PIECEWISE AFFINE SYSTEMS CONTROLLABILITY AND HYBRID OPTIMAL CONTROL

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Abstract: We consider a particular class of hybrid systems, defined by a piecewise affine dynamic over non-overlapping regions of the state space. We want to control their behaviors so that it reaches a target by minimizing a given cost. We provide a new numerical algorithm under-approximating the controllable domain under the given hybrid dynamic. Given an optimal sequence of states of the hybrid automaton, we are then able to traverse the automaton till the target, locally insuring optimality.

1 INTRODUCTION

Aerospace engineering, automatics and other industries provide a lot of optimization problems, which can be described by optimal control formulations: change of satellites orbits, flight planning, motion coordination (Fierro et al., 2001; Pesch, 1994). Since the years 1950-1970, the optimal control theory has been extensively developed and provides us with powerful results like dynamic programming (Bellman, 1957) or the maximum principle (Pontryagin et al., 1974). However resolutions are mainly numerical.
Now, in “real-life”, optimal control problems are fully nonlinear. There are today two main classes of numerical methods: the first one uses a discrete version of the dynamical principle (Bertsekas, 1984; Bardi and Capuzzo-Dolcetta, 1997). But those algorithms are very expensive in high dimension. The second is based on the Pontryagin Maximum Principle (Pontryagin et al., 1974), (Bryson and Ho, 1975), which provides a pseudo-Hamiltonian formulation of optimal control problems. However, the main difficulty is actually the synthesis of optimal feedback, even not solved for linear systems, except in some very special cases as time-optimal problems (Bryson and Ho, 1975; Pinch, 1993; Pesch, 1994).

In this paper, we consider a particular class of hybrid systems, defined by a piecewise affine dynamic over non-overlapping regions of the state space:

$$\dot{X}(t) = A_q X(t) + B_q u(t) + c_q, \quad \text{for } X(t) \in D_q \quad (1)$$

We present a hybrid algorithm controlling the system (1) from an initial state X_0 at time $t = 0$ to a final state $X_f = 0$ at an unspecified time t_f. To reach this state, we allow the admissible control functions u to take values in a convex and compact polyhedral set $\mathbb{U}_m$ of $\mathbb{R}^m$, in such a way that: $J(X, u(.)) = \int_0^{t_f} l(X(t), u(t))dt$ is minimized.

Piecewise affine models has become a relevant and powerful tool in the approximation of general smooth nonlinear systems (Johansson, 1999). They usually manage to capture many features of general physical systems, and enable a tractable mathematical analysis. Where usual numerical methods suffers from the curse of the dimension (and with the expansion of aerospace, today algorithms in the control theory have to deal with dimension 6 or 7), the analytical approach by piecewise affine models must allow to improve approximations (Girard, 2004): the level of details allows to reach a compromise between quantitative quality of the approximation and the computational time. Such studies has already be done e.g. for biological systems, where simplifications in relation to real data and in regard of simulations are possible, see (Dumas and Rondepierre, 2003).
Here, we provide a full implementation for the analysis of polyhedral piecewise affine control systems in every dimension. In particular, we develop a

new efficient numerical method to compute an under-approximation of the controllable domain. We also propose some promising directions towards generic algorithms for solving piecewise affine optimal control problems.
The paper is organized as follows. In section 2, we define hybrid systems and formulate the hybrid optimal control problem. In section 3, we provide a numerical controllability analysis and then, in section 4, an algorithmic resolution of the hybrid optimal control problem. Some examples are presented in section 5.

2 Hybrid Optimal Control Problem

Let us start defining our hybrid problem. The control domain $\mathbb{U}_m$ is a polytope in $\mathbb{R}^m$, defined as the convex hull of a finite number of points: $\mathbb{U}_m = Conv(s_1, \ldots, s_p)$, such that: $0 \in \mathbb{U}_m$. The points s_i are assumed to be the vertices of $\mathbb{U}_m$.

Let $(D_q)_q$ be a polyhedral partition of the state space $\mathbb{R}^n$. We thus introduce the hybrid automaton $\mathcal{H} = (\mathcal{Q}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{R})$ defined as follows:

1. $\mathcal{Q}$ the countable set of indices of the simplexes $\mathcal{D}_q$.
2. $\mathcal{D} = \{D_q \ / \ q \in \mathcal{Q}\}$ the collection of domains over the state space: $\forall q \in \mathcal{Q}$, D_q is a polyhedron of $\mathbb{R}^n$

   $$\forall (q, q') \in \mathcal{Q}^2, [int(D_q) \cap int(D_{q'}) \neq \emptyset \Rightarrow D_q = D_{q'}]$$
3. $\mathcal{E} = \{(q, q') \in \mathcal{Q} \times \mathcal{Q} / \ \partial D_q \cap \partial D_{q'} \neq \emptyset\}$ the transition set.
4. $\mathcal{F} = \{f_q \ / \ q \in \mathcal{Q}\}$ the collection of affine field vectors:

   $$\boxed{\begin{array}{llll} f_q : & \mathcal{D}_q \times \mathbb{U}_m & \rightarrow & \mathbb{R}^n \\ & (X, u) & \rightarrow & A_q X + B_q u + c_q \end{array}}$$

   such that: $[\ 0 \in D_q \Rightarrow c_q = 0\ ]$.
5. $\mathcal{G} = \{G_e \ / \ e \in \mathcal{E}\}$ the collection of the guards: $\forall e = (q, q') \in \mathcal{E}, G_e = \partial D_q \cap \partial D_{q'}$
6. $\mathcal{R} = \{R_e / e \in \mathcal{E}\}$ the collection of Reset functions: $\forall e = (q, q') \in \mathcal{E}, \forall x \in G_e, R_e(x) = \{x\}$ (Here, we do not need to reinitialize the continuous variable x, since the D_q are adjacent).

Remark 1 *The assumption* $[\ 0 \in D_q \Rightarrow c_q = 0\ ]$ *ensures that the target* 0 *is an equilibrium point of our hybrid dynamic for* $u = 0$.

From now on, the hybrid automaton $\mathcal{H}$ is assumed not Zeno[1].

[1]Zeno executions correspond to an infinite number of switch in a finite time. That often involves problems in the simulation of hybrid system. Indeed the transition times come closer and closer and in simulations, we can not differentiate them any more (see (Girard, 2004; Zhang et al., 2001)).

In this paper, we focus on optimal control problems ($\mathcal{P}_\mathcal{H}$) associated to the hybrid automaton $\mathcal{H}$; we consider the hybrid dynamic induced by $\mathcal{H}$:

$$\dot{X}(t) = A_q X(t) + B_q u(t) + c_q, \quad \text{for } X(t) \in D_q \quad (2)$$

and want to control (2) from an initial state X_0 to a target $X_f = 0$ at an unspecified time t_f. To reach this state, we allow the admissible control functions u to take values in the polytope $\mathbb{U}_m$, in such a way that: $J(X, u(.)) = \int_0^{t_f} l(X(t), u(t))dt$ is minimized.

3 Hybrid System Controllability

In this section, we want to compute the set of controllable points in $\mathbb{R}^n$, i.e. the set of initial points for which the hybrid problem ($\mathcal{P}_\mathcal{H}$) admits a solution. The idea is, by time reversal, to come down to the computation of the attainable set from 0 and to guarantee the controllability of given initial points.
In (Dumas and Rondepierre, 2005, §3.1), an algorithm is proposed to compute an under-approximation in time T of the controllable set for linear systems without state constraints. In this paper, we propose an extension of this algorithm to piecewise affine systems. First we present our under-approximating algorithm over one given cell of the space state. This enables us then to build an under-approximation of the controllable set over a path of cells.

3.1 Under-Approximation of the Controllable set in a given cell

Let q be a discrete mode satisfying: $0 \in D_q$. We want to compute an under-approximation of the controllable set inside the cell $D_q \subset \mathbb{R}^n$ under the control constraints: $u \in \mathbb{U}_m = Conv(s_1, \ldots, s_p)$.

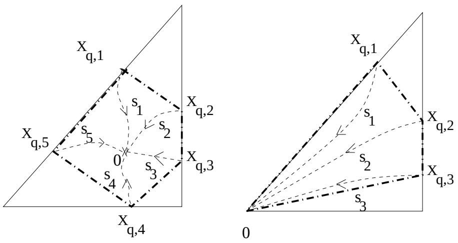
Definition 1 (Controllable set in D_q when $0 \in D_q$) *$X \in D_q$ is controllable iff there exist $T \geq 0$ and $u : [0, T] \rightarrow \mathbb{U}_m$ admissible, such that:*
i. $X = -\int_0^T e^{-A_q \omega}[B_q u(\omega) + c_q]d\omega$
ii. $\forall t \in [0, T], -\int_0^t e^{-A_q \omega}[B_q u(\omega) + c_q]d\omega \in D_q$

In the next, $X_q[0, s_i](.)$ denotes the trajectory according to $u = s_i$ that goes through 0 ; then, by time reversal, $X_{q,i}$ denotes the first intersection of $X_q[0, s_i](.)$ with ∂D_q. Let $F_{q,i}$ be the encountered face:

$$X_{q,i} = X[0, s_i](T_i) \in F_{q,i}$$

where: $T_i = \sup\{t < 0 / \ X[0, s_i](t) \in \partial D_q\}$ as shown on figure 1.

Notation 1 $(X_{q,i}, F_{q,i}) := OutCell(q, 0, s_i)$
By convention, when $X_q[0, s_i](.)$ goes out of D_q, we state: $(X_{q,i}, F_{q,i}) = (\emptyset, \emptyset)$.



(a) (b)

Figure 1: Under-approximation in state q of the controllable set when (a) $O \in int(D_q)$ (b) $0 \in \partial D_q$.

By linearity of the hybrid dynamic in the mode q, the controllable domain in D_q is convex and (Dumas and Rondepierre, 2005, Proposition 3) can be applied in our context:

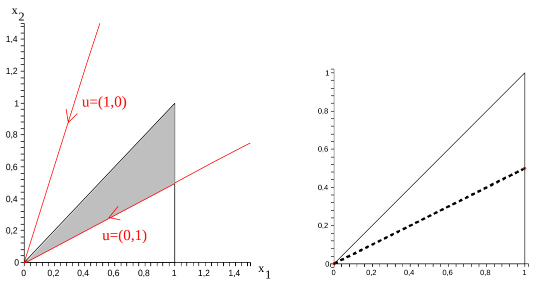
Proposition 1 *$Conv(X_{q,1},\ldots,X_{q,k})$ is an under-approximation of the controllable set in D_q.*

However the quality of the resulting under-approximation is very poor, especially when most of trajectories do not evolve inside D_q.

Example 1 $\dot{X}(t) = \begin{bmatrix} -1 & -2 \\ -3 & -1 \end{bmatrix} u(t)$

where $u(t) \in Conv([0,0],[1,0],[0,1])$ and $X(t) \in D = Conv([0,0],[1,0],[0,1])$.

As shown on figure 2-(a), the trajectory according $u = (1,0)$ evolves outside D, so that there is no valid intersection point with the boundary of D. Our approximation is actually insufficient (see figure 2-(b)).



(a) (b)

Figure 2: (a) Exact controllable set in grey (b) Under-approximation inside the cell D in **...**

To improve our under-approximation, we so have to compute more controllable points on the boundary of D_q. We then propose a new algorithm based on the discretization of the edges of the control set and on the following lemma:

Lemma 1 *Let u be a constant control in $]s_i, s_j[$, then: $\forall t \leq 0,\ X[0,u](t) \in\ ]X[0,s_i](t), X[0,s_j](t)[$*

Likewise, if $u \in int(Conv(s_1,\ldots,s_k))$, then:
$X[0,u](t) \in int(Conv(X[0,s_i](t); i = 1\ldots k))$

Let $[s_i, s_j]$ be an edge of $\mathbb{U}_m$. The principle of the algorithm 1 is the following: let us state:

$$(X_{q,k}, F_{q,k}) := OutCell(q, 0, s_k),\ k \in \{i,j\}$$

i. If $F_{q,i} = F_{q,j}$ ($\neq \emptyset$), any refinement is required. Indeed, if $u \in]s_i, s_j[$, then the trajectory $X[0,u](.)$ evolves between $X[0,s_i](.)$ and $X[0,s_j](.)$, so that its intersection with ∂D_q already is in the under-approximation (see e.g. $X_{q,2}$ and $X_{q,3}$ on figure 1).

ii. Otherwise, by dichotomy, we introduce the control $u_{i,j} = \frac{u_i+u_j}{2}$ and $(X,F) := OutCell(q, 0, u_{i,j})$ to recursively complete the under-approximation. The principle is illustrated on figure 3.

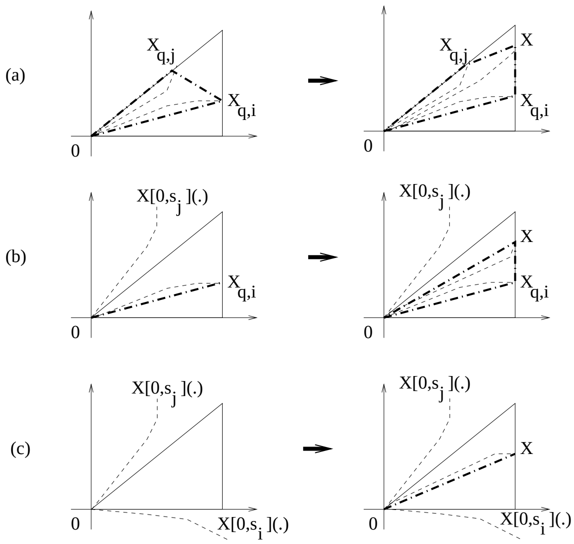


Figure 3: Principle of the $DiscreteEdge$ algorithm
(a) Recursive call for the controls $[u_{i,j}, s_j]$
(b) Recursive call for the controls $[u_{i,j}, s_j]$
(c) Recursive call for the controls $[s_i, u_{i,j}]$ and $[u_{i,j}, s_j]$

We so have a complete algorithm to under-approximate the controllable set in a given state cell in any dimension.

3.2 Controllability in a given cells path

Let $\Pi = (q_i)_{i=0\ldots r}$ be a given sequence of discrete modes of the hybrid automaton $\mathcal{H}$, such that:

$$\begin{cases} 0 \in D_{q_0} \\ \forall i \in [|0, r-1|],\ G_{(q_i,q_{i+1})} \neq \emptyset \end{cases}$$

Now, we want to build an under-approximation of the controllable set over the sequence of adjacent states D_{q_i}. The principle is to start by computing the under-approximation of the controllable set from 0 in the

Algorithm 1 DiscreteEdge

Require: q, X_f (target point), u_i, u_j, $h > 0$ the discretization step.$\{$In §3.1 $X_f := 0\}$
$(X_i, F_i) := OutCell(q, X_f, u_i)$;
$(X_j, F_j) := OutCell(q, X_f, u_j)$;
Ensure: Λ set of controllable points.

$\Lambda := \emptyset$;
if $distance(u_i, u_j) \geq h$ and $(F_i \neq F_j$ or $F_i = F_j = \emptyset)$ $\{$see i.$\}$ **then**
  $(X, F) := OutCell(q, X_f, \frac{u_i+u_j}{2})$;
  if $X \neq \emptyset$ **then**
    $\Lambda := \{X\}$;$\{X$ belongs to the under-approximation$\}$
  end if
  if $X_i \neq \emptyset$ and $X_j \neq \emptyset$ $\{$Case (a) figure 3$\}$ **then**
    if $F \neq F_i$ **then**
      $\Lambda := \Lambda \cup DiscreteEdge(q, X_f, u_i, \frac{u_i+u_j}{2}, h)$;
    end if
    if $F \neq F_j$ **then**
      $\Lambda := \Lambda \cup DiscreteEdge(q, X_f, \frac{u_i+u_j}{2}, u_j, h)$;
    end if
  else
    $\{$Case (b) or (c) figure 3$\}$
    if $F_i = \emptyset$ or $X \neq \emptyset$ **then**
      $\Lambda := \Lambda \cup DiscreteEdge(q, X_f, u_i, \frac{u_i+u_j}{2}, h)$;
    end if
    if $F_j = \emptyset$ or $X \neq \emptyset$ **then**
      $\Lambda := \Lambda \cup DiscreteEdge(q, X_f, \frac{u_i+u_j}{2}, u_j, h)$;
    end if
  end if
end if
Return Λ;

cell q_0 as previously explained. Then, from its intersection with the guard $G_{(q_0,q_1)}$, we pursue the under-approximation, the same way. The only difference is that the reverse starting point is not 0 any more, but the extremal points of the intersection between the guard and the current under-approximation. The algorithm stops when this intersection is empty or when the last state q_r is reached.

4 Solving the Hybrid Optimal Control Problem

This section deals with the algorithmic solving of hybrid control problems: first we focus on the controllability of given initial points. Then, a method is proposed to solve local affine optimal control problems in each cell of the automaton $\mathcal{H}$. Lastly, we detail a generic algorithm for solving the whole hybrid problem.

4.1 Controllability of the initial point

Let X_0 be a given initial point in $\mathbb{R}^n$. Now we want to define the controllability of X_0. This leads us to introduce the notion of solution of our hybrid problem:

Definition 2 *$(X(.), u(.))$ is a solution of the hybrid control problem ($\mathcal{P}_{\mathcal{H}}$) (i.e. X_0 controllable) if there exists a finite execution $\chi = (\tau, \Pi, X)$ satisfying:*
i. $(\Pi(\tau_0), X(\tau_0)) = (q_0, X_0)$ such that: $X_0 \in D_{q_0}$.
ii. $\forall i, X(.)$ is continuously differentiable, $\Pi(t) = q_i$ and $X(t) \in D_{q_i}$ over $]\tau_i, \tau_{i+1}[$ $(\tau_i < \tau_{i+1})$.
iii. $\forall i = 1, \ldots, r, X(\tau_i) \in G_{(q_{i-1},q_i)}$.
iv. $(\Pi(\tau_{r+1}), X(\tau_{r+1})) = (q_r, 0)$.
where: $\tau = (\tau_i)_{i=0\ldots r+1}$ ($\tau_0 = 0$) and $\Pi = (q_i)_{0 \leq i \leq r}$.

Notation 2 *For a given sequence of discrete modes $\Pi = (q_i)_{i=0\ldots r}$, we define a successor function $succ$ as follows: $succ_\Pi(q) = q_{i+1}$ if $q = q_i$ $(i < r)$*

From this definition, the difficulty is to determine the optimal sequence of modes. Some directions to solve this problem include numerical pre-simulations as done in (Bonnans and Maurin, 2000) or a variable change $ds = l(X(t), u(t))dt$ to come down to a time optimal control problem. From now on, we then consider the following assumption:

Hypothesis 1 *let $\Pi = (q_i)_{i=1\ldots r}$ be a given admissible sequence of discrete modes i.e. there exists (τ, X), such that $\chi = (\tau, \Pi, X)$ is a (non optimal) finite execution of the hybrid automaton $\mathcal{H}$ that steers the initial point X_0 to the target 0.*

Under this hypothesis, the $UnderApproximation$ algorithm tests if the initial point X_0 is reachable by time reversal from the 0 in the given path of cells.

4.2 Local Optimal Solutions

In this section, we analyze the dynamic behavior of our hybrid system $\mathcal{H}$ in one given mode q. Let us define our local affine optimal control problem $\mathcal{P}_q$:
Minimize the cost function $J(X_q, u(.)) = \int_0^{t_f} l(X(t), u(t))dt$ with respect to the control $u(.)$ under the dynamic:

$$\begin{cases} \dot{X}(t) = A_q X(t) + B_q u(t) + c_q \\ X(0) = X_q \end{cases}$$

and the constraints: $\forall t \in [0, t_f], X(t) \in D_q,\ u(t) \in \mathbb{U}_m$, where the final time t_f is unspecified.

So, in the mode q, we have to solve a state constrained optimal control problem $\mathcal{P}_q$. The main difficulty is then the choice of the target, when 0 is not in the considered cell D_q. Indeed, in this case, two possible tactics could be considered:

$\mathbf{X(t_f) = 0}$. If $0 \notin D_q$, $\mathcal{P}_q$ is solved as an affine optimal control problem without state constraints. As soon as the so computed optimal trajectory reaches a guard $G_{(q,q')}$ of the cell D_q, the system switches to the mode q' with a new problem $\mathcal{P}_{q'}$. Methods and algorithms have been developed in (Dumas and Rondepierre, 2005) to solve affine optimal control problems via their Hamiltonian formulations. Unfortunately, the convergence of trajectories towards the origin is not guaranteed.

$\mathbf{X(t_f)} \in \mathbf{G_{(q,q_{succ_{\Pi}(q)})}}$. As defined in hypothesis 1, we are given a sequence $\Pi = (q_i)_{i=0\ldots r}$ of discrete modes in our hybrid automaton, for which the initial point X_0 is controllable. The strategy is then to reach the guard between the current mode and its successor towards Π. We so compute *local* optimal trajectories for the given path in the state space.

From now on, we choose these final conditions.

Optimal control under state inequality constraints is a hard and subtle problem. Indeed, for some special conditions like bounded target curves, there can be no generic solving methods, see e.g. (Pinch, 1993, §5,Optimal Control to target curve). Let us show that our problem can be solved via the Pontryagin maximum principle: we consider the affine optimal control problem $\mathcal{P}_q$ with the final condition: $\mathbf{X(t_f)} \in \mathbf{G_{(q,q_{succ_{\Pi}(q)})}}$. The state constraints induced by D_q are affine in the state variable, so that we can state:

$$\boxed{L_q X + M_q \leq 0 \quad \text{State constraints in mode } q}$$

Under this constraints, the above final condition is to reach the hyperplan containing the face $G_{(q,q_{succ_{\Pi}(q)})}$.

We then introduce the Hamiltonian function: $H_q(X,u,\lambda) = l(X,u) + (\lambda^T + \mu^T L_q)(A_q X + B_q u + c_q)$ The Pontryagin principle (Bryson and Ho, 1975; Clarke, 1990) provides then us the following optimization problem: "Minimize the Hamiltonian function H with respect to the control variable $u \in \mathbb{U}_m = Conv(s_1,\ldots,s_p)$ under the constraints:

- $\dot{X}(t) = \frac{\partial H}{\partial \lambda}(X(t),u(t),\lambda(t),\mu(t))$
- $\dot{\lambda}^T(t) = \frac{\partial H}{\partial X(t)}(X(t),u(t),\lambda(t),\mu(t))$
- $H(X^\star(t),u^\star(t),\lambda^\star(t),\mu^\star(t)) = 0$ along the optimal trajectory
- Transversality condition: $<\lambda(t_f), n_{q,succ_{\Pi}(q)}> = 0$ where $n_{q,succ_{\Pi}(q)}$ is the normal vector to the face $G_{(q,q_{succ_{\Pi}(q)})}$".

The parameter μ is a Lagrange multiplier verifying:

$$\forall i,\ \mu_i(t) \begin{cases} = 0 & \text{if } (L_q X + M_q)_i < 0 \\ > 0 & \text{if } (L_q X + M_q)_i = 0 \end{cases}$$

4.3 Hybrid Solver

In regard of previous sections, we can now describe the HybridSolving algorithm:

Let X_0 be the initial point and $\Pi = (q_i)_{i=0\ldots r}$ a given sequence of discrete modes as expressed in hypothesis 1. The principle is to replace the hybrid problem $\mathcal{P}_{\mathcal{H}}$ by $(r+1)$ state constrained affine optimal control problems $(\mathcal{P}_{q_i})_{i=0\ldots r}$ as defined in section 4.2 and to compute cells by cells a local piecewise optimal solution of our initial hybrid problem $\mathcal{P}_{\mathcal{H}}$. For each problem $\mathcal{P}_{q_i}$, we respectively define the initial condition: $X(0) = X_{q_i}$ (in mode q_i) where:

$$\begin{cases} X_{q_0} = X_0 \\ X_{q_{i+1}} = X[X_{q_i}, u^\star](.) \cap G_{(q_i,q_{i+1})},\ i = 0\ldots r-1 \end{cases}$$

Algorithm 2 HybridSolving

Require: X_0, $\mathcal{H}$, $\Pi = (q_i)_{i=1..r}$ a sequence of discrete modes s.t. $X_0 \in D_{q_0}$ and $0 \in D_{q_r}$.
Ensure: (τ, X, u), $V(X_0)$ where (τ, Π, X) is a local optimal execution of H, $V(X_0)$.

1: **if** $X_0 \notin$ UnderApproximation(H,q) **then**
2: Return "X_0 may not be controllable".
3: **end if**
4: $\tau_0 := 0; V := 0;$
{*Piecewise Affine Resolution*}
5: **for all** time step i (from 1 to r) **do**
6: Solve the affine problem $\mathcal{P}_{q_i} \rightarrow (X(.),u(.),t_f,V_f)$
7: $X_0 := X(t_f); \tau_{i+1} := \tau_i + t_f; V := V + V_f;$
8: **end for**
9: Return (τ, X, u, V).

5 Under-Approximation Examples

5.1 In dimension 2

We consider the linear system:

$$\dot{X}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X(t) + \begin{bmatrix} -1 & -2 \\ -3 & -1 \end{bmatrix} u(t)$$

where $u(t) \in \mathbb{U}_2 = Conv([0,0],[1,0],[0,1])$. The under-approximation algorithm is performed on the simplex: $D = Conv([0,0],[1,0],[0,1])$. On figure 4, we show the successive steps to build a good under-approximation of the controllable domain.

5.2 In dimension 3

We now consider the system for $q \in \mathbb{N}$:

$$\dot{X}(t) = \begin{bmatrix} q & 0 & q \\ 0 & 3q & 0 \\ 0 & q & q \end{bmatrix} X(t) + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} u(t)$$

with $u(t) \in Conv([0,0,0],[1,0,0],[0,1,0],[0,0,1])$. The under-approximation is performed on two adjacent cubes (see figure 5) in modes $(q = 0, q = 6)$.

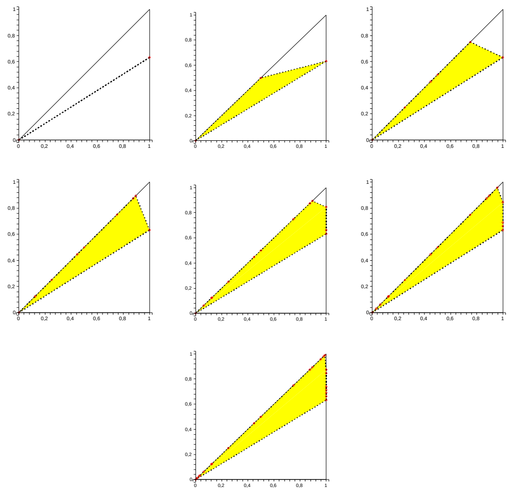


Figure 4: Different steps in the under-approximation for respectively $h = 2, 1, 0.5, 0.25, 0.1, 0.05, 0.01$

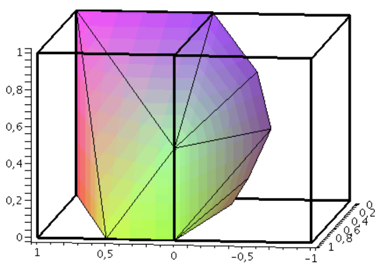


Figure 5: Under-Approximation in 3d

6 CONCLUSION

In this paper, we addressed the optimal control problem for piecewise affine systems. We first provided a full algorithm to compute an under-approximation of the controllable domain. Then the resolution of the hybrid problem is reduced to several explicit resolutions of state constraints affine optimal control problems.

This algorithm however guarantees only a local optimization. Next step will be to give a way to find an optimal sequence of cells containing an optimal trajectory. Several directions to solve this problem include: exploration of different sequences of states, partial numerical simulations to obtain some informations on the localization of optimal trajectories and thus reduce the exploration. Another way could be to replace $l(X(t), u)$, the cost function with an admissible variable change $ds = l$, so that the problem comes down to a time optimal control problem. Finding the optimal sequence would then be to minimize the time to reach the target.

Further developments are also a study of the approximation error and a rigorous proof of the convergence of our under-approximation towards the real controllable set. Future works will include the analysis of nonlinear dynamics: $\dot{X}(t) = f(X(t), u(t))$ by piecewise affine models. The hybrid approximant is build by linear interpolation of f at the vertices of a mesh of $\mathbb{R}^n \times \mathbb{U}_m$. In consequence, in each cell of the resulting automaton, the system is subject to mixed affine inequalities constraints in both state and control[2].

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[2]Work in progress, see (Rondepierre and Dumas, 2005)