operator in free space:

$$G(x,y) = \frac{1}{2\pi\gamma_0(x)} \log|x-y|.$$

Using the definition of the Green's function $N(x, y)$, we obtain:

$$\begin{aligned} r_\varepsilon(x) &= \int_D \mathrm{div}_y(\gamma_0(y)\nabla_y N(x,y)) r_\varepsilon(y)\,\mathrm{d}y, \\ &= -\int_D \gamma_0(y)\nabla r_\varepsilon(y)\cdot\nabla_y N(x,y)\,\mathrm{d}y, \\ &= -\int_D \gamma_\varepsilon(y)\nabla r_\varepsilon(y)\cdot\nabla_y N(x,y)\,\mathrm{d}y + \int_{\omega_{\sigma,\varepsilon}} (\gamma_1-\gamma_0)(y)\nabla r_\varepsilon(y)\cdot\nabla_y N(x,y)\,\mathrm{d}y. \end{aligned}$$

Now "using $y \mapsto N(x, y)$ as test function" in the variational formulation for r_ε, we get:

$$\int_D \gamma_\varepsilon(y)\nabla r_\varepsilon(y)\cdot\nabla_y N(x,y)\,\mathrm{d}y = -\frac{1}{\varepsilon}\int_{\omega_{\sigma,\varepsilon}} (\gamma_1-\gamma_0)(y)\nabla u_0(y)\cdot\nabla_y N(x,y)\,\mathrm{d}y,$$

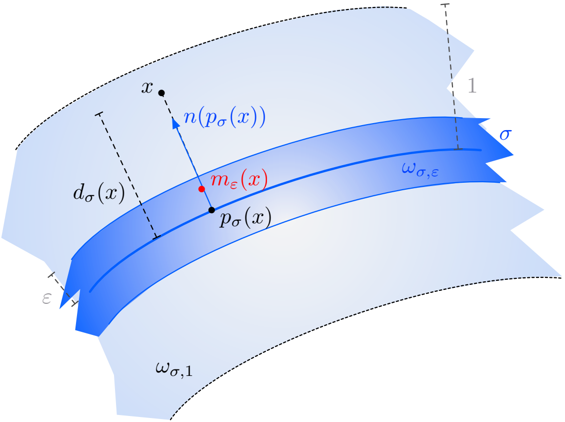
and so:

$$\begin{aligned} r_\varepsilon(x) = \frac{1}{\varepsilon}\int_{\omega_{\sigma,\varepsilon}} &(\gamma_1-\gamma_0)(y)\nabla u_0(y)\cdot\nabla_y N(x,y)\,\mathrm{d}y \\ &+ \int_{\omega_{\sigma,\varepsilon}} (\gamma_1-\gamma_0)(y)\nabla r_\varepsilon(y)\cdot\nabla_y N(x,y)\,\mathrm{d}y. \end{aligned}$$

The main result: sketch of proof (IV)

We rescale the thin tube $\omega_{\sigma,\varepsilon}$ into that $\omega_{\sigma,1}$ with unit size, thanks to the mapping:

$$m_\varepsilon : \omega_{\sigma,1} \to \omega_{\sigma,\varepsilon}, \quad m_\varepsilon(x) := p_\sigma(x) + \varepsilon d_\sigma(x) n(p_\sigma(x)).$$



The main result: sketch of proof (V)

A change of variables now yields immediately:

$$r_\varepsilon(x) = \int_{\omega_{\sigma,\mathbf{1}}} \frac{1+\varepsilon d_\sigma \kappa}{1+d_\sigma \kappa}((\gamma_1-\gamma_0)\circ m_\varepsilon)((\nabla u_0)\circ m_\varepsilon)\cdot \nabla_y N(x, m_\varepsilon(y))\,\mathrm{d}y$$

$$+\int_{\omega_{\sigma,\mathbf{1}}} (\gamma_1-\gamma_0)\circ m_\varepsilon \left(\varepsilon \frac{\partial s_\varepsilon}{\partial \tau}\frac{\partial N}{\partial \tau_y}(x, m_\varepsilon(y)) + \frac{1+\varepsilon d_\sigma \kappa}{1+d_\sigma \kappa}\frac{\partial s_\varepsilon}{\partial n}\frac{\partial N}{\partial n_y}(x, m_\varepsilon(y))\right)\,\mathrm{d}y,$$

where

- $\kappa : \sigma \to \mathbb{R}$ is the curvature of σ,
- $s_\varepsilon = r_\varepsilon \circ m_\varepsilon \in H^1(\omega_{\sigma,1})$ is the profile of r_ε inside the rescaled inclusion $\omega_{\sigma,1}$.

Step 2: We get information about the behavior of the rescaled error s_ε inside $\omega_{\sigma,1}$.

The couple $(r_\varepsilon, s_\varepsilon)$ is the solution to the two-scale minimization problem:

$$\min_{(u,v)\in V_\varepsilon} F_\varepsilon(u,v),$$

where the space V_ε is defined by:

$$V_\varepsilon = \left\{(u,v)\in H^1_{\Gamma_D}(D)\times H^1(\omega_{\sigma,1}),\ \forall x\in\sigma,\ \left\{\begin{array}{l} v(x+n(x)) = u(x+\varepsilon n(x)) \\ v(x-n(x)) = u(x-\varepsilon n(x)) \end{array}\right.\right\},$$

and the two-scale energy $F_\varepsilon(u,v)$ reads:

$$F_\varepsilon(u,v) := \frac{1}{2}\int_{D\setminus\overline{\omega_{\sigma,\varepsilon}}} \gamma_0|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\omega_{\sigma,1}} (\gamma_1\circ m_\varepsilon)|\mathrm{det}\nabla m_\varepsilon|(\nabla m_\varepsilon^{-1}\nabla m_\varepsilon^{-T})\nabla v\cdot\nabla v\,\mathrm{d}x$$
$$+\frac{1}{\varepsilon}\int_{\omega_{\sigma,1}} ((\gamma_1-\gamma_0)\circ m_\varepsilon)|\mathrm{det}\nabla m_\varepsilon|(\nabla u_0)\circ m_\varepsilon\cdot(\nabla m_\varepsilon^{-T}\nabla v)\,\mathrm{d}x.$$

The main result: sketch of proof (VII)

An elementary calculation yields:

$$F_\varepsilon(u,v) := \frac{1}{2}\int_{D\setminus\overline{\omega_{\sigma,\varepsilon}}} \gamma_0|\nabla u|^2\,\mathrm{d}x + \frac{1}{2\varepsilon}\int_{\omega_{\sigma,\mathbf{1}}} (\gamma_1\circ m_\varepsilon)\left(\frac{1+\varepsilon d_\sigma\kappa}{1+d_\sigma\kappa}\right)\left(\frac{\partial v}{\partial n}\right)^2\,\mathrm{d}x$$

$$+\frac{\varepsilon}{2}\int_{\omega_{\sigma,\mathbf{1}}} (\gamma_1\circ m_\varepsilon)\left(\frac{1+d_\sigma\kappa}{1+\varepsilon d_\sigma\kappa}\right)\left(\frac{\partial v}{\partial \tau}\right)^2\,\mathrm{d}x + \int_{\omega_{\sigma,\mathbf{1}}}((\gamma_1-\gamma_0)\circ m_\varepsilon)\left(\frac{\partial u_0}{\partial\tau}\circ m_\varepsilon\right)\frac{\partial v}{\partial \tau}\,\mathrm{d}x$$

$$+\frac{1}{\varepsilon}\int_{\omega_{\sigma,\mathbf{1}}}((\gamma_1-\gamma_0)\circ m_\varepsilon)\left(\frac{1+\varepsilon d_\sigma\kappa}{1+d_\sigma\kappa}\right)\left(\frac{\partial u_0}{\partial n}\circ m_\varepsilon\right)\frac{\partial v}{\partial n}\,\mathrm{d}x.$$

Idea: The behavior of s_ε should be dictated by the minimization of the highest-order terms in this energy:

$$s_\varepsilon \approx \arg\min_{v\in H^{\mathbf{1}}(\omega_{\sigma,\mathbf{1}})} \widetilde{F}(v), \text{ where}$$

$$\widetilde{F}(v) := \frac{1}{2}\int_{\omega_{\sigma,\mathbf{1}}}(\gamma_1\circ p_\sigma)\left(\frac{1}{1+d_\sigma\kappa}\right)\left(\frac{\partial v}{\partial n}\right)^2\,\mathrm{d}x$$

$$+\int_{\omega_{\sigma,\mathbf{1}}}((\gamma_1-\gamma_0)\circ p_\sigma)\left(\frac{1}{1+d_\sigma\kappa}\right)\left(\frac{\partial u_0}{\partial n}\circ p_\sigma\right)\frac{\partial v}{\partial n}\,\mathrm{d}x.$$

The main result: sketch of proof (VIII)

Writing down the corresponding Euler-Lagrange equations, we obtain that the minimizer $v \in H^1(\omega_{\sigma,1})$ of $\widetilde{F}(v)$ satisfies:

$$\frac{\partial v}{\partial n}(p + tn(p)) = -\frac{1}{\gamma_1(p)}(\gamma_1(p) - \gamma_0(p))\frac{\partial u_0}{\partial n}(p), \quad p \in \sigma,$$

which is all that we need for the following.

The main result: sketch of proof (IX)

Step 3: We pass to the limit in the representation formula.

$$
\begin{aligned}
\lim_{\varepsilon\to 0} r_\varepsilon(x) \quad = \quad & \int_{\omega_{\sigma,1}} \frac{1}{1+d_\sigma\kappa}((\gamma_1-\gamma_0)\circ p_\sigma)((\nabla u_0)\circ p_\sigma)\cdot\nabla_y N(x,p_\sigma(y))\,\mathrm{d}y \\
& + \int_{\omega_{\sigma,1}} (\gamma_1-\gamma_0)\circ p_\sigma \frac{1}{1+d_\sigma\kappa}\frac{\partial v}{\partial n}\frac{\partial N}{\partial n_y}(x,p_\sigma(y))\,\mathrm{d}y,
\end{aligned}
$$

We now employ the coarea formula (as a curvilinear version of the Fubini theorem) to rewrite integrals over $\omega_{\sigma,1}$ as nested integrals over $\sigma\times(-1,1)$.

Proposition 2.

For any function $\varphi\in L^1(\omega_{\sigma,1})$, it holds:

$$
\int_{\omega_{\sigma,1}} \varphi(x)\,\mathrm{d}x = \int_\sigma \left(\int_{-1}^{1}(1+t\kappa(p))f(p+tn(p))\,\mathrm{d}t\right)\mathrm{d}\ell(p).
$$

Eventually, a simple calculation yields

$$\begin{aligned}
\lim_{\varepsilon\to 0} r_\varepsilon(x) &= 2\int_\sigma (\gamma_1-\gamma_0)(p)\nabla u_0(p)\cdot\nabla_y N(x,p)\,\mathrm{d}\ell(p) \\
&\qquad + \int_\sigma (\gamma_1-\gamma_0)(p)\left(\int_{-1}^{1}\frac{\partial v}{\partial n}(p+tn(p))\,\mathrm{d}t\right)\frac{\partial N}{\partial n_y}(x,p)\,\mathrm{d}\ell(p), \\
&= 2\int_\sigma (\gamma_1-\gamma_0)(p)\frac{\partial u_0}{\partial \tau}(p)\frac{\partial N}{\partial \tau_y}(x,p)\,\mathrm{d}\ell(p) \\
&\qquad +2\int_\sigma \gamma_0(p)\left(1-\frac{\gamma_0(p)}{\gamma_1(p)}\right)\frac{\partial u_0}{\partial n}(p)\frac{\partial N}{\partial n_y}(x,p)\,\mathrm{d}\ell(p),
\end{aligned}$$

which is the desired expression. □

Derivative of an observable (I)

The previous result allows to calculate the derivative of the observable

$$J_\sigma(\varepsilon) = \int_D j(u_\varepsilon)\,\mathrm{d}x.$$

Proposition 3.

The function $J_\sigma(\varepsilon)$ is differentiable at $\varepsilon = 0$, with derivative:

$$J'_\sigma(0) = \int_\sigma \mathcal{M}\nabla u_0 \cdot \nabla p_0 \,\mathrm{d}\ell,$$

where $\mathcal{M}$ is the polarization tensor, and the adjoint state $p_0 \in H^1_{\Gamma_D}(D)$ is the unique solution to the equation:

$$\begin{cases} -\mathrm{div}(\gamma_0 \nabla p_0) = -j'(u_0) & \text{in } D, \\ p_0 = 0 & \text{on } \Gamma_D, \\ \gamma_0 \frac{\partial p_0}{\partial n} = 0 & \text{on } \partial D \setminus \overline{\Gamma_D}. \end{cases}$$

Derivative of an observable (II)

Sketch of proof: At first, the dominated convergence theorem implies that

$$J'_\sigma(0) = \lim_{\varepsilon \to 0} \frac{J_\sigma(\varepsilon) - J_\sigma(0)}{\varepsilon} = \lim_{\varepsilon \to 0} \int_D \frac{j(u_\varepsilon) - j(u_0)}{\varepsilon} \, \mathrm{d}x = \int_D j'(u_0) u_1 \, \mathrm{d}x.$$

Then, using the integral formula for u_1 with Fubini's theorem, we get

$$\begin{aligned} J'_\sigma(0) &= \int_D \int_\sigma j'(u_0)(x) \mathcal{M}(y) \nabla u_0(y) \cdot \nabla_y N(x,y) \, \mathrm{d}\ell(y) \, \mathrm{d}x \\ &= \int_\sigma \mathcal{M}(y) \nabla u_0(y) \cdot \nabla_y \left(\int_D j'(u_0)(x) N(x,y) \, \mathrm{d}x \right) \mathrm{d}\ell(y). \end{aligned}$$

Finally, the definition of the adjoint state and the properties of $N(x,y)$ entail

$$p_0(y) = \int_D j(u_0)(x) N(x,y) \, \mathrm{d}x,$$

and the desired result follows.

$\square$

A similar (albeit more technical) result holds in the context of 2d elasticity.

Theorem 4.

For an arbitrary point $x \in D \setminus \sigma$, the following asymptotic expansion holds:

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + \mathrm{o}(\varepsilon), \text{ where } u_1(x) = \int_\sigma \mathcal{M}(y)e(u_0) : e_y(N(x,y))\, \mathrm{d}\ell(y).$$

The polarization tensor $\mathcal{M}(y)$ reads, for any symmetric 2×2 matrix $e \in \mathcal{S}_2(\mathbb{R})$:

$$\mathcal{M}(y)e = \alpha_T(y)\mathrm{tr}(e)\mathrm{I} + \beta_T(y)e + \gamma_T(y)(e\tau \cdot \tau)\tau \otimes \tau + \delta_T(y)(en \cdot n)n \otimes n,$$

where the coefficients $\alpha_T, \beta_T, \gamma_T$ and δ_T are given by:

$$\alpha_T = 2(\lambda_1 - \lambda_0)\frac{\lambda_0 + 2\mu_0}{\lambda_1 + 2\mu_1}, \;\; \beta_T = 4(\mu_1 - \mu_0)\frac{\mu_0}{\mu_1},$$

and

$$\gamma_T = 4(\mu_1 - \mu_0)\left(\frac{2\lambda_1 + 2\mu_1 - \lambda_0}{\lambda_1 + 2\mu_1} - \frac{\mu_0}{\mu_1}\right), \;\; \delta_T = 4(\mu_1 - \mu_0)\frac{\mu_1\lambda_0 - \mu_0\lambda_1}{\mu_1(\lambda_1 + 2\mu_1)}.$$

Diametrically small inhomogeneities

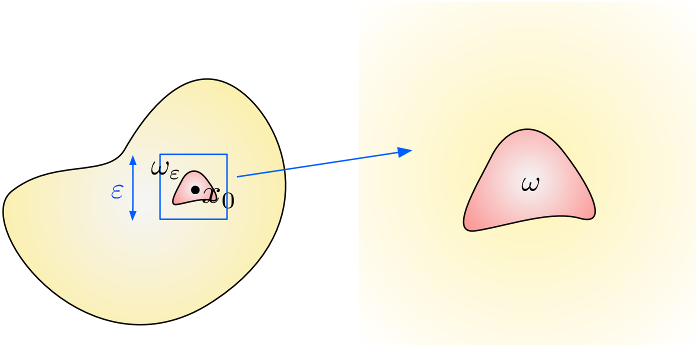
- The previous formal energy argument allows to deal with diametrically small inhomogeneities

$$\omega_\varepsilon = x_0 + \varepsilon\omega, \text{ where } \omega \Subset \mathbb{R}^d.$$

- The "classical" formulas are recovered:

$$u_\varepsilon(x) = u_0(x) + \varepsilon^d u_1(x) + \mathrm{o}(\varepsilon^d), \text{ where } u_1(x) := \mathcal{M}\nabla u_0(x_0) \cdot \nabla_y N(x, x_0),$$

and the polarization tensor $\mathcal{M}$ involves the solution to an exterior problem, posed on the rescaled configuration $\omega \cup (\mathbb{R}^d \setminus \overline{\omega})$.



3d tubular inhomogeneities (I)

- In the 3d conductivity case, a similar expansion holds

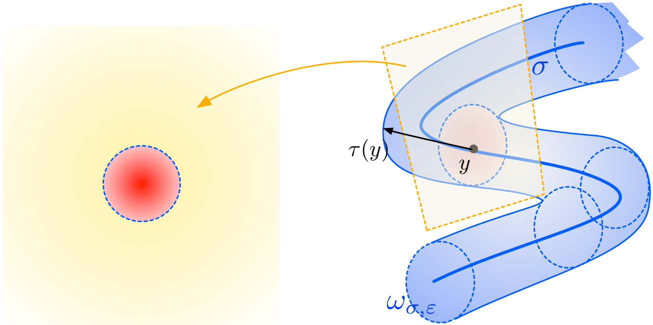
$$u_\varepsilon(x) = u_0(x) + \varepsilon^2 u_1(x) + o(\varepsilon^2), \text{ where } u_1(x) := \int_\sigma \mathcal{M}(y) \nabla u_0(y) \cdot \nabla_y N(x,y) \, d\ell(y).$$

- For $y \in \sigma$, the polarization tensor $\mathcal{M}(y) \in \mathbb{R}^{3\times 3}$ is defined by:

$$\mathcal{M}(y) = \begin{pmatrix} \pi(\gamma_1 - \gamma_0)(y) & 0 \\ 0 & \mathcal{M}_{NN}(y) \end{pmatrix},$$

as expressed in a basis made from $\tau(y)$ and the normal plane to $\tau(y)$.

- $\mathcal{M}_{NN}(y)$ is the polarization tensor for a 2d disk-shaped small inclusion.



3d tubular inhomogeneities (II)

- The derivative of the quantity of interest

$$J_\sigma(\varepsilon) = \int_D j(u_\varepsilon)\,\mathrm{d}x$$

can be calculated as in the 2d case:

$$J_\sigma(\varepsilon) = J_\sigma(0) + \varepsilon^2 J'_\sigma(0) + \mathrm{o}(\varepsilon^2)$$

- Finally, similar (but much more complicated) expressions hold in the context of 3d linear elasticity.

The general strategy to add a tube to a shape

In order to graft a tube $\omega_{\sigma,\varepsilon}$ to a given shape $\Omega \subset D$,

1. We convert the elasticity problem in Ω into a two-phase elasticity problem in D thanks to the ersatz material method.

2. We calculate the ersatz material approximations u_η, p_η of u_Ω, p_Ω.

3. For "many" curve configurations σ (segments), we calculate the quantity

$$J'_\sigma(0) = \int_\sigma \mathcal{M}e(u_\Omega) : e(p_\Omega) \, \mathrm{d}\ell,$$

   measuring the sensitivity of adding a tube (a bar) with direction σ to Ω.

4. The curve σ realizing the largest negative value of $J'_\sigma(0)$ yields the "optimal" tube (bar) to be added to Ω.

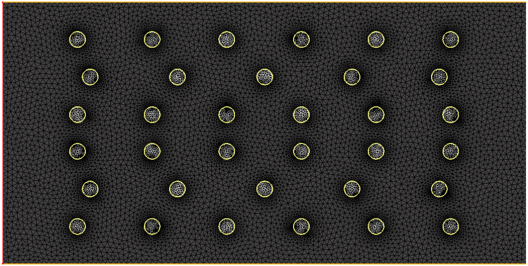
Optimal insertion of a bar in the course of a shape evolution (I)

- We minimize the compliance of a shape Ω under a volume constraint:

$$\min_{\Omega} C(\Omega) \text{ s.t. } \mathrm{Vol}\,(\Omega) \leq V_T,$$

$$\text{where } C(\Omega) := \int_{\Omega} Ae(u_{\Omega}) : e(u_{\Omega})\,\mathrm{d}x, \text{ and } \mathrm{Vol}(\Omega) = \int_{\Omega} \mathrm{d}x.$$

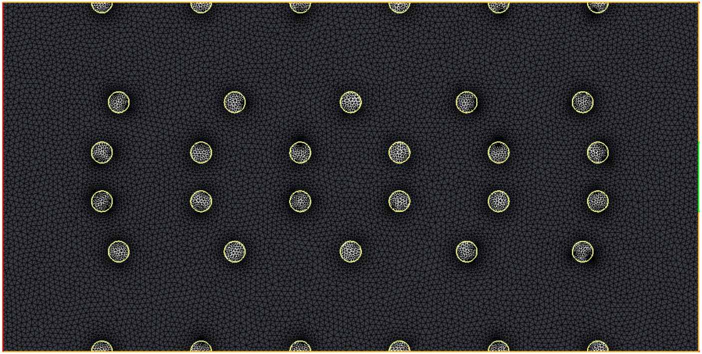
- We rely on the level set based mesh evolution method from [AlDaFre].
- Like with any boundary variation algorithm, the optimized shape is prone to falling into local minima with trivial topologies.



- To remedy this, we periodically interrupt the optimization process to insert bars.

Optimal insertion of a bar in the course of a shape evolution (II)

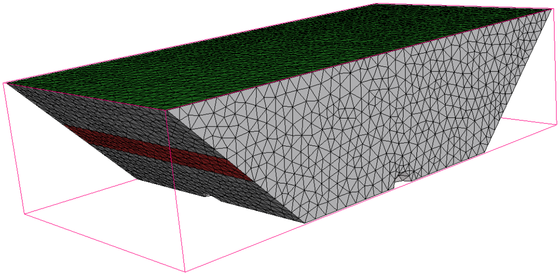
The "benchmark" 2d cantilever test case is considered.



- The shape Ω is optimized with a boundary variation algorithm.
- Every now and then, the process in interrupted and a bar is added to Ω at an "optimal location".

Optimal insertion of a bar in the course of a shape evolution (III)

The optimization of a 3d bridge Ω is considered.



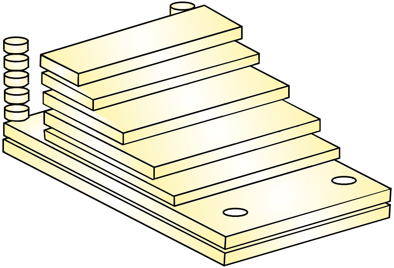
- We minimize the compliance of Ω

$$C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\,\mathrm{d}x.$$

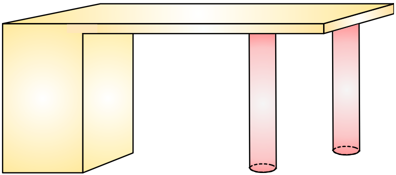
- A volume constraint is enforced.
- Every now and then, a bar is added to Ω at an "optimal location".

Optimization of supports in additive manufacturing (I)

- Additive manufacturing processes feature a layer by layer assembly of the shape Ω.
- Most of these technologies experience difficulties dealing with overhang regions.
- One remedy is to erect a scaffold structure S with Ω, such that:
  - The compliance of the total structure $\Omega \cup S$ has minimum value.
  - S has minimum volume and... it does not itself present overhangs!
- To achieve this, we propose to
  1. Incrementally add vertical pillars to Ω, made of a different material.
  2. (Optionally) Optimize S further via a more "classical" algorithm.



Layer by Layer construction of a structure by additive manufacturing



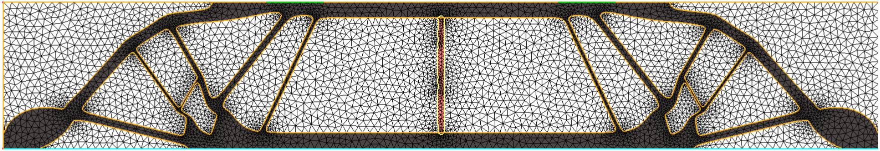
Supporting pillars for an overhang feature

The scaffold structure S of a fixed MBB beam Ω is optimized.

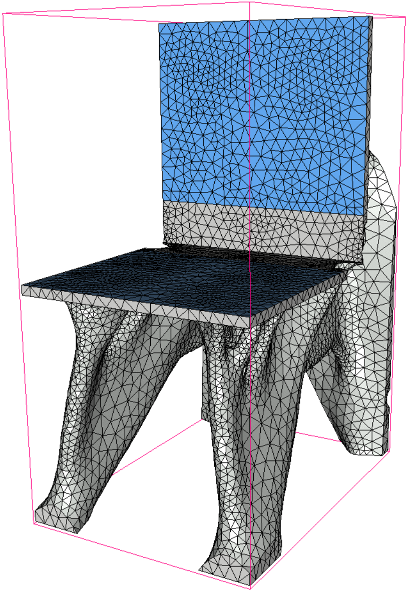
- We minimize the compliance of the total structure $\Omega \cup S$

$$C(S) = \int_{\Omega \cup S} Ae(u_{\Omega \cup S}) : e(u_{\Omega \cup S}) \, \mathrm{d}x.$$

- A constraint is imposed on the volume $\mathrm{Vol}(S)$ of supports.



Optimization of supports in additive manufacturing (III)



The scaffold structure S of a 3d chair Ω is optimized.

- The compliance of the total structure $\Omega \cup S$ is minimized:

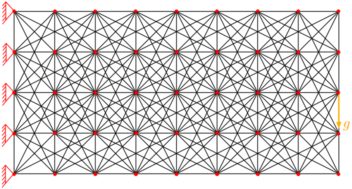
$$C(S) := \int_{\Omega \cup S} Ae(u_{\Omega \cup S}) : e(u_{\Omega \cup S}) \, \mathrm{d}x.$$

- A constraint on the volume $\mathrm{Vol}(S)$ of supports is enforced.

A "clever" initialization for truss structures (I)

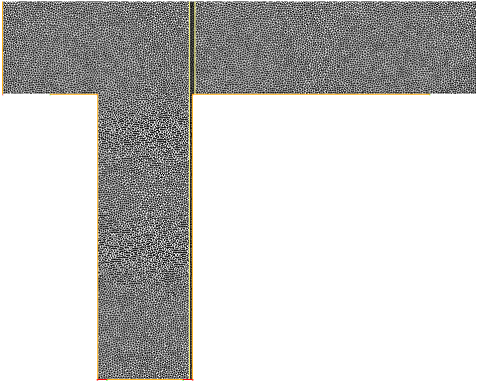
- Truss structures are collections of bars.
- Many truss optimization methods rely on the ground structure approach: an initial, dense network of bars is iteratively decimated.
- We propose instead to start from void and
  1. Incrementally add bars to the structure.
  2. (Optionally) Take on the optimization with a more "classical" boundary-variation algorithm.

Example of a truss structure



Initialization of a truss optimization algorithm by the ground structure approach

A “clever” initialization for truss structures (II)



We consider the optimization of the shape of a 2d crane Ω.

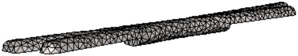
- The compliance

  $$C(\Omega) := \int_\Omega Ae(u_\Omega) : e(u_\Omega)\,\mathrm{d}x$$

  is minimized.

- A volume constraint is enforced.

A "clever" initialization for truss structures (III)



The shape of a 3d mast Ω is optimized.

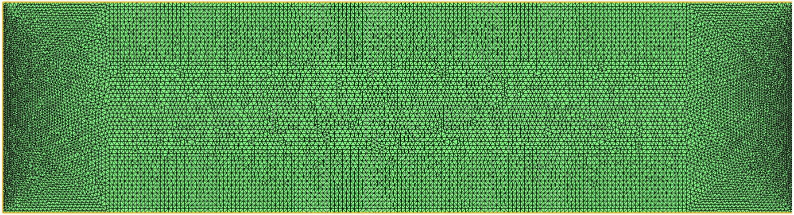
- The compliance

$$C(\Omega) := \int_{\Omega} Ae(u_{\Omega}) : e(u_{\Omega})\, \mathrm{d}x$$

  of Ω is minimized.

- A constraint is imposed on the volume of Ω.

Thank you!



References I

[AlDaFre] G. Allaire, C. Dapogny, and P. Frey, *Shape optimization with a level set based mesh evolution method*, Comput Methods Appl Mech Eng, 282 (2014), pp. 22–53.

[ASe] H. Ammari and J. K. Seo, *An accurate formula for the reconstruction of conductivity inhomogeneities*, Advances in Applied Mathematics, 30 (2003), pp. 679–705.

[AmKa] H. Ammari and H. Kang, *Reconstruction of small inhomogeneities from boundary measurements*, Springer, 2004.

[AmVoVo] H. Ammari, M. S. Vogelius, and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter ii. the full maxwell equations*, Journal de mathématiques pures et appliquées, 80 (2001), pp. 769–814.

[BeFran] E. Beretta and E. Francini, *An asymptotic formula for the displacement field in the presence of thin elastic inhomogeneities*, SIAM journal on mathematical analysis, 38 (2006), pp. 1249–1261.

References II

[BeBoFranMa] E. Beretta, E. Bonnetier, E. Francini, and A. L. Mazzucato, *Small volume asymptotics for anisotropic elastic inclusions*, Inverse Problems and Imaging, 6 (2012), pp. 1–23.

[BeFranVo] E. Beretta, E. Francini, and M. S. Vogelius, *Asymptotic formulas for steady state voltage potentials in the presence of thin inhomogeneities. a rigorous error analysis*, Journal de mathématiques pures et appliquées, 82 (2003), pp. 1277–1301.

[BeCapGoFran] E. Beretta, Y. Capdeboscq, F. De Gournay, and E. Francini, *Thin cylindrical conductivity inclusions in a three-dimensional domain: a polarization tensor and unique determination from boundary data*, Inverse Problems, 25 (2009), p. 065004.

[CapGrieKno] Y. Capdeboscq, R. Griesmaier, and M. Knöller, *An asymptotic representation formula for scattering by thin tubular structures and an application in inverse scattering*, Multiscale Modeling & Simulation, 19 (2021), pp. 846–885.

References III

[CapVo] Y. Capdeboscq and M. S. Vogelius, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, ESAIM: M2AN, 37 (2003), pp. 159–173.

[CeMoVo] D. Cedio-Fengya, S. Moskow, and M. Vogelius, *Identification of conductivity imperfections of small diameter by boundary measurements. continuous dependence and computational reconstruction*, Inverse problems, 14 (1998), p. 553.

[Da1] C. Dapogny, *A connection between topological ligaments in shape optimization and thin tubular inhomogeneities*, Comptes Rendus Mathématique, 358(2), (2020), pp. 119–127.

[Da2] C. Dapogny, *The topological ligament in shape optimization: a connection with thin tubular inhomogeneities*, submitted, (2020), `Hal` preprint: `https://tel.archives-ouvertes.fr/LJK-GI-CVGI/hal-02924929v1`.

[Grie] R. Griesmaier, *A general perturbation formula for electromagnetic fields in presence of low volume scatterers*, ESAIM: Mathematical Modelling and Numerical Analysis, 45 (2011), pp. 1193–1218.

References IV

[HenPi] A. Henrot and M. Pierre, *Shape Variation and Optimization*, EMS Tracts in Mathematics Vol. 28, 2018.

[KheZri] A. Khelifi and H. Zribi, *Asymptotic expansions for the voltage potentials with two-dimensional and three-dimensional thin interfaces*, Mathematical Methods in the Applied Sciences, 34 (2011), pp. 2274–2290.

[NaSo] S. Nazarov and J. Sokolowski, *The topological derivative of the dirichlet integral due to formation of a thin ligament*, Siberian Mathematical Journal, 45 (2004), pp. 341–355.

[NoSo] A. A. Novotny and J. Sokołowski, *Topological derivatives in shape optimization*, Springer Science & Business Media, 2012.