Improving methods based on spectrogram zeros using unsupervised classification

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Agenda:

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Introduction

Short-time Fourier transform (STFT)

Let $x(t)$ be a real -or complex- signal and its STFT be defined as:

$$V_x^g(t,f) := \int_{-\infty}^{+\infty} x(u)\,\overline{g(u-t)}\,e^{-i2\pi uf}\,du,$$

with $g(t) = 2^{1/4} e^{-\pi t^2}$. Then, its **spectrogram** is defined as:

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- $S_x^g(t,f)$ can be *interpreted* as a time-frequency distribution:

$$\iint_{\mathbb{R}^2} S_x^g(t,f)dt\, df = E_x$$

- *Largest values* of $S_x^g(t,f)$ contain more information about $x(t)$.

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- *Largest values* of $S_x^g(t,f)$ contain more information about $x(t)$.
  $\rightarrow$ Classical TF paradigm.

Introduction

In the following, we will consider signal and noise mixtures, in which:

- $\xi(t)$ is a zero-mean white Gaussian noise (WGN) satisfying:

$$\mathbb{E}\left\{\xi(t)\overline{\xi(t-\tau)}\right\} = \gamma_0^2 \delta(\tau),$$

  where γ_0^2 is the noise variance.

- The Signal-to-Noise Ratio (SNR) between a signal x and ξ as:

$$\mathrm{SNR}(x, \xi) = 10 \log_{10}\left(\frac{P_x}{\gamma_0^2}\right) \text{ (dB)},$$

  where P_x is the power of the signal.

The zeros of the spectrogram

Considering $z = t + if$, then the STFT can be written as[1]

$$V_x^g(t,f) = \mathcal{F}_x(z)\exp\left(-|z|^2\right)\exp(-i\pi tf),$$

where $\mathcal{F}_x(z)$ is the *Bargmann transform*.

[1] Karlheinz Gröchenig. *Foundations of time-frequency analysis*. Springer Science & Business Media, 2001.

[2] Patrick Flandrin. "Time–frequency filtering based on spectrogram zeros". In: *IEEE Signal Processing Letters* 22.11 (2015), pp. 2137–2141.

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where $\mathcal{F}_x(z)$ is the *Bargmann transform*.
$\mathcal{F}_x(z)$ admits a Haddamard-Weierstrass factorization[2]:

$$\mathcal{F}_x(z) = z^m e^{Q(z)} \prod_n \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{z^2}{2z_n^2}\right),$$

where z_n are the zeros of $\mathcal{F}_x(z)$, m is the order of a (possible) zero at the origin, and $Q(z)$ is a quadratic polynomial.

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where z_n are the zeros of $\mathcal{F}_x(z)$, m is the order of a (possible) zero at the origin, and $Q(z)$ is a quadratic polynomial.
$\rightarrow$ $S_x^g(t,f)$ is characterized by the distribution of its zeros.

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[2] Flandrin, "Time–frequency filtering based on spectrogram zeros".

The zeros of the spectrogram of white noise

- The zeros of the spectrogram of complex white Gaussian noise (CWGN) are homogeneously distributed.
- Its distribution corresponds to that of the roots of a *planar Gaussian Analytic Function* (planar GAF)[3].
- The expected number of zeros in the TF plane can be rigorously deduced from the properties of the planar GAF.
- For a discrete signal with N time samples, the expected number of zeros of the spectrogram is N.

[3] Rémi Bardenet, Julien Flamant, and Pierre Chainais. "On the zeros of the spectrogram of white noise". In: *Applied and Computational Harmonic Analysis* 48.2 (2020), pp. 682–705.

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- When a signal is present, the zeros *surround* the signal domain.
  $\rightarrow$ Larger-than-expected regions without zeros are created.

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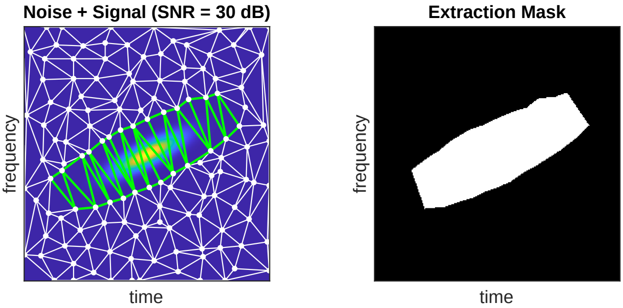
Signal estimation based on spectrogram zeros

Using Delaunay Triangulation (Flandrin, 2015)



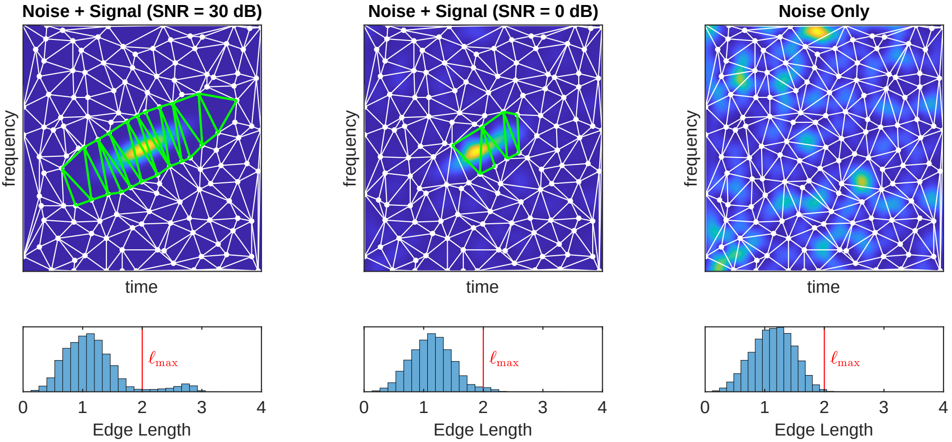
1. Compute the Delaunay triangulation on the SZ.
2. Find triangles with at least one edge length larger than ℓ_{max}.
3. Approximate the signal's TF domain $\mathcal{D}_x$.
4. Estimate $x(t)$ using:

$$\tilde{x}(t) = \frac{1}{g(0)} \int_{-\infty}^{+\infty} V_y^g(t,f) \mathbb{1}_{\mathcal{D}_x}(t,f) e^{i2\pi ft} df.$$

Signal estimation based on spectrogram zeros

Limitations

- Low SNR makes more difficult to identify the triangles.
- ℓ_{max} depends on the Signal-to-Noise Ratio (SNR).



Signal estimation based on spectrogram zeros

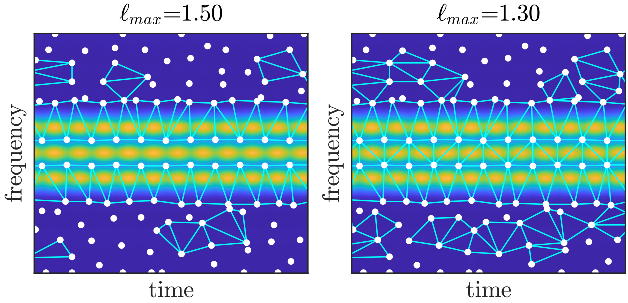
A pathological case

- Consider a signal given by:

$$x(t) = s_{f_1}(t) + s_{f_2}(t) + s_{f_3}(t),$$

where $s_{f_i}(t) = \cos(2\pi f_i t)$, and $f_1 < f_2 < f_3$.

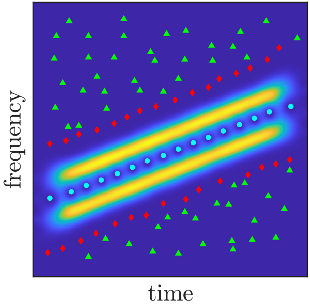
- If $\Delta f_{1,2} = \Delta f_{2,3} = \frac{3}{\sqrt{2\pi}}$, the longest edge of the Delaunay triangles covering the middle tone is $\ell \approx 1.46$.



Agenda:

Three Kinds of SZ

1. Signal-Signal Zeros (SS): these zeros are generated by the interference between signal components.
2. Signal-Noise Zeros (SN) : these are produced by the interference between signal components and noise, and surround the signal domain.
3. Noise-Noise Zeros (NN): these zeros are generated by the noise only, and can be viewed as the result of interference between randomly located Gaussian *logons*.



Three Kinds of SZ

Destructive Interference

$$|V^g_{x_1+x_2}(t,f)|^2 = |V^g_{x_1}(t,f)|^2 + |V^g_{x_2}(t,f)|^2 + 2|V^g_{x_1}(t,f)V^g_{x_2}(t,f)|\cos\left(\Phi^g_{x_2}(t,f) - \Phi^g_{x_1}(t,f)\right),$$

where $\Phi^g_x(t,f)$ is the phase of the complex valued $V^g_x(t,f)$.
Then, $|V^g_{x_1+x_2}(t,f)|^2 = 0$ if and only if:

1. $\Phi_{x_1}(t,f)$ and $\Phi_{x_2}(t,f)$ differ by an odd factor of π.
2. The modulus of $V^g_{x_1}(t,f)$ and $V^g_{x_2}(t,f)$ are equal.

Signal-Signal Spectrogram Zeros

Let x_1 and x_2 be deterministic signals. SS zeros appear where x_1 and x_2 fulfill the conditions of destructive interference.

Three Kinds of SZ

Noise-Noise Spectrogram Zeros

The spectrogram of (complex) white Gaussian noise can be expressed as[a]:

$$S_{\xi}^{g}(t,f) \approx \left| \sum_{k} V_{h_k}^{g}(t,f) e^{i\varphi_k} \right|^2 .$$

where $h_k(t) = a_k \, g(t - t_k) \, e^{i(2\pi f_k + \varphi_k)}$, also called *logons*, with random amplitude a_k, position (t_k, f_k), and phase φ_k.

[a] Patrick Flandrin. *Explorations in time-frequency analysis*. Cambridge University Press, 2018.

Three Kinds of SZ

Signal-Noise Spectrogram Zeros

Let x_1 be a deterministic signal, and x_2 be a realization of noise.
If x_2 is real white Gaussian noise:

$$\mathbb{E}\left\{|V_{x_2}^{g}(t,f)|^2\right\} = \gamma_0^2. \tag{1}$$

We can then define a level curve

$$\Gamma = \left\{(t,f):\ S_{x_1}^{g}(t,f) = \gamma_0^2\right\}. \tag{2}$$

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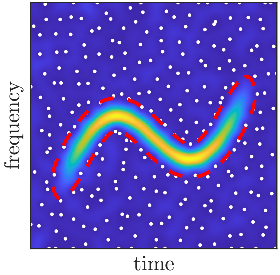
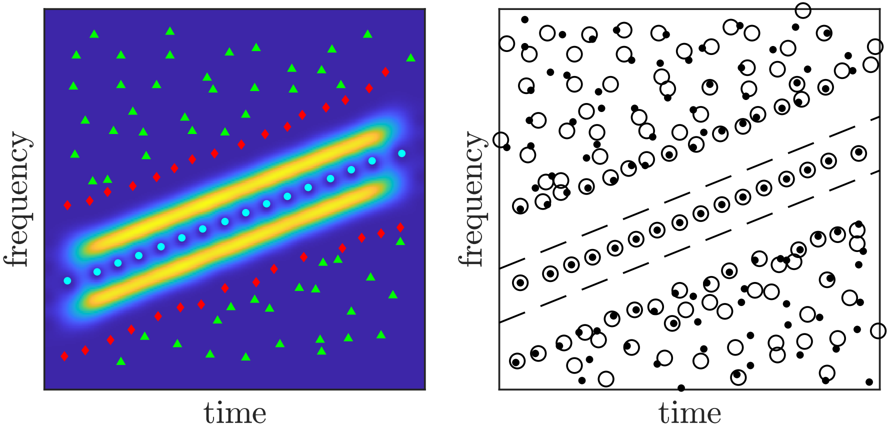


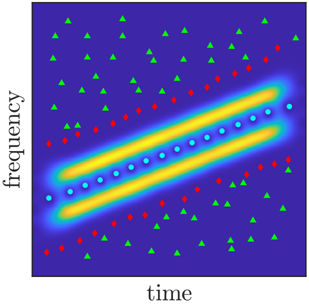
Agenda:

Stability to Noise Addition



2D Histograms of the position of zeros

1. Get a new mixture $y_j(t) = x(t) + \xi(t) + \eta_j(t)$, where $\eta_j(t)$ is white Gaussian noise with variance $\gamma_j^2 = \beta\hat{\gamma}_0^2$, and:
   $\hat{\gamma}_0 = \frac{\sqrt{2}}{0.6745}$ median $\left(\left|\Re\left\{V_y^g(t,f)\right\}\right|\right)$.
2. Repeat this procedure for $j = 1, ..., J$ independent noise realizations.



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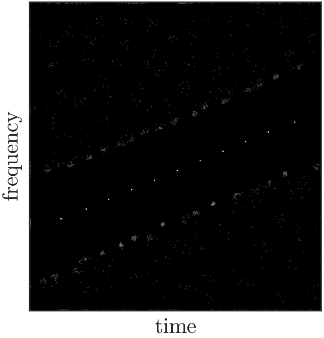
3. A 2D histogram $G[n, m]$ can then be obtained by counting the number of zeros that have fallen in each TF position (n, m).



$(J = 64)$

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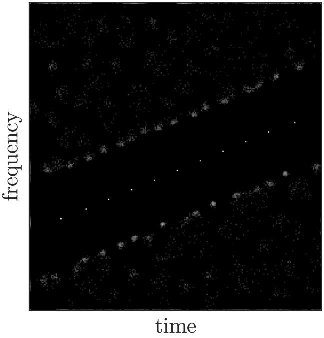

$(J = 128)$

2D Histograms of the position of zeros

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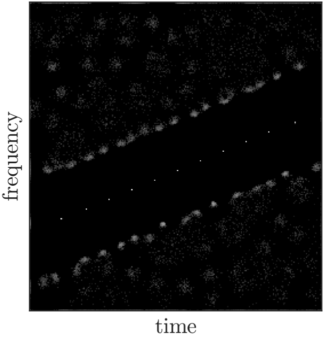

$(J = 256)$

2D Histograms of the position of zeros

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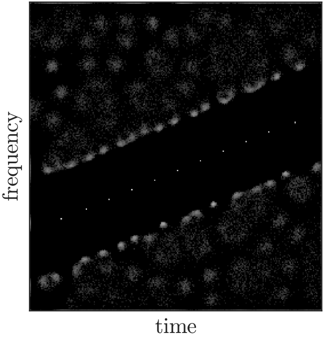

$(J = 512)$

2D Histograms of the position of zeros

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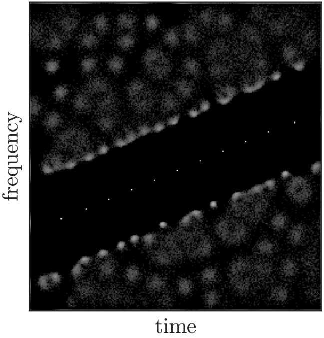
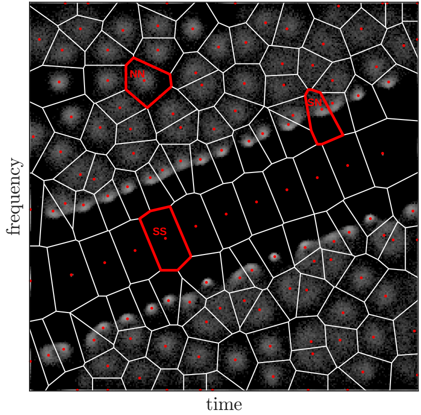


$(J = 1024)$

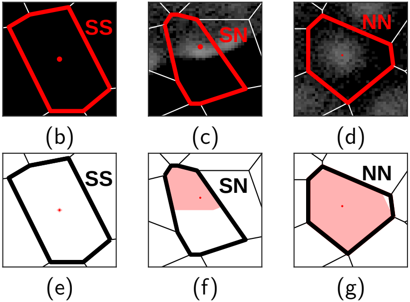
Agenda:

Local features for SZ classification

- Extracting features describing the local distribution of $G[n, m]$.



(a) Voronoi tessellation superimposed on the 2D histogram of spectrogram zeros.



Local features for SZ classification

Convex Hull & Voronoi Cell Area Ratio

For each $\mathbf{z} = (n_z, m_z)$ compute the area ratio:

$$\mathrm{AR}(\mathbf{z}) = \frac{A_{\mathrm{CH}}(\mathbf{z})}{A_{\mathcal{V}(\mathbf{z})}(\mathbf{z})},$$

where

1. $\mathcal{V}(\mathbf{z})$ is the Voronoi cell corresponding to $\mathbf{z}$.
2. CH is the convex hull of $\{(n, m) : (n, m) \in \mathcal{V}(\mathbf{z}) \wedge G[n, m] > 0\}$.

Maximum in Voronoi Cell

For each $\mathbf{z}$, compute the maximum of the histogram in $\mathcal{V}(\mathbf{z})$:

$$\mathrm{Max}(\mathbf{z}) = \max_{(n,m)\in\mathcal{V}(\mathbf{z})} \frac{G[n, m]}{J}.$$

Unsupervised Classification

- Based on first *clustering*, and then *labeling*.
- We use Gaussian Mixture Models (GMM) for clustering in the feature space spanned by AR($\mathbf{z}$) and Max($\mathbf{z}$).
- The number K of clusters to search must be determined. For example, using the GAP criterion[4]:

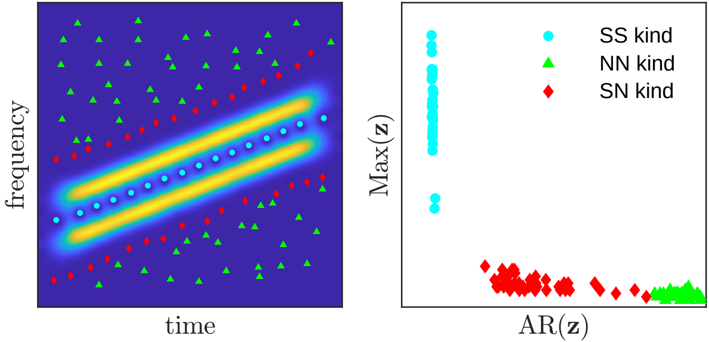
$$K = \underset{k \in \{1,2,3\}}{\arg\max} \ \mathbb{E}\{\log(W_k^{\star})\} - \log(W_k), \tag{3}$$

  - $W_k = \sum_{r=1}^{k} \frac{1}{N_r} D_r$
  - D_r is the sum of all the pairwise distances for the points in the r-th cluster.
  - N_r the number of elements in the cluster.
  - W_k^* is computed using Monte-Carlo simulations.

[4] Robert Tibshirani, Guenther Walther, and Trevor Hastie. "Estimating the number of clusters in a data set via the gap statistic". In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 63.2 (2001), pp. 411–423.

Unsupervised Classification

- The clusters are then labeled according to the value of AR($\mathbf{z}$).



Algorithm Summary

Require: A noisy signal y, J, β.

Compute $V_y^g(t,f)$, the set $\mathcal{Z}_0$ of original SZs and Voronoi tessellation.

Compute $\hat{\gamma}_0 = \frac{\sqrt{2}}{0.6745}$ median $\left(\left|\Re\left\{V_y^g(t,f)\right\}\right|\right)$.

Compute the 2D histograms using J realizations and $\gamma_j^2 = \beta\gamma_0^2$.

Extract AR($\mathbf{z}$) and Max($\mathbf{z}$).

Apply GMM clustering, for $K \in \{1,2,3\}$.

Decide whether $K = 1$, 2 or 3.

Find the centroids C_i, $i = 1, \ldots, K$ of the detected clusters.

Sort clusters $C_{(i)}$ in ascending order of AR($\mathbf{z}$).

If $K = 1$, $C_{(1)} \to$ NN.

If $K = 2$, $(C_{(1)}, C_{(2)}) \to (\mathrm{SN}, \mathrm{NN})$.

If $K = 3$, $(C_{(1)}, C_{(2)}, C_{(3)}) \to (\mathrm{SS}, \mathrm{SN}, \mathrm{NN})$.

return K and the labels of each SZ in $\mathcal{Z}_0$

Signal domain estimation based on classified zeros

Different criteria to select Delaunay triangles

- Based on the classification of the spectrogram zeros, one can select Delaunay triangles that satisfy any of the following criteria:
  1. All its vertices are zeros of the SN kind (i.e. zeros located at the border of the signal domain).
  2. At least one of its vertices is a zero of the SS kind.

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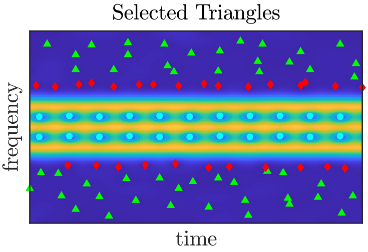
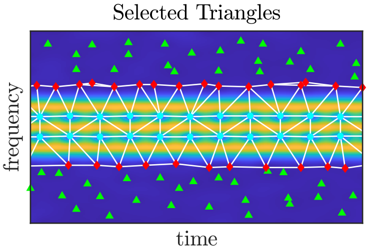
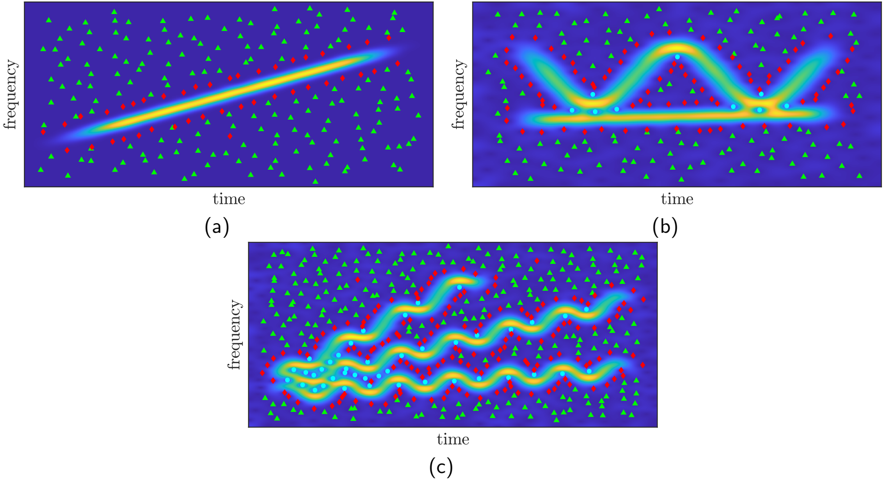




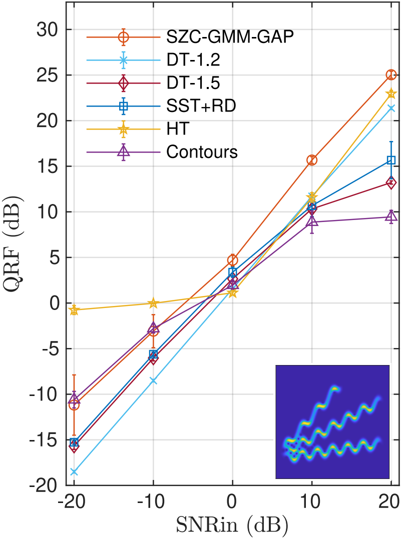
Agenda:

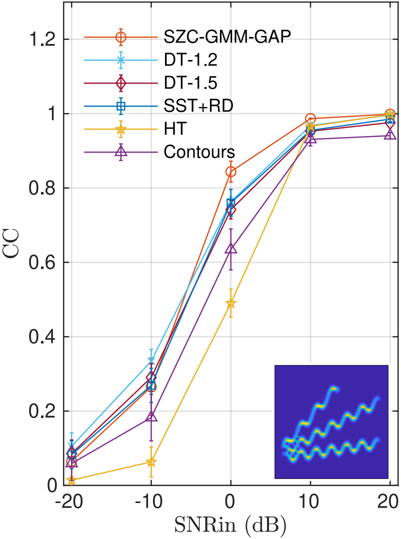
Results - Spectrogram Zeros

- Spectrogram zeros classification (using $\beta = 1.0$ and $J = N/4$):



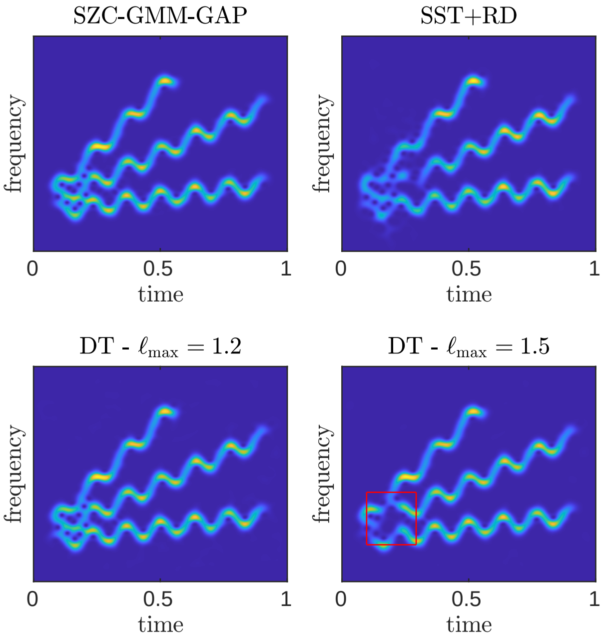
Results - Signal estimation





$$\mathrm{QRF} := 10\log_{10}\left(\frac{\|x\|_2^2}{\|x-\tilde{x}\|_2^2}\right)\ (\mathrm{dB}) \qquad \mathrm{CC} := \frac{\langle x,\tilde{x}\rangle}{\|x\|_2\|\tilde{x}\|_2}$$

Results - Signal estimation



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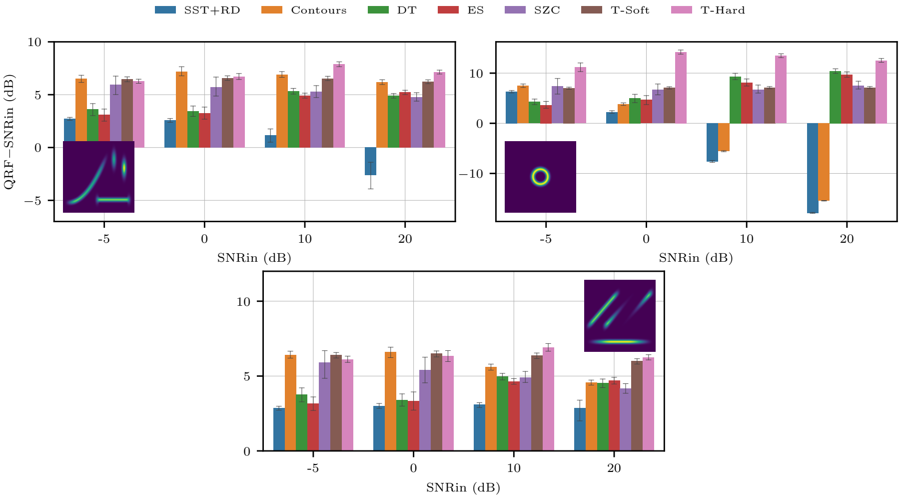
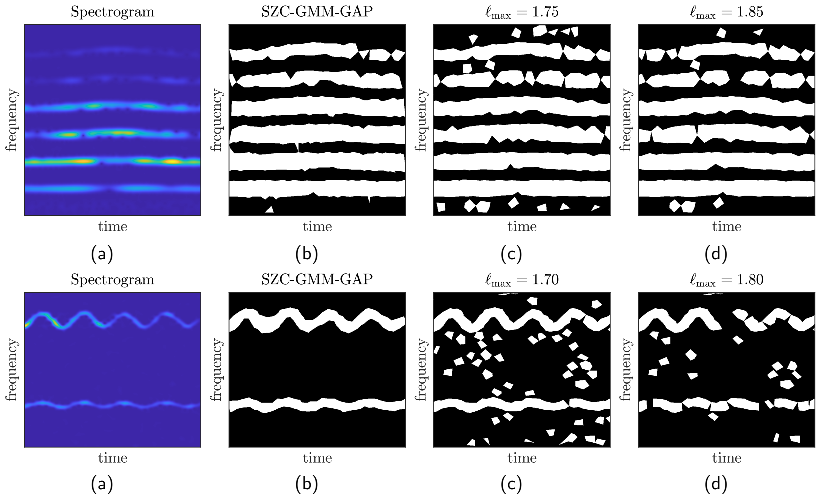


Results - Audio signals



Results

- Average execution time (in seconds):

Method	Execution Time	
	$N = 512$	$N = 1024$
SZC-GMM-GAP-$J = 256$	6.69	18.12
SZC-GMM-CaHa†-$J = 256$	1.94	9.87
SZC-KMEANS-GAP-$J = 256$	2.54	10.76
SZC-KMEANS-CaHa-$J = 256$	1.92	9.82
DT - $\ell_{\max} = 1.2$	0.16	1.13

† CaHa stands for Calinsky-Harabasz criterion.

Agenda:

Conclusions

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- We proposed a study of the spectrogram zeros from the perspective of interference in the TF.
- The classification of the spectrogram zeros provides different criteria to estimate the signal domain using the Delaunay triangulation.
- The approach works better in low SNR scenarios and in the presence of interference.
- Improvement comes with the cost of an increased computation time, mainly due to the 2D histograms.
- The noise is assumed to be broadband.

Future work

- Zeros of the SS kind implies strong interference between components.
  - Detecting *time-frequency bubbles*.
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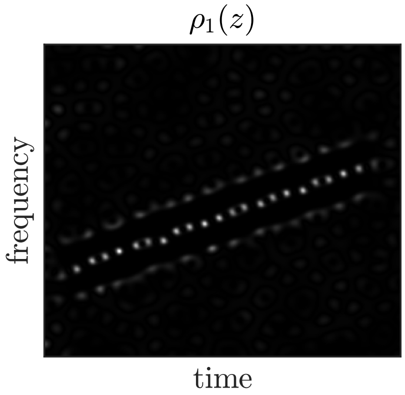
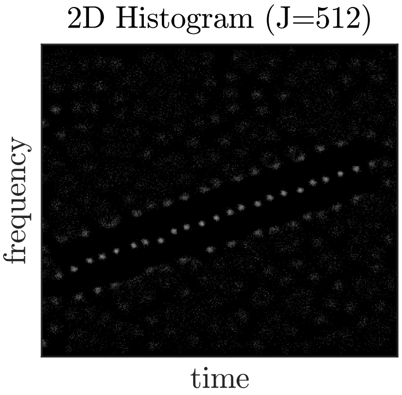
  - Detecting *time-frequency bubbles*.
  - Improving window selection to reduce interference between signal components.
- The introduced 2D histograms are interesting objects by themselves.
  - Describing/ modeling their underlying distribution.
  - Reducing the number of noise realizations needed for computation.

Future work



($\approx$ 0.01 s)



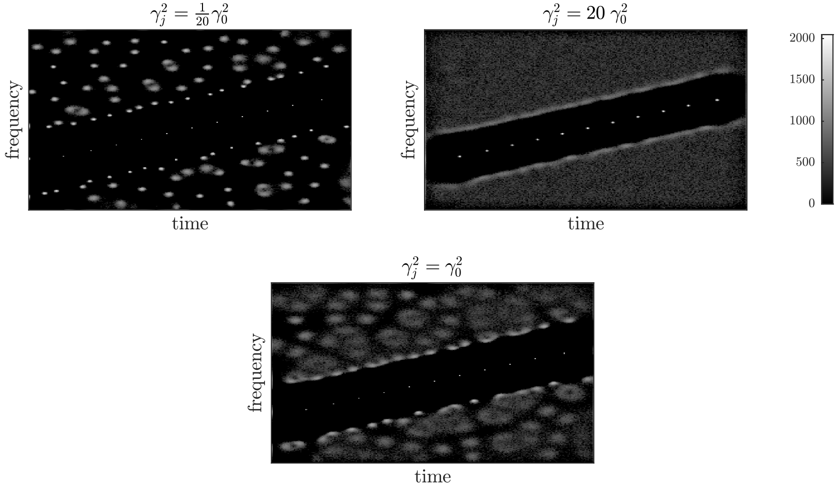
($\approx$ 2 s)

Thank you for your attention!

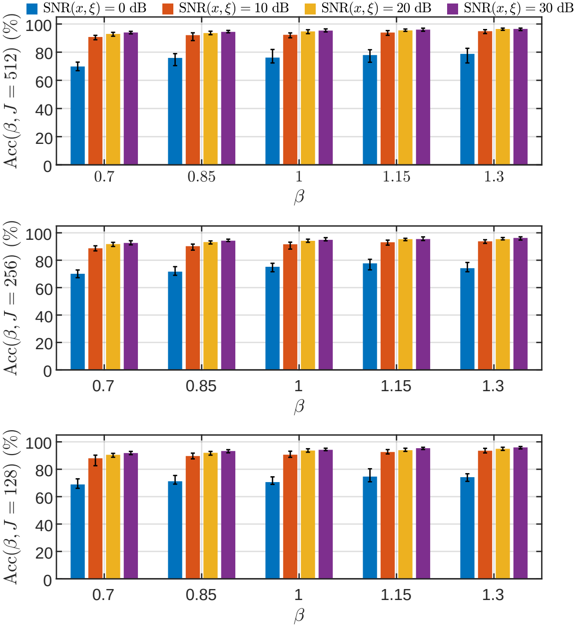
`https://github.com/jmiramont/spectrogram-zeros-classification`

Juan M. Miramont, François Auger, Marcelo A. Colominas, Nils Laurent, and Sylvain Meignen. *"Unsupervised classification of the spectrogram zeros with an application to signal detection and denoising*", Signal Processing (2023).

2D Histograms of the position of zeros



Setting parameters



Variations of the proposed approach