Small perturbations in the type of boundary conditions for an elliptic operator

(Part I)

Éric Bonnetier[1], Charles Dapogny[2], Michael Vogelius[3]

[1] Institut Fourier, Université Grenoble Alpes, Grenoble, France
[2] Laboratoire Jean Kuntzmann, Université Grenoble Alpes, Grenoble, France
[3] Department of Mathematics, Rutgers University, USA

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Foreword: "small" inhomogeneities

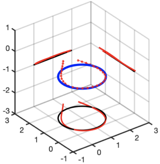
- Many analyses have been devoted to the effect of inhomogeneities occupying a "small" subset ω_ε of an ambient medium $\Omega \subset \mathbb{R}^d$.

- One typically looks for asymptotic formulas of the "physical field" u_ε when ω_ε vanishes:

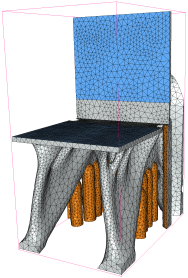
$$u_\varepsilon = u_0 + |\omega_\varepsilon| \Big(\dots \Big) + \mathrm{o}(|\omega_\varepsilon|).$$

- In practice, such formulas can be used to
    - detect small defects inside Ω,
    - optimize the placement of small bodies made of a different material.

- We investigate a variant of these problems, where the boundary conditions on u_ε are perturbed on a "small" subset $\omega_\varepsilon \subset \partial\Omega$.



Reconstruction of a "thin" electromagnetic toroidal scatterer, from [CapGrieKno].



Optimization of "thin" vertical pillars to sustain a chair, from [Da].

Small inhomogeneities: generalities (I)

To set ideas, let us consider a model problem in the conductivity setting.

- $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, filled by a material with smooth conductivity $\gamma_0 \in \mathcal{C}^\infty(\overline{\Omega})$.
- A smooth current g is applied on $\partial\Omega$ such that $\int_{\partial\Omega} g \, \mathrm{d}s = 0$.
- The "background" voltage potential u_0 is the unique $H^1_0(\Omega)$ solution such that $\int_\Omega u_0 \, \mathrm{d}x = 0$ to the boundary-value problem

$$\left\{ \begin{array}{rl} -\mathrm{div}(\gamma_0 \nabla u_0) = 0 & \text{in } \Omega, \\ \gamma_0 \frac{\partial u_0}{\partial n} = g & \text{on } \partial\Omega. \end{array} \right.$$

- In a perturbed situation, Ω contains inhomogeneities with conductivity $\gamma_1 \in \mathcal{C}^\infty(\mathbb{R}^d)$, occupying a "small" subset $\omega_\varepsilon \Subset \Omega$.
- The perturbed potential $u_\varepsilon \in H^1(\Omega)$ satisfies $\int_\Omega u_\varepsilon \, \mathrm{d}x = 0$ and

$$\left\{ \begin{array}{rl} -\mathrm{div}(\gamma_\varepsilon \nabla u_\varepsilon) = 0 & \text{in } \Omega, \\ \gamma_0 \frac{\partial u_\varepsilon}{\partial n} = g & \text{on } \partial\Omega, \end{array} \right. \quad \text{where } \gamma_\varepsilon(x) := \left\{ \begin{array}{ll} \gamma_1(x) & \text{if } x \in \omega_\varepsilon, \\ \gamma_0(x) & \text{otherwise.} \end{array} \right.$$

Small inhomogeneities: generalities (II)

- A general representation formula for u_ε in the low-volume limit $|\omega_\varepsilon| \to 0$ was derived in [CapVo]: for $x \in \partial\Omega$, and a subsequence of the ε,

$$u_\varepsilon(x) = u_0(x) + |\omega_\varepsilon| \int_\Omega (\gamma_1 - \gamma_0)(y)\mathcal{M}(y)\nabla u_0(y) \cdot \nabla_y N(x,y)\, \mathrm{d}\mu(y) + \mathrm{o}(|\omega_\varepsilon|),$$

where

  - The probability measure μ describes the "limiting" position of the subsets ω_ε.
  - The polarization tensor $\mathcal{M}(y)$ accounts for the "limiting behavior" of a rescaled version of the field u_ε inside ω_ε.
  - $N(x,y)$ is the Neumann function of the background problem.

- The relevant quantity to measure the "smallness" of ω_ε is the volume $|\omega_\varepsilon|$.

- This formula can be refined further when particular geometries are assumed for ω_ε.

Y. Capdeboscq and M.S. Vogelius, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, ESAIM: M2AN, 37(1), (2003), pp. 159–173.

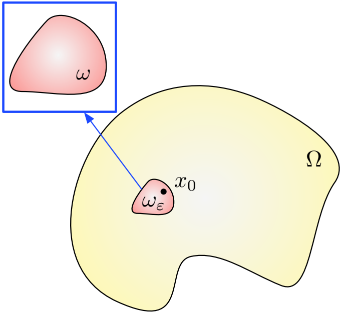
Small inhomogeneities: examples

❶ Diametrically small inhomogeneities read:

$$\omega_\varepsilon = x_0 + \varepsilon\omega,$$

where $x_0 \in \Omega$ and ω is a bounded subset of $\mathbb{R}^d$.

- μ is a multiple of δ_{x_0},
- $\mathcal{M}$ involves the solution to an exterior problem posed on ω and $\mathbb{R}^d \setminus \overline{\omega}$.
- *References:* [CeMoVo] [ASe]

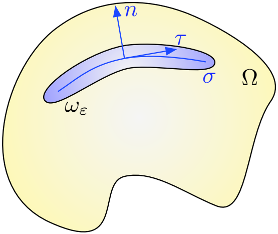


❷ Thin inhomogeneities are of the form

$$\omega_\varepsilon = \left\{ x \in \mathbb{R}^d, \;\; d(x,\sigma) < \varepsilon \right\},$$

where $\sigma \Subset \Omega$ is a (open or closed) hypersurface.

- μ is the integration measure on σ,
- $\mathcal{M}$ is diagonal in a local basis $(\tau_1, \ldots, \tau_{d-1}, n)$ attached to σ.
- *References:* [BeFranVo] [KheZri]



Small inhomogeneities: extensions and applications

- These questions have been considered in various more challenging physical settings, such as
  - that of the linearized elasticity equations [BeFran, BeBoFranMa];
  - that of the Maxwell system [AmVoVo, Grie].
- These asymptotic formulas pave the way to multiple numerical methods for the detection or the reconstruction of small inhomogeneities [AmKa].
- They also allow for the optimization of the placement and shape of inhomogeneities:
  - Topological derivatives in shape optimization [NoSo].
  - Optimization of the placement of tubular inhomogeneities [Da].

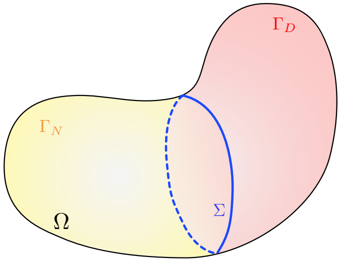
Setting of the present work (I)

We study a variant of the above framework: the boundary conditions attached to an elliptic operator are modified on a “small” subset ω_ε of the boundary $\partial\Omega$.

Interpretation:

- When Ω is a dielectric medium, this allows to study the impact of replacing a region of $\partial\Omega$ where the domain is insulated by a “ground”, and vice-versa.
- When Ω is an elastic structure, this accounts for the effect of adding a new clamping zone within a traction-free region of $\partial\Omega$ (or the other way around).

Setting of the present work (II)



- Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain.
- The boundary $\partial\Omega$ is decomposed as

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \quad \Gamma_D \cap \Gamma_N = \emptyset,$$

and $\Sigma = \overline{\Gamma_D} \cap \overline{\Gamma_N}$ denotes the interface between Γ_D and Γ_N.

- Ω is filled with a material with smooth conductivity $\gamma \in \mathcal{C}^\infty(\overline{\Omega})$, satisfying:

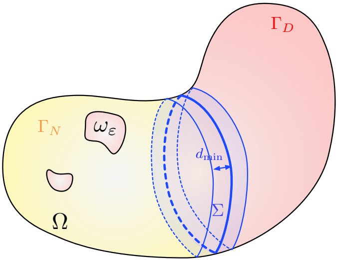
$$\forall x \in \Omega, \quad \alpha \leq \gamma(x) \leq \beta, \text{ for some fixed constants } 0 < \alpha \leq \beta.$$

- A smooth external source $f \in \mathcal{C}^\infty(\overline{\Omega})$ is at play.

The "background" potential u_0 is then the unique $H^1(\Omega)$ solution to the problem

$$\begin{cases} -\operatorname{div}(\gamma \nabla u_0) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u_0}{\partial n} = 0 & \text{on } \Gamma_N. \end{cases} \qquad \text{(BG)}$$

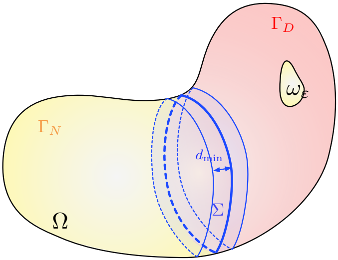
- ω_ε is a "small" Lipschitz subset of the Neumann region Γ_N.
- ω_ε is "well-separated" from Γ_D.



The "perturbed" voltage potential u_ε is the unique $H^1(\Omega)$ solution to:

$$\begin{cases} -\mathrm{div}(\gamma \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_D \cup \omega_\varepsilon, \\ \gamma \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon}. \end{cases} \qquad (\mathrm{D}_\varepsilon)$$

- ω_ε is a "small" Lipschitz subset of the Dirichlet region Γ_D.
- ω_ε is "well-separated" from Γ_N.



The "perturbed" voltage potential u_ε is the unique $H^1(\Omega)$ solution to:

$$\begin{cases} -\mathrm{div}(\gamma \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_D \setminus \overline{\omega_\varepsilon}, \\ \gamma \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \cup \omega_\varepsilon. \end{cases} \qquad (\mathrm{N}_\varepsilon)$$

Foreword

Objectives of this work:

1. **(Part I: C.D.)** Find a general representation formula

$$u_\varepsilon = u_0 + \rho(\omega_\varepsilon)\Big(\dots\Big) + \mathrm{o}(\rho(\omega_\varepsilon))$$

under minimal assumptions on ω_ε, except that it be "small".

$\Rightarrow$ What is the relevant quantity $\rho(\omega_\varepsilon)$ to measure the "smallness" of ω_ε?

2. **(Part II: Eric Bonnetier)** Derive *explicit* representation formulas in specific situations as regards the geometry of ω_ε.

A few related references:

- The case where $\Gamma_D = \emptyset$ and ω_ε is a "small disk" is referred to as the "Narrow escape problem", see [HoSchu] for an overview and [CheFrie, CheWaStrau] [AmKaKaLee, Li] for asymptotic formulas.
- The case where $\Gamma_N = \emptyset$ and ω_ε is a "small disk" has applications in the theory of metasurfaces, see [KaDuFinLe] about the physical context and [AmIWu, Ga] for mathematical analyses.

Sobolev spaces on the boundary $\partial\Omega$

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain; for any real number $0 < s < 1$ [McLea],

- The Sobolev space $H^s(\partial\Omega)$ is associated to the norm

$$||v||^2_{H^s(\partial\Omega)} = ||v||^2_{L^2(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{d-1+2s}} \, \mathrm{d}s(x)\mathrm{d}s(y).$$

- The Sobolev space $H^{-s}(\partial\Omega)$ is the topological dual of $H^s(\partial\Omega)$; it is equipped with the norm

$$||w||_{H^{-s}(\partial\Omega)} = \sup_{\substack{v \in H^s(\partial\Omega) \\ ||v||_{H^s(\partial\Omega)} = 1}} \langle w, v \rangle.$$

W. C. H. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge university press, 2000.

Sobolev spaces on a Lipschitz subset $\Gamma \subset \partial\Omega$

Let $\Gamma \subset \partial\Omega$ be a Lipschitz subset, i.e. a finite collection of disjoint open, Lipschitz subdomains of $\partial\Omega$.

For $-1 < s < 1$, we distinguish two different Sobolev spaces on Γ:

- $H^s(\Gamma)$ is the space of the restrictions of $H^s(\partial\Omega)$ functions to Γ. It is equipped by the quotient norm induced by $||\cdot||_{H^s(\partial\Omega)}$.
- $\widetilde{H}^s(\Gamma)$ is the space of distributions in $H^s(\partial\Omega)$ with compact support inside $\overline{\Gamma}$. It is equipped with the norm $||\cdot||_{H^s(\partial\Omega)}$.

For any such real number s, $\widetilde{H}^{-s}(\Gamma)$ can be identified with the dual space of $H^s(\Gamma)$:

$$\forall u \in \widetilde{H}^{-s}(\Gamma), \;\; v \in H^s(\Gamma),$$

$$\langle u, v\rangle_{\widetilde{H}^{-s}(\Gamma),H^s(\Gamma)} = \langle \underbrace{\widetilde{u}}_{\substack{\textbf{Extension of } u \textbf{ by 0} \\ \textbf{from } \Gamma \textbf{ to } \partial\Omega \; (\in H^{-s}(\partial\Omega))}} , \underbrace{w}_{\substack{\textbf{Any } w\in H^s(\partial\Omega) \\ \textbf{s.t. } w|_\Gamma = v}} \rangle_{H^{-s}(\partial\Omega),H^s(\partial\Omega)}.$$

The background potential u_0

- The background potential is the unique solution $u_0 \in H^1(\Omega)$ to the mixed boundary value problem

$$\left\{ \begin{array}{ll} -\mathrm{div}(\gamma \nabla u_0) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u_0}{\partial n} = 0 & \text{on } \Gamma_N. \end{array} \right.$$

- Its variational formulation reads:

$$\forall v \in H^1(\Omega) \text{ s.t. } v = 0 \text{ on } \Gamma_D, \quad \int_\Omega \gamma \nabla u_0 \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x.$$

- Owing to elliptic regularity, u_0 is smooth in a vicinity V of every point $x \in \overline{\Omega} \setminus \Sigma$:

$$\text{For all } \; m \in \mathbb{N}, \quad u_0 \in H^{m+2}(\Omega), \text{ and } ||u_0||_{H^{m+2}(\Omega)} \leq C_m ||f||_{H^m(\Omega)}.$$

- The trace of u_0 vanishes on Γ_D, so that $u_0|_{\Gamma_N} \in \widetilde{H}^{1/2}(\Gamma_N)$.
- The normal derivative $\gamma \frac{\partial u_0}{\partial n}$ vanishes as an element in $H^{-1/2}(\Gamma_N)$ and so $\gamma \frac{\partial u_0}{\partial n} \in \widetilde{H}^{-1/2}(\Gamma_D)$.

The fundamental solution to the background equation

- The fundamental solution $N(x, y)$ to the background equation satisfies: for $x \in \Omega$, the function $y \mapsto N(x, y)$ is the solution to

$$\begin{cases} -\mathrm{div}_y(\gamma(y)\nabla_y N(x, y)) = \delta_{y=x} & \text{in } \Omega\ , \\ N(x, y) = 0 & \text{for } y \in \Gamma_D\ , \\ \gamma(y)\frac{\partial N}{\partial n_y}(x, y) = 0 & \text{for } y \in \Gamma_N\ . \end{cases}$$

- Equivalently, $x \mapsto N(x, y)$ satisfies the following "variational formulation": for any function $\varphi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$ with $\varphi = 0$ on Γ_D,

$$\varphi(x) = \int_\Omega \gamma(y)\nabla\varphi(y) \cdot \nabla_y N(x, y)\, \mathrm{d}y, \quad x \in \overline{\Omega}.$$

- It is symmetric in its arguments: $N(x, y) = N(y, x)$ for $x, y \in \Omega$, $x \neq y$.

- $N(x, y)$ can be constructed from the Green's function for the Laplace equation.

The capacity of a subset: definition

The relevant quantity to measure the "smallness" of ω_ε when it accounts for the replacement of Neumann B.C with Dirichlet B.C is that of capacity [HenPi, Lan].

Definition 1.

The capacity $\mathrm{cap}(E)$ *of an arbitrary subset* $E \subset \mathbb{R}^d$ *is defined by:*

$$\mathrm{cap}(E) = \inf \left\{ ||v||^2_{H^1(\mathbb{R}^d)}, \;\; v(x) \geq 1 \textit{ a.e. on an open neighborhood of } E \right\}.$$

<u>*Intuition:*</u> Loosely speaking, $\mathrm{cap}(E)$ is the energy of the function v such that

- v equals 1 on E;
- v "decreases at ∞";
- v is harmonic in $\mathbb{R}^d \setminus E$.

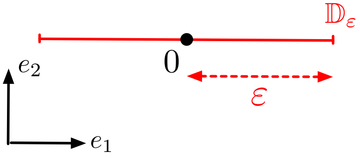
A. Henrot and M. Pierre, *Shape Variation and Optimization*, EMS Tracts in Mathematics, Vol. 28, 2018.
N. S. Landkof, *Foundations of modern potential theory*, vol. 180, Springer, 1972.

Let the subset $\mathbb{D}_\varepsilon \subset \mathbb{R}^d$ be defined by:

$$\mathbb{D}_\varepsilon := \left\{ x = (x_1, \ldots, x_{d-1}, 0) \in \mathbb{R}^d, \ |x| < \varepsilon \right\},$$

i.e.

- $\mathbb{D}_\varepsilon$ is a segment with length 2ε if $d = 2$;
- $\mathbb{D}_\varepsilon$ is a planar disk with radius ε if $d = 3$.

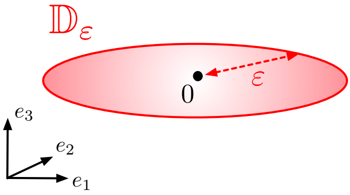


$\mathbb{D}_\varepsilon$ when $d = 2$

The capacity of $\mathbb{D}_\varepsilon$ satisfies:

- If $d = 2$, $\mathrm{cap}(\mathbb{D}_\varepsilon) \le \frac{C_2}{|\log \varepsilon|}$,
- If $d = 3$, $\mathrm{cap}(\mathbb{D}_\varepsilon) \le C_3 \varepsilon$,

where C_2 and C_3 are universal constants.

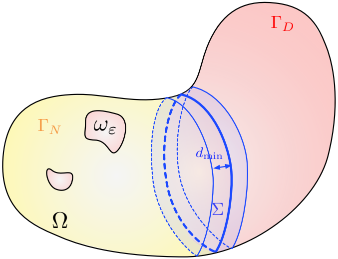


$\mathbb{D}_\varepsilon$ when $d = 3$

The Dirichlet case: setting

- The ω_ε are open Lipschitz subsets of $\partial\Omega$.
- They are all contained in Γ_N, and stay well-separated from Σ:

$$\exists\, d_{\min} > 0 \text{ s.t. } \forall \varepsilon > 0 \quad \mathrm{dist}(\omega_\varepsilon, \Sigma) \geq d_{\min} \quad (\mathrm{S})$$



- The background and perturbed potentials u_0 and $u_\varepsilon \in H^1(\Omega)$ are the solutions to:

$$\begin{cases} -\mathrm{div}(\gamma \nabla u_0) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u_0}{\partial n} = 0 & \text{on } \Gamma_N, \end{cases} \quad \text{and} \quad \begin{cases} -\mathrm{div}(\gamma \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_D \cup \omega_\varepsilon, \\ \gamma \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon}. \end{cases}$$

Theorem 1 (Representation formula in the Dirichlet case).

Assume that $\mathrm{cap}(\omega_\varepsilon) \to 0$*. Then there exists a subsequence, still denoted by* ε*, and a Radon measure* μ *on* $\partial\Omega$ *such that:*

$$u_\varepsilon(x) = u_0(x) - \mathrm{cap}(\omega_\varepsilon) \int_{\partial\Omega} u_0(y)\gamma(y)N(x,y)\,\mathrm{d}\mu(y) + \mathrm{o}(\mathrm{cap}(\omega_\varepsilon)) \text{ for } x \in \Omega.$$

Here,

- *The measure* μ *is non trivial and non negative; it depends only on the subsequence* ω_ε*,* Ω*, and* Γ_N*;*
- *The support of* μ *lies inside any compact subset* $K \subset \partial\Omega$ *containing all the* ω_ε *for* $\varepsilon > 0$ *small enough;*
- *The term* $\mathrm{o}(\mathrm{cap}(\omega_\varepsilon))$ *is uniform when* x *is confined on compact subsets of* Ω*.*

Sketch of the proof.

The error

$$r_\varepsilon := u_\varepsilon - u_0$$

between the perturbed and the background potentials is the unique $H^1(\Omega)$ solution to

$$\begin{cases} -\mathrm{div}(\gamma \nabla r_\varepsilon) = 0 & \text{in } \Omega \ , \\ r_\varepsilon = -u_0 & \text{on } \omega_\varepsilon \ , \\ r_\varepsilon = 0 & \text{on } \Gamma_D \ , \\ \gamma \frac{\partial r_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon} \ . \end{cases}$$

The proof is divided into seven steps.

Step 1: Construction of a suitable "capacity function".

Let χ_ε be the unique solution in $H^1(\Omega)$ to the problem:

$$\begin{cases} -\Delta\chi_\varepsilon = 0 & \text{in } \Omega \ , \\ \chi_\varepsilon = 1 & \text{on } \omega_\varepsilon \ , \\ \chi_\varepsilon = 0 & \text{on } \Gamma_D \ , \\ \frac{\partial \chi_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon} \ , \end{cases}$$

or, under variational form: $\chi_\varepsilon \in H^1(\Omega)$ is such that $\chi_\varepsilon = 0$ on Γ_D, $\chi_\varepsilon = 1$ on ω_ε and

$$\forall v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_D \cup \omega_\varepsilon, \quad \int_\Omega \nabla\chi_\varepsilon \cdot \nabla v \, \mathrm{d}x = 0 \ .$$

Lemma 2.

There exist two constants $0 < m \leq M$ which are independent of ω_ε such that

$$m \, \mathrm{cap}(\omega_\varepsilon) \leq ||\chi_\varepsilon||^2_{H^1(\Omega)} \leq M \, \mathrm{cap}(\omega_\varepsilon) \ .$$

The representation formula: Step 2 (I)

Step 2: H^1 a priori estimates and improved L^2 estimates.

We now consider the solution $v_\varepsilon \in H^1(\Omega)$ to the boundary value problem:

$$\begin{cases} -\mathrm{div}(\gamma \nabla v_\varepsilon) = 0 & \text{in } \Omega, \\ v_\varepsilon = g & \text{on } \omega_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon}\ , \end{cases}$$

where g is a given function in $C^1(\overline{\Omega})$.

Lemma 3.

There exists a constant M which is independent of ω_ε such that:

$$||v_\varepsilon||_{H^1(\Omega)} \leq M||g||_{\mathcal{C}^1(\overline{\Omega})} \mathrm{cap}(\omega_\varepsilon)^{\frac{1}{2}}.$$

In addition, v_ε satisfies the improved L^2 estimate

$$||v_\varepsilon||_{L^2(\Omega)} \leq M||g||_{\mathcal{C}^1(\overline{\Omega})} \mathrm{cap}(\omega_\varepsilon)^{\frac{3}{4}}.$$

The representation formula: Step 2 (II)

The variational formulation for v_ε reads: $v_\varepsilon = g$ on ω_ε, $v_\varepsilon = 0$ on Γ_D, and:

$$\forall w \in H^1(\Omega) \text{ s.t. } w = 0 \text{ on } \Gamma_D \cup \omega_\varepsilon, \quad \int_\Omega \gamma \nabla v_\varepsilon \cdot \nabla w \, \mathrm{d}x = 0.$$

Proof of the H^1 estimate: We simply remark that $(v_\varepsilon - g\chi_\varepsilon)$ vanishes on ω_ε, and so

$$\int_\Omega \gamma \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, \mathrm{d}x = \int_\Omega \gamma \nabla v_\varepsilon \cdot \nabla (g\chi_\varepsilon) \, \mathrm{d}x.$$

The result follows easily from the estimate $m \, \mathrm{cap}(\omega_\varepsilon) \leq ||\chi_\varepsilon||^2_{H^1(\Omega)} \leq M \, \mathrm{cap}(\omega_\varepsilon)$.

Proof of the improved L^2 estimate: We rely on the Aubin-Nitsche trick [Au].

- Let $w_\varepsilon \in H^1(\Omega)$ be the solution to

$$\begin{cases} -\mathrm{div}(\gamma \nabla w_\varepsilon) = v_\varepsilon & \text{in } \Omega, \\ w_\varepsilon = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial w_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N. \end{cases}$$

  Let $\eta \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ be a smooth cutoff function such that

$$\eta = 1 \text{ on all the } \omega_\varepsilon \text{ and } \eta = 0 \text{ on an open set } U \text{ with } \Gamma_D \Subset U.$$

The representation formula: Step 2 (III)

- The following estimate stems from elliptic regularity applied to w_ε:

$$\eta w_\varepsilon \in H^3(\Omega), \text{ with } ||\eta w_\varepsilon||_{H^3(\Omega)} \subset C||v_\varepsilon||_{H^1(\Omega)}.$$

- We then calculate

$$\begin{aligned} \int_\Omega v_\varepsilon^2 \,\mathrm{d}x &= \int_\Omega \gamma \nabla w_\varepsilon \cdot \nabla v_\varepsilon \,\mathrm{d}x, \\ &= \int_\Omega \gamma \nabla(\eta \chi_\varepsilon w_\varepsilon) \cdot \nabla v_\varepsilon \,\mathrm{d}x, \end{aligned}$$

  where the last line uses the fact that $(1 - \eta\chi_\varepsilon) w_\varepsilon = 0$ on $\Gamma_D \cup \omega_\varepsilon$.

- Finally, using the Sobolev imbedding theorem,

$$\begin{aligned} ||v_\varepsilon||_{L^2(\Omega)}^2 &\leq C||v_\varepsilon||_{H^1(\Omega)} ||\chi_\varepsilon||_{H^1(\Omega)} ||\eta w_\varepsilon||_{\mathcal{C}^1(\overline{\Omega})} \\ &\leq C||v_\varepsilon||_{H^1(\Omega)} ||\chi_\varepsilon||_{H^1(\Omega)} ||\eta w_\varepsilon||_{H^3(\Omega)} \\ &\leq C \mathrm{cap}(\omega_\varepsilon)^{\frac{1}{2}} ||v_\varepsilon||_{H^1(\Omega)}^2 \end{aligned}$$

  and we conclude thanks to the H^1 estimate.

The representation formula: Step 3

Step 3: Representation of r_ε in terms of the fundamental solution $N(x,y)$.

Introducing $N(x,y)$ and integrating by parts twice, we obtain for any $x \in \Omega$

$$\begin{aligned} r_\varepsilon(x) &= \int_\Omega r_\varepsilon(y)(-\mathrm{div}_y(\gamma(y)\nabla_y N(x,y)))\,\mathrm{d}y \\ &= \int_\Omega \gamma(y)\nabla r_\varepsilon(y)\cdot\nabla_y N(x,y)\,\mathrm{d}y - \int_{\partial\Omega}\gamma(y)\frac{\partial N}{\partial n_y}(x,y)r_\varepsilon(y)\,\mathrm{d}s(y) \\ &= \int_{\partial\Omega}\gamma(y)\frac{\partial r_\varepsilon}{\partial n}(y)N(x,y)\,\mathrm{d}s(y) - \int_{\partial\Omega}\gamma(y)\frac{\partial N}{\partial n_y}(x,y)r_\varepsilon(y)\,\mathrm{d}s(y)\,. \end{aligned}$$

Since

- $y \mapsto \frac{\partial N}{\partial n_y}(x,y)$ vanishes on Γ_N (i.e. as an element in $H^{-1/2}(\Gamma_N)$),
- r_ε vanishes on Γ_D (i.e., $r_\varepsilon \in \widetilde{H}^{1/2}(\Gamma_N)$),

the second integral in the above right-hand side equals 0, and so

$$\boxed{r_\varepsilon(x) = \int_{\partial\Omega}\gamma(y)\frac{\partial r_\varepsilon}{\partial n}(y)N(x,y)\,\mathrm{d}s(y), \quad x \in \Omega.}$$

The representation formula: Step 4 (I)

Step 4: Compensated compactness. This step is inspired by [CapVo, MuTar].

- We evaluate the effect of integrating the normal derivative $\gamma \frac{\partial r_\varepsilon}{\partial n}$ against a general function $\phi \in \mathcal{C}^1(\partial\Omega)$, i.e. we transform the expression

$$\int_{\partial\Omega} \gamma(y) \frac{\partial r_\varepsilon}{\partial n}(y) \phi(y) \, \mathrm{d}s(y).$$

- Let $\psi \in \mathcal{C}^1(\overline{\Omega})$ be such that $\psi = \phi$ on $\partial\Omega$ and $||\psi||_{\mathcal{C}^1(\overline{\Omega})} \leq C||\phi||_{\mathcal{C}^1(\partial\Omega)}$.

- Since $\gamma \frac{\partial r_\varepsilon}{\partial n} = 0$ in $H^{-1/2}(\Gamma_N \setminus \overline{\omega_\varepsilon})$ and $\chi_\varepsilon = \begin{cases} 1 & \text{on } \omega_\varepsilon, \\ 0 & \text{on } \Gamma_D, \end{cases}$ it holds:

$$\int_{\partial\Omega} \gamma \frac{\partial r_\varepsilon}{\partial n} \phi \, \mathrm{d}s = \int_{\partial\Omega} \gamma \frac{\partial r_\varepsilon}{\partial n} \chi_\varepsilon \psi \, \mathrm{d}s.$$

Y. Capdeboscq and M.S. Vogelius, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, ESAIM: M2AN, 37(1), (2003), pp. 159–173. .

F. Murat and L. Tartar, *H-convergence*, in Topics in the mathematical modelling of composite materials, Springer, (2018), pp. 21–43.

The representation formula: Step 4 (II)

- An integration by parts yields:

$$\int_{\partial\Omega} \gamma \frac{\partial r_\varepsilon}{\partial n} \phi \, \mathrm{d}s = \int_\Omega \gamma \psi \nabla r_\varepsilon \cdot \nabla \chi_\varepsilon \, \mathrm{d}x + \int_\Omega \gamma \chi_\varepsilon \nabla r_\varepsilon \cdot \nabla \psi \, \mathrm{d}x$$
$$= \int_\Omega \gamma \psi \nabla r_\varepsilon \cdot \nabla \chi_\varepsilon \, \mathrm{d}x + \mathcal{O}(\mathrm{cap}(\omega_\varepsilon)^{\frac{5}{4}}) ||f||_{H^m(\Omega)} ||\phi||_{\mathcal{C}^1(\partial\Omega)},$$

where we have used the H^1 estimate for r_ε and the improved L^2 estimate for χ_ε.

- By the same token,

$$\int_{\partial\Omega} \gamma \frac{\partial r_\varepsilon}{\partial n} \phi \, \mathrm{d}s = \int_\Omega \nabla(\gamma \psi r_\varepsilon) \cdot \nabla \chi_\varepsilon \, \mathrm{d}x + \mathcal{O}(\mathrm{cap}(\omega_\varepsilon)^{\frac{5}{4}}) ||f||_{H^m(\Omega)} ||\phi||_{\mathcal{C}^1(\partial\Omega)}.$$

- By another integration by parts,

$$\int_{\partial\Omega} \gamma \frac{\partial r_\varepsilon}{\partial n} \phi \, \mathrm{d}s = \int_{\partial\Omega} \gamma \psi \frac{\partial \chi_\varepsilon}{\partial n} r_\varepsilon \, \mathrm{d}s + \mathcal{O}(\mathrm{cap}(\omega_\varepsilon)^{\frac{5}{4}}) ||f||_{H^m(\Omega)} ||\phi||_{\mathcal{C}^1(\partial\Omega)}.$$

Since $r_\varepsilon = -u_0\chi_\varepsilon$ on $\Gamma_D \cup \omega_\varepsilon$ and $\frac{\partial \chi_\varepsilon}{\partial n} = 0$ on $\Gamma_N \setminus \overline{\omega_\varepsilon}$, we end up with the expression:

$$\int_{\partial\Omega} \gamma \frac{\partial r_\varepsilon}{\partial n} \phi \,\mathrm{d}s = -\int_{\partial\Omega} \frac{\partial \chi_\varepsilon}{\partial n} \chi_\varepsilon u_0 \gamma \phi \,\mathrm{d}s \; + \; \mathcal{O}(\mathrm{cap}(\omega_\varepsilon)^{\frac{5}{4}}) \|f\|_{H^m(\Omega)} \|\phi\|_{C^1(\partial\Omega)} \; .$$

The representation formula: Step 5

Step 5: Definition of the limiting distribution μ.

- Let us recall the following estimates over the "capacity function" χ_ε:

$$m\,\mathrm{cap}(\omega_\varepsilon) \leq ||\chi_\varepsilon||^2_{H^1(\mathbb{R}^d)} \leq M\,\mathrm{cap}(\omega_\varepsilon).$$

- It follows that, for any $\phi \in \mathcal{C}^1(\partial\Omega)$

$$\left| \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \int_{\partial\Omega} \frac{\partial \chi_\varepsilon}{\partial n} \chi_\varepsilon \phi \,\mathrm{d}s \right| = \left| \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \int_{\Omega} \nabla \chi_\varepsilon \cdot \nabla(\chi_\varepsilon \psi) \,\mathrm{d}y \right| \leq C ||\phi||_{\mathcal{C}^1(\partial\Omega)} .$$

- From the Banach-Alaoglu theorem, up to a subsequence (still labelled by ε), there exists a bounded linear functional μ on $\mathcal{C}^1(\partial\Omega)$ such that:

$$\forall \phi \in \mathcal{C}^1(\partial\Omega), \quad \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \int_{\partial\Omega} \frac{\partial \chi_\varepsilon}{\partial n} \chi_\varepsilon \phi \,\mathrm{d}s \xrightarrow{\varepsilon \to 0} \mu(\phi) .$$

- For now, μ is only a distribution of order 1 on $\partial\Omega$.

Step 6: Conclusion

Let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ be a smooth cut-off function such that

$$\eta \equiv 0 \text{ on } \left\{x,\ \mathrm{dist}(x,\Gamma_D) < \frac{d_{\min}}{3}\right\} \text{ and } \eta \equiv 1 \text{ on } \left\{x,\ \mathrm{dist}(x,\Gamma_D) > \frac{d_{\min}}{2}\right\}.$$

We return to the representation formula for $r_\varepsilon(x)$ in terms of $N(x,y)$, and use the above result with the smooth function $\phi(\cdot) = N(x,\cdot)\eta(\cdot)$:

$$\begin{aligned} r_\varepsilon(x) &= \int_{\partial\Omega} \gamma(y)\frac{\partial r_\varepsilon}{\partial n}(y)N(x,y)\,\mathrm{d}s(y), \\ &= \int_{\partial\Omega} \gamma(y)\frac{\partial r_\varepsilon}{\partial n}(y)N(x,y)\eta(y)\,\mathrm{d}s(y), \\ &= -\int_{\partial\Omega} \frac{\partial \chi_\varepsilon}{\partial n}(y)\chi_\varepsilon(y)u_0(y)\gamma(y)N(x,y)\eta(y)\,\mathrm{d}s(y) \\ &\qquad + \mathcal{O}(\mathrm{cap}(\omega_\varepsilon)^{\frac{5}{4}})\|f\|_{H^m(\Omega)}\|N(x,\cdot)\eta(\cdot)\|_{C^1(\partial\Omega)} . \end{aligned}$$

Finally, we obtain the representation formula

$$\boxed{r_\varepsilon(x) = -\mathrm{cap}(\omega_\varepsilon)\mu_y\left[\eta(y)u_0(y)\gamma(y)N(x,y)\right] + \mathrm{o}(\mathrm{cap}(\omega_\varepsilon)) .}$$

The representation formula: Step 7 (I)

Step 7: Refined properties of the limiting distribution μ.

- The distribution μ is defined by the limit:

$$\forall \phi \in \mathcal{C}^1(\partial\Omega), \quad \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \int_{\partial\Omega} \frac{\partial \chi_\varepsilon}{\partial n} \chi_\varepsilon \phi \, \mathrm{d}s \xrightarrow{\varepsilon \to 0} \mu(\phi) \; .$$

  A priori, μ is only a distribution of order 1 on $\partial\Omega$: it may depend on the derivatives of ϕ.

- Using again the estimates over χ_ε

$$m \, \mathrm{cap}(\omega_\varepsilon) \leq ||\chi_\varepsilon||^2_{H^1(\mathbb{R}^d)} \leq M \, \mathrm{cap}(\omega_\varepsilon).$$

  we obtain that:

$$\mu(1) = \lim_{\varepsilon \to 0} \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \int_{\partial\Omega} \frac{\partial \chi_\varepsilon}{\partial n} \chi_\varepsilon \, \mathrm{d}s \geq m$$

  and so μ is non-trivial.

- Since $\chi_\varepsilon = 0$ on Γ_D and $\frac{\partial \chi_\varepsilon}{\partial n} = 0$ on $\Gamma_N \setminus \overline{\omega_\varepsilon}$, the support of μ is included in any compact subset $K \subset \partial\Omega$ containing all the ω_ε.

The representation formula: Step 7 (II)

We now prove that μ is actually a non negative Radon measure on $\partial\Omega$.

- Since Ω is smooth, there exists an extension operator $E : \mathcal{C}^1(\partial\Omega) \to \mathcal{C}^1(\overline{\Omega})$:

$$\forall \phi \in \mathcal{C}^1(\partial\Omega), \quad E(\phi) = \phi \text{ on } \partial\Omega, \text{ and } ||E(\phi)||_{\mathcal{C}^0(\overline{\Omega})} = ||\phi||_{\mathcal{C}^0(\partial\Omega)}.$$

- We use Green's formula to transform the defining expression for μ:

$$\int_{\partial\Omega} \frac{\partial \chi_\varepsilon}{\partial n} \chi_\varepsilon \phi \, \mathrm{d}s = \int_\Omega (\nabla \chi_\varepsilon \cdot \nabla \chi_\varepsilon) E(\phi) \, \mathrm{d}x + \int_\Omega (\nabla \chi_\varepsilon \cdot \nabla(E(\phi))) \chi_\varepsilon \, \mathrm{d}x \ .$$

  Using the improved L^2 estimate over χ_ε, we obtain:

$$\lim_{\varepsilon \to 0} \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \int_\Omega (\nabla \chi_\varepsilon \cdot \nabla(E(\phi))) \chi_\varepsilon \, \mathrm{d}x = 0 \ ,$$

  and as a consequence:

$$\forall \phi \in \mathcal{C}^1(\partial\Omega), \quad \mu(\phi) = \lim_{\varepsilon \to 0} \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \int_\Omega (\nabla \chi_\varepsilon \cdot \nabla \chi_\varepsilon) E(\phi) \, \mathrm{d}x.$$

The representation formula: Step 7 (III)

- On the other hand, using the estimate $||\chi_\varepsilon||^2_{H^1(\Omega)} \leq M\,\mathrm{cap}(\omega_\varepsilon)$, it holds

$$\forall \psi \in \mathcal{C}^0(\overline{\Omega}), \;\; \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \left| \int_\Omega (\nabla\chi_\varepsilon \cdot \nabla\chi_\varepsilon)\psi \,\mathrm{d}x \right| \leq M||\psi||_{\mathcal{C}^0(\overline{\Omega})}.$$

The Banach-Alaoglu theorem implies that there exists a subsequence of the ε's and a non negative Radon measure ν on $\overline{\Omega}$ such that

$$\forall \psi \in \mathcal{C}^0(\overline{\Omega}), \;\; \frac{1}{\mathrm{cap}(\omega_\varepsilon)} \int_\Omega (\nabla\chi_\varepsilon \cdot \nabla\chi_\varepsilon)\psi \,\mathrm{d}x \to \int_\Omega \psi \,\mathrm{d}\nu \;.$$

- By uniqueness of the limit, we conclude that, for any $\phi \in C^1(\partial\Omega)$,

$$\mu(\phi) = \int_\Omega E(\phi)\,\mathrm{d}\nu,$$

and so

$$|\mu(\phi)| = \left| \int_\Omega E(\phi)\,\mathrm{d}\nu \right| \leq M\|E(\phi)\|_{C^0(\overline{\Omega})} = M||\phi||_{C^0(\partial\Omega)}.$$

Hence, μ is a Radon measure on $\partial\Omega$, whose non negativity follows from that of ν.

Example: the compliance

- The representation formula allows to appraise the asymptotic behavior of the compliance (or power consumption) of Ω,

$$\int_\Omega f u_\varepsilon \,\mathrm{d}x.$$

- Indeed, assuming for simplicity that f has compact support inside Ω, it holds:

$$\int_\Omega f u_\varepsilon \mathrm{d}x = \int_\Omega f u_0 \mathrm{d}x - \mathrm{cap}(\omega_\varepsilon) \int_\Omega f(x) \int_{\partial\Omega} u_0(y) f(y) N(x,y) \,\mathrm{d}\mu(y) \mathrm{d}x + \mathrm{o}(\mathrm{cap}(\omega_\varepsilon)) \,.$$

Due to the symmetry of the fundamental solution $N(x,y)$, it follows:

$$u_0(y) = \int_\Omega N(x,y) f(x) \,\mathrm{d}x \,,$$

and so

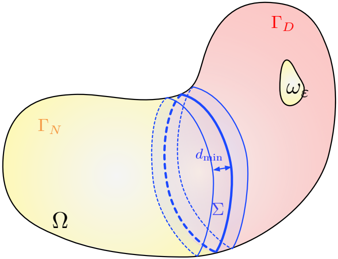
$$\int_\Omega f u_\varepsilon \,\mathrm{d}x = \int_\Omega f u_0 \,\mathrm{d}x - \mathrm{cap}(\omega_\varepsilon) \int_{\partial\Omega} \gamma(x) u_0^2(x) \,\mathrm{d}\mu(x) + \mathrm{o}(\mathrm{cap}(\omega_\varepsilon)) \,.$$

- As expected, the emergence of a small Dirichlet region within the homogeneous Neumann zone Γ_N always decreases the value of the compliance.

The Neumann case: setting

- The ω_ε are open Lipschitz subsets of $\partial\Omega$;
- They are all contained in Γ_D, and stay well-separated from Γ_N:

$$\exists d_{\min} > 0 \text{ s.t. } \forall \varepsilon > 0 \quad \mathrm{dist}(\omega_\varepsilon, \Sigma) \geq d_{\min}. \quad (S)$$



- The background and perturbed potentials u_0 and $u_\varepsilon \in H^1(\Omega)$ are the solutions to:

$$\begin{cases} -\mathrm{div}(\gamma \nabla u_0) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma_D, \\ \gamma \frac{\partial u_0}{\partial n} = 0 & \text{on } \Gamma_N, \end{cases} \quad \text{and} \quad \begin{cases} -\mathrm{div}(\gamma \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_D \setminus \overline{\omega_\varepsilon}, \\ \gamma \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \cup \omega_\varepsilon. \end{cases}$$

The "Neumann capacity" (I)

The relevant means to quantify the "smallness" of ω_ε is a sort of "Neumann capacity".

Definition 2.

Let $\omega \subset \mathbb{R}^d$ be a finite collection of disjoint Lipschitz hypersurfaces; we define:

$$e(\omega) = \max_{\substack{\kappa \in \mathcal{C}_c^\infty(\mathbb{R}^d), \\ \kappa(x) = \pm 1 \text{ for } x \in \overline{\omega}}} \left\{ \int_{\mathbb{R}^d} (z^2 + |\nabla z|^2)\,\mathrm{d}x, \right.$$

$$\left. z \in H^1(\mathbb{R}^d \setminus \overline{\omega}) \text{ s.t. } \left\{ \begin{array}{ll} -\Delta z + z = 0 & \text{in } \mathbb{R}^d \setminus \overline{\omega}, \\ \frac{\partial z}{\partial n} = \kappa & \text{on } \omega \end{array} \right. \right\}.$$

In the above expression,

- n is any smooth unit normal vector field on each connected component of ω.
- $e(\omega)$ does not depend on the choice of an orientation for n, due to the presence of the maximum in its definition.

The "Neumann capacity" (II)

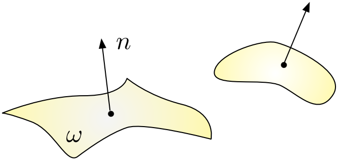
Interpretation:

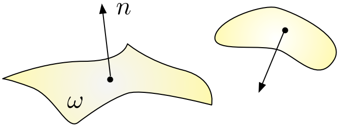
- When ω has only one connected component, $e(\omega)$ is the Dirichlet energy of the unique $H^1(\mathbb{R}^d \setminus \overline{\omega})$ solution z to the equation

$$\begin{cases} -\Delta z + z = 0 & \text{in } \mathbb{R}^d \setminus \overline{\omega}\,, \\ \frac{\partial z}{\partial n} = 1 & \text{on } \omega. \end{cases}$$

  The orientation choice for n only affects the sign of z and not the value of $e(\omega)$.

- When ω has several connected components, an orientation for n can be set independently on each such component. The possible choices are indexed by κ in the maximum and $e(\omega)$ captures the maximum energy of all associated functions z.





Two different orientations for n in the case where ω is not connected, leading to non proportional functions z.

The "Neumann capacity" (III)

The "Neumann capacity" can be compared to more explicit quantities, involving only the geometry of ω.

Lemma 4.

Let ω be an open Lipschitz subset of Γ_D, which is well-separated from Γ_N, i.e. (S) holds. There exists a constant $C > 0$, depending only on Ω, Γ_D and $d_{\min}$ such that

$$e(\omega) \leq C\, D(\omega), \text{ where } D(\omega) := \int_\omega \frac{1}{\rho_\omega(x)}\, \mathrm{d}s(x)$$

and $\rho_\omega(x)$ denotes the weight function defined by

$$\forall x \in \omega, \quad \rho_\omega(x) := \int_{\partial\Omega \setminus \overline{\omega}} \frac{1}{|x-y|^d}\, \mathrm{d}s(y) \; .$$

The "Neumann capacity" (IV)

In turn, $D(\omega)$ can be estimated by more explicit quantities in some important cases.

Proposition 5.

Let ω be a geodesically convex, open Lipschitz subset of $\partial\Omega$. Then

$$c \int_\omega \mathrm{dist}(x, \partial\omega)\, \mathrm{d}s(x) \le D(\omega) \le C \int_\omega \mathrm{dist}(x, \partial\omega)\, \mathrm{d}s(x) \ ,$$

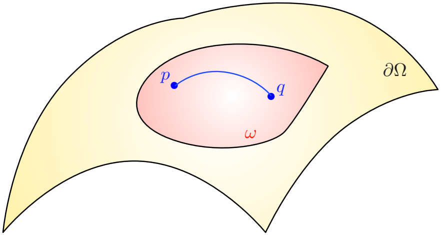
where the positive constants c and C depend only on $\partial\Omega$.

Example: The Neumann capacity of the set

$$\mathbb{D}_\varepsilon := \left\{ x = (x_1, \dots, x_{d-1}, 0) \in \mathbb{R}^d, \ |x| < \varepsilon \right\}$$

satisfies

$$e(\mathbb{D}_\varepsilon) \le C\varepsilon^d.$$



A geodesically convex subset $\omega \subset \partial\Omega$ is such that any minimizing geodesic segment between any two points $p, q \in \omega$ lies entirely inside ω.

Theorem 6 (Representation formula in the Neumann case).

Let ω_ε be such that $e(\omega_\varepsilon) \to 0$ as $\varepsilon \to 0$. Then there exists a subsequence, still denoted by ε, and a Radon measure μ on $\partial\Omega$ such that:

$$u_\varepsilon(x) = u_0(x) + e(\omega_\varepsilon) \int_{\partial\Omega} \frac{\partial u_0}{\partial n}(y)\gamma(y)\frac{\partial N}{\partial n_y}(x,y)\,\mathrm{d}\mu(y) + \mathrm{o}(e(\omega_\varepsilon)) \text{ for } x \in \Omega.$$

Here,

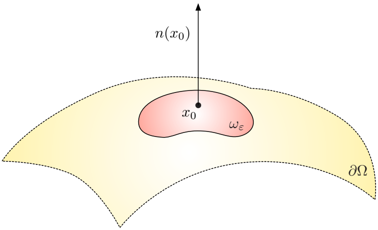
- *The measure μ is non negative and non trivial; it depends only on the subsequence ω_ε, Ω, and Γ_N;*
- *The support of μ lies inside any compact subset $K \subset \partial\Omega$ containing the ω_ε for $\varepsilon > 0$ small enough;*
- *The term $\mathrm{o}(e(\omega_\varepsilon))$ is uniform when x is confined in compact subsets of Ω.*

The proof of this result is fairly similar to that in the Dirichlet case.

Explicit asymptotic formulas when ω_ε is a surfacic ball

We now consider an interesting particular case as regards the definition of ω_ε.

- ω_ε is a small "surfacic ball" around a point $x_0 \in \partial\Omega$.
- It is contained either in Γ_N (Dirichlet case) or in Γ_D (Neumann case).



The perturbed potential u_ε is the unique $H^1(\Omega)$ solution to

$$\left\{ \begin{array}{rcl} -\text{div}(\gamma\nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_D \cup \omega_\varepsilon, \\ \gamma\frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \setminus \overline{\omega_\varepsilon}, \end{array} \right. \qquad \left\{ \begin{array}{rcl} -\text{div}(\gamma\nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \Gamma_D \setminus \overline{\omega_\varepsilon}, \\ \gamma\frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \Gamma_N \cup \omega_\varepsilon. \end{array} \right.$$

(D_ε) (Dirichlet case) $\qquad$ (N_ε) (Neumann case)

We search for an explicit asymptotic expansion of u_ε.

The result of interest in the case where Neumann boundary conditions are replaced by Dirichlet boundary conditions on the "small" surfacic ball $\omega_\varepsilon \subset \Gamma_N$ is:

Theorem 7.

The following asymptotic expansion holds at any point $x \in \overline{\Omega}$, $x \notin \Sigma \cup \{0\}$:

$$u_\varepsilon(x) = u_0(x) - \frac{\pi}{|\log \varepsilon|}\gamma(x_0)u(x_0)N(x, x_0) + \mathrm{o}\left(\frac{1}{|\log \varepsilon|}\right), \quad \text{if } d = 2,$$

and:

$$u_\varepsilon(x) = u_0(x) - 4\varepsilon\gamma(x_0)u(x_0)N(x, x_0) + \mathrm{o}(\varepsilon), \quad \text{if } d = 3.$$

The result of interest in the case where Dirichlet boundary conditions are replaced by Neumann boundary conditions on the "small" surface ball $\omega_\varepsilon \subset \Gamma_D$ is:

Theorem 8.

Let $d = 2$ or 3 and let $x \in \overline{\Omega}$, $x \notin (\Sigma \cup \{0\})$. The following asymptotic expansion holds:

$$u_\varepsilon(x) = u_0(x) + a_d \varepsilon^d \gamma(x_0) \frac{\partial u_0}{\partial n}(x_0) \frac{\partial N}{\partial n_y}(x, x_0) + \mathrm{o}(\varepsilon^d),$$

where the constant a_d equals:

$$a_d = \begin{cases} \frac{\pi}{2} & \text{if } d = 2, \\ \frac{1}{3} & \text{if } d = 3. \end{cases}$$

A little teasing...

For a proof of these results... be sure to attend the next presentation by Éric Bonnetier!

Thank you for your attention!

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