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Aix Marseille université Socialement engagée

Time-Scale Synthesis of Non-Stationary Signals

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Outline

Stationarity

Definition (Stationarity)

A random process X is said to be second-order stationary if :

- $\mathbb{E}\{X(t)\} = m_X, \ \forall t$,
- $\mathbb{E}\{X(t)X(\tau)\} = k_X(t-\tau), \ \forall (t,\tau)$.

Spectrum :

- Gives the distribution over frequencies of the power of X.
- Many methods to estimate the spectrum from a single realization of the stationary process X (e.g. Welch method).

Nonstationarity

The stationarity assumption is often irrelevant to study real-life signals, such as audio signals, or physiological signals.

⇒ **Questions :**

1. Which classes of nonstationarity should we consider ?
2. How should we analyze nonstationarity ? In particular, how to extend spectral estimation to nonstationary signals ?

Broken stationarity : a class of nonstationarity [1]

Two key ingredients :

1. A zero-mean stationary random process X.
2. A deformation operator that breaks stationarity $\mathcal{T}$.

We observe the "deformed" process Y given by :

$$Y = \mathcal{T}X \ .$$

$\Rightarrow$ We limit ourselves to some physically relevant forms of operators.

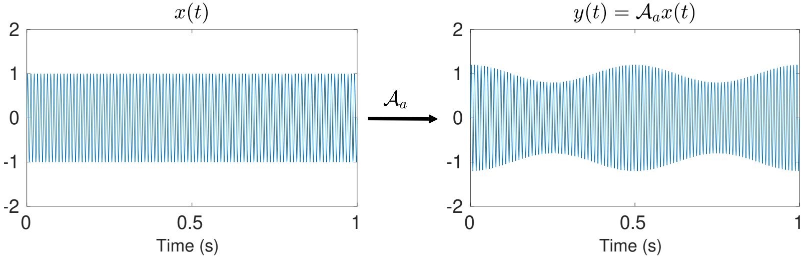
1. H. Omer. *Modèles de déformation de processus stochastiques généralisés. Application à l'estimation des non stationnarités dans les signaux audio.* PhD thesis, Aix-Marseille Université, 2015

Deformation operators

- Amplitude modulation

$$\mathcal{A}_\alpha : \qquad \mathcal{A}_\alpha x(t) = \alpha(t) x(t) \ ,$$

with $\alpha \in C^1$ such that $\forall t,\ 0 < c_\alpha \leq \alpha(t) \leq C_\alpha < \infty$.



- Frequency modulation

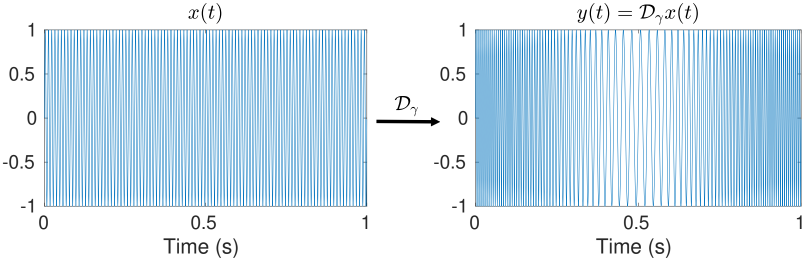
Deformation operators

- Time warping

$$\mathcal{D}_\gamma : \qquad \mathcal{D}_\gamma x(t) = \sqrt{\gamma'(t)}x(\gamma(t)) \ ,$$

where $\gamma \in C^2$ is a strictly increasing function such that

$$0 < c_\gamma \leq \gamma'(t) \leq C_\gamma < \infty, \forall t \ .$$



- Any combination of the above deformations

Goal : spectral estimation of nonstationary signals

From a single realization of the nonstationary random process Y, we aim at estimating simultaneously :

- The spectrum $\mathscr{S}_X$ of the underlying stationary random process X,
- The deformation operator $\mathcal{T}$.

Model and goal

- **Model** :

$$Y = \mathscr{A}_\alpha \mathscr{D}_\gamma X \ .$$

where X is a stationary process.

  - Relevant to model physical phenomena, such as Doppler effect, speed variations, or animal vocalizations.

- **Goal** : From a single realization of the process Y, estimate simultaneously :
  - the spectrum $\mathscr{S}_X$ of the underlying stationary process X,
  - the deformation functions α and γ.

Wavelet transform

Definition (Wavelet transform)

$$\mathcal{W}_x(s,\tau) = \langle x, \psi_{s\tau} \rangle \quad \text{avec} \quad \psi_{s\tau}(t) = 2^{-s/2}\psi\left(2^{-s}(t-\tau)\right) \ ,$$

where ψ is the analysis wavelet.

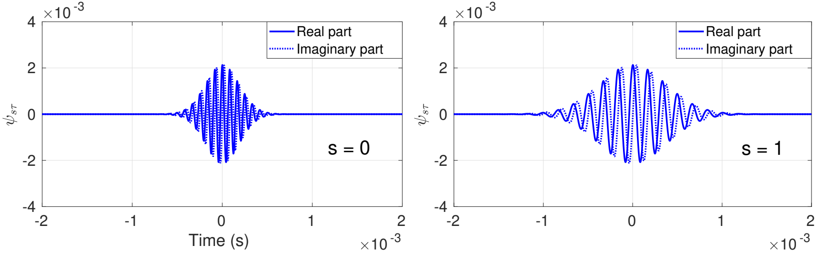
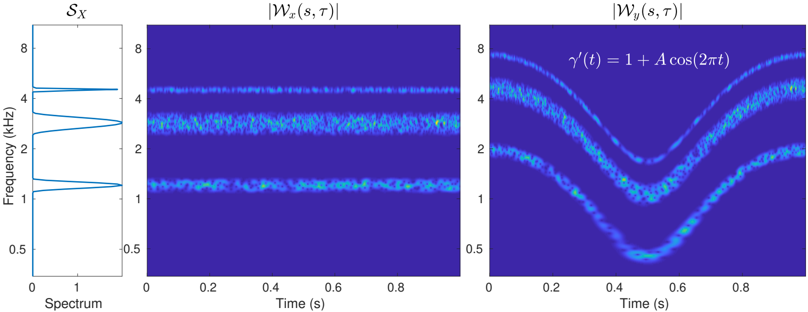


FIGURE – "Sharp wavelet" for two different values of s.

Approximated behavior



Approximation theorem

The wavelet transforms of X and Y are approximately related by :

$$\mathcal{W}_Y(s,\tau) \approx \widetilde{\mathcal{W}}_Y(s,\tau) = \alpha(\tau)\mathcal{W}_X\left(s + \log_2(\gamma'(\tau)), \gamma(\tau)\right) \ .$$

The error term $\epsilon = \mathcal{W}_Y - \widetilde{\mathcal{W}}_Y$ is a zero-mean random process, whose variance $\mathbb{E}\left\{|\epsilon(s,\tau)|^2\right\}$ depends on the regularity of α and γ', and the speed of decay of ψ.

Estimation procedure

Fix $\tau \Rightarrow$ Unknown parameters :

- $\mathscr{S}_X$
- $\theta_1 = \alpha(\tau)^2$
- $\theta_2 = \log_2(\gamma'(\tau))$

Assumption : X is a zero-mean stationary Gaussian process.
$\Rightarrow$ Each column of the wavelet transform of Y : $\mathbf{w}_{Y,\tau} \sim \mathcal{CN}_c(0, \mathbf{C}(\boldsymbol{\Theta}))$, with covariance matrix :

$$\mathbf{C}(\Theta)_{ij} = \theta_1 2^{(s_i+s_j+2\theta_2)/2} \int_0^\infty \mathscr{S}_X(\xi)\overline{\hat{\psi}}(2^{s_i+\theta_2}\xi)\hat{\psi}(2^{s_j+\theta_2}\xi)d\xi\,.$$

$\Rightarrow$ The log-likelihood is given by

$$\mathscr{L}(\Theta) = -\frac{1}{2}\ln|\det(\mathbf{C}(\Theta))| - \frac{1}{2}\mathbf{C}(\Theta)^{-1}\mathbf{w}_{y,\tau}\cdot\mathbf{w}_{y,\tau}\ .$$

Estimation algorithm : JEFAS

The JEFAS (*Joint Estimation of Frequency, Amplitude and Spectrum*) algorithm consists in an alternating estimation.

Initializations :

- Initialize the power spectrum estimate.
- Initialize the amplitude modulation by a constant.
- $k \leftarrow 1$

while stopping criterion = FALSE **do**

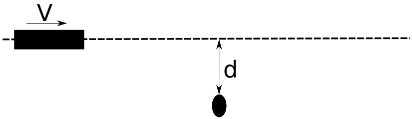
- Time warping : Estimate $\tilde{\alpha}^{(k)}$ by ML, $\forall \tau$.
- Amplitude modulation : Estimate $\tilde{\gamma}^{(k)}$ by ML, $\forall \tau$.
- Spectrum : Estimate $\tilde{\mathscr{S}}_X^{(k)}$ from the "rectified" wavelet transform.
- $k \leftarrow k + 1$

end while

Doppler effect

Assumptions :

- A source emits a stationary sound.
- The source follows a uniform linear motion, at speed V.
- From a fixed station, we record the sound emitted by the source.

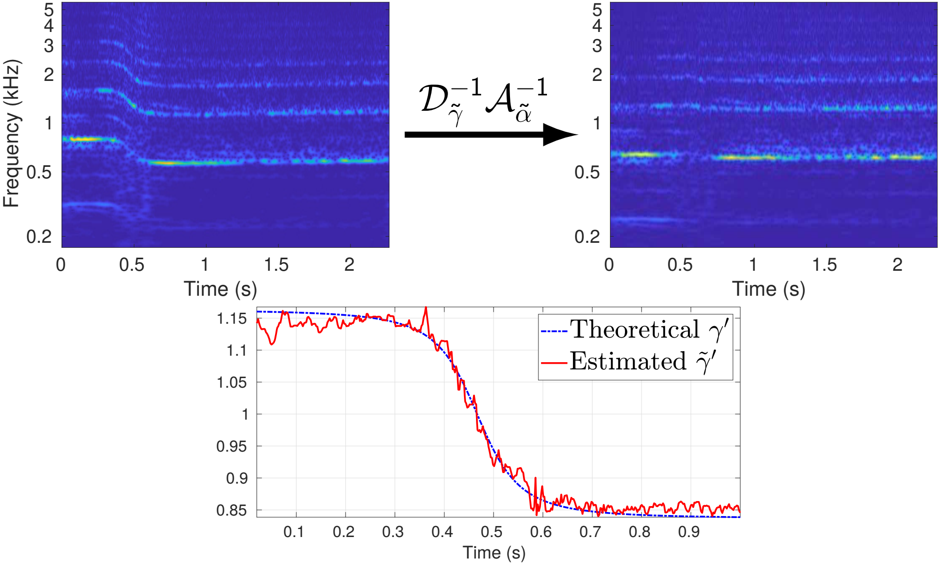


$\Rightarrow$ Due to the Doppler effect, the sound we receive is time-warped, with :

$$\gamma'(t) = \frac{c^2}{c^2 - V^2}\left(1 - \frac{V^2 t}{\sqrt{d^2(c^2 - V^2) + (cVt)^2}}\right).$$

An example : 

Doppler effect



Comparison to the theoretical function with : $d = 5$ m and $V = 54$ m/s.

Spectral analysis of a broadband wind sound

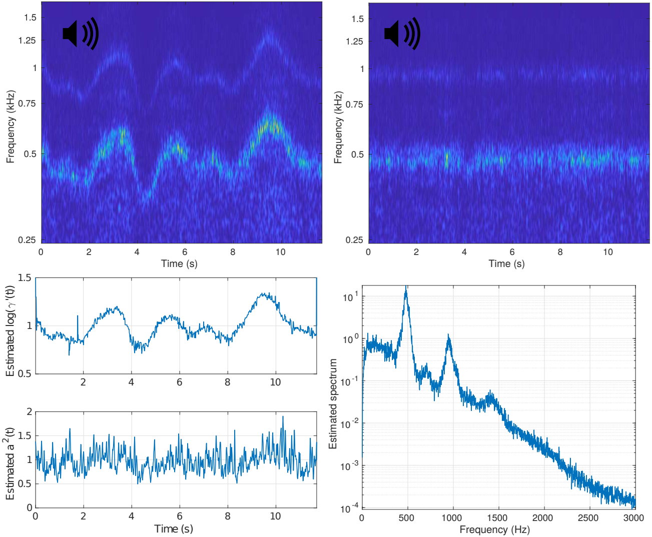


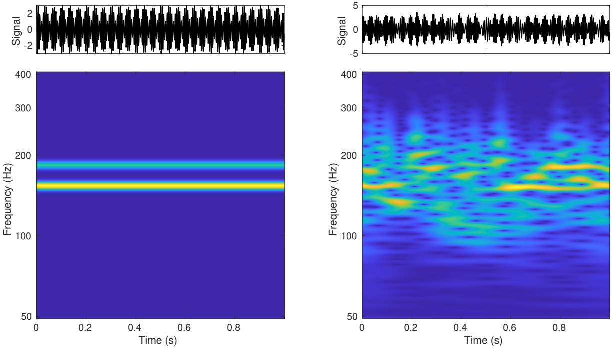
FIGURE – Top : Scalograms of the original signal (left) and the estimated stationary signal (right). Bottom left : estimated time warping and amplitude modulation. Bottom right : estimated spectrum.

Locally harmonic signal

Signal of the form :

$$y(t) = A_1 \cos(2\pi\xi_1\gamma(t)) + A_2 \cos(2\pi\xi_2\gamma(t)) \ ,$$

where γ' is the **fast varying instantaneous frequency**.



- JEFAS : Interference patterns on the wavelet transform.
  ⇒ Approximated behavior does not hold.
  ⇒ JEFAS diverges.

Spectral estimation : Analysis vs. Synthesis

1. Analysis-based approach $\Rightarrow$ JEFAS and JEFAS-BSS

   Model of nonstationarity :
   locally time-warped signals, of the form :

$$Y = \mathscr{D}_{\gamma} X \ ,$$

where X is an arbitrary stationary process.

Spectral estimation : Analysis vs. Synthesis

1. Analysis-based approach ⇒ JEFAS and JEFAS-BSS

Model of nonstationarity :
locally time-warped signals, of the form :

$$= \mathscr{D}_{\gamma} X \;,$$

where X is an arbitrary stationary process.

Spectral estimation : Analysis vs. Synthesis

2. Synthesis-based approach $\Rightarrow$ JEFAS-S

Synthesis model $\Leftrightarrow$ Reconstruction formula :

$$y(t) = \text{Re}\left(\sum_s (\psi_s * W_s)(t)\right) + \epsilon(t) \ ,$$

where $W_s(t)$ are random time-scale coefficients, and $\epsilon(t)$ is a noise.

- Discretization of the problem :

$$\mathbf{y} = \text{Re}\left(\sum_{n=1}^{N} \boldsymbol{\Psi}_n \mathbf{w}_n\right) + \boldsymbol{\epsilon} \ ,$$

where $\mathbf{w}_n$ is the n-th column of the time-scale representation.

Bayesian inference

- Likelihood : $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$

$$p(\mathbf{y}|\mathbf{w}_n) = \mathcal{N}\left(\mathrm{Re}\left(\sum_{n=1}^{N} \boldsymbol{\Psi}_n \mathbf{w}_n\right), \sigma^2 \mathbf{I}\right)$$

- Prior : on the synthesis coefficients $\mathbf{w}_n$
  - Uncorrelated vectors such that :

$$\mathbf{w}_n \sim \mathcal{CN}_c(0, \mathbf{C}_n) \ .$$

  - Covariance matrices $\mathbf{C}_n$ are translated versions of reference covariance function c :

$$[\mathbf{C}_n]_{mm'} = [\mathbf{C}(\theta_n)]_{mm'} = c(s_m + \theta_n, s_{m'} + \theta_n) \ ,$$

where $\theta_n \in \mathbb{R}$ is the shift parameter.

Estimation strategy

- Expectation Maximization (EM) principle where :
  - $\boldsymbol{\theta}$ is the parameter,
  - $\mathbf{w}_n$ is the latent variable.

EM steps

The update at iteration k relies on the following two steps :

1. Time-scale representation update

   $\Leftrightarrow$

   Maximum *a posteriori* estimation :

$$\tilde{\mathbf{w}}_n^{(k)} = \frac{1}{2}\mathbf{C}\left(\tilde{\theta}_n^{(k-1)}\right)\boldsymbol{\Psi}_n^H\mathbf{C}_y\left(\tilde{\boldsymbol{\theta}}^{(k-1)}\right)^{-1}\mathbf{y}\ .$$

2. Nonstationarity parameter update :

$$\tilde{\theta}_n^{(k)} = \arg\max_{\theta_n}\ \mathscr{L}(\theta_n) - \frac{1}{2}\mathrm{Tr}\left(\mathbf{C}(\theta_n)^{-1}\boldsymbol{\Gamma}_n\left(\tilde{\boldsymbol{\theta}}^{(k-1)}\right)\right)\ ,$$

Algorithm : JEFAS-S

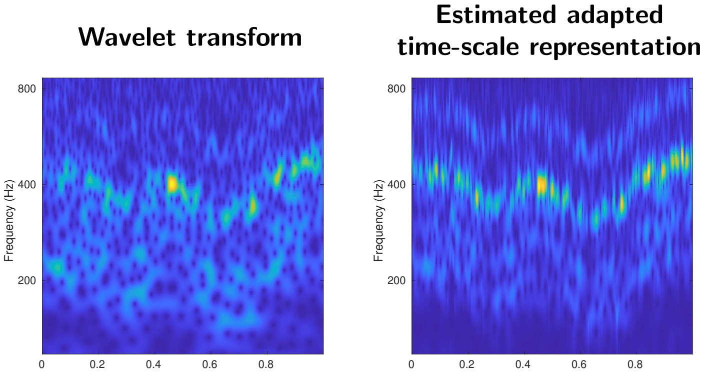
Initialization : estimate $\tilde{\theta}^{(0)}$ and $\tilde{\mathscr{S}}^{(0)}$ using JEFAS.
- $k \leftarrow 1$

while stopping criterion = FALSE **do**
- Time-scale representation estimation : $\tilde{\mathbf{w}}_n^{(k)}$.
- Time-warping parameter estimation : $\tilde{\theta}^{(k)}$.
- Spectrum estimation : $\tilde{\mathscr{S}}_X^{(k)}$.
- $k \leftarrow k + 1$.

end while

- Alternating estimation ⇒ Similar to JEFAS.
- Convergence ensured by the EM principle.
- Additional estimation of the time-scale representation ⇒ JEFAS-S is slower to converge than JEFAS.

Broadband synthetic signal



- Time-warping parameter estimation : not improved with respect to JEFAS.
- Allows denoising : improvement of **7.06 dB** of the Signal to Noise Ratio between the measurements $\mathbf{y}$ and the reconstructed signal $\tilde{\mathbf{y}}_0$.

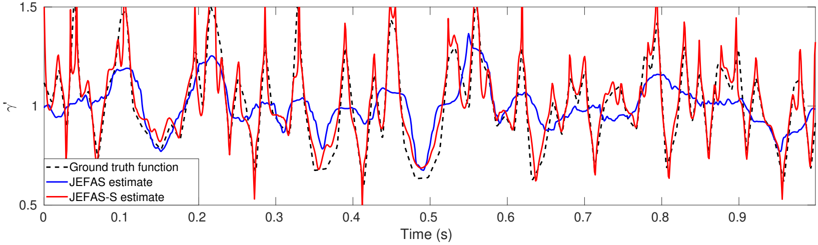
Locally harmonic signal

Signal of the form :

$$y(t) = A_1 \cos(2\pi\xi_1\gamma(t)) + A_2 \cos(2\pi\xi_2\gamma(t)) \ ,$$

where γ' is the (normalized) **fast varying instantaneous frequency**.

JEFAS-S : Prior of uncorrelation between $\mathbf{w}_n \Rightarrow$ No interference $\Rightarrow$ JEFAS-S converges.

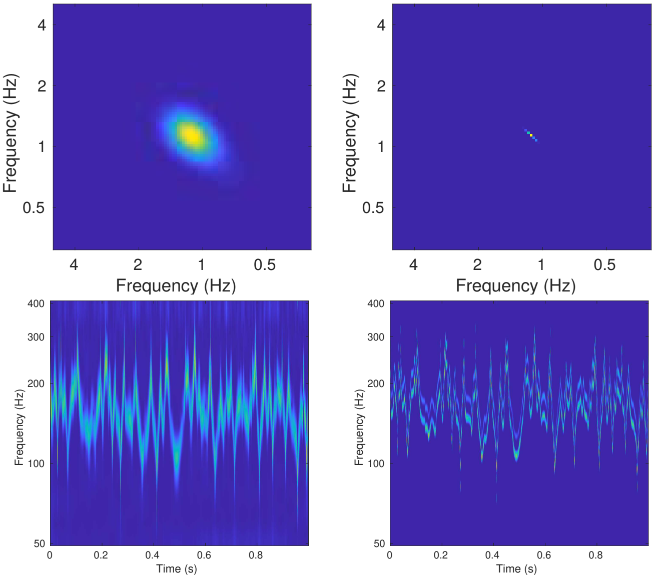


Locally harmonic signal

Wavelet-like prior | **Sharp prior**

Covariance matrices

Time-scale representations



Conclusion

Summary :

- Broken stationarities :

$$Y = \mathcal{T}X$$

  - Locally deformed signals
  - Multivariate locally deformed signals
  - Locally harmonic signals
- Spectral estimation : Simultaneous estimation of the spectrum $\mathscr{S}_X$ and the deformation operator $\mathcal{T}$.

Two estimation strategies :

- Analysis-based approaches
- Synthesis-based approaches

Advertising

GitHub

https://github.com/AdMeynard/JEFAS

$y_1(t) = \alpha_1(t) x_1(\gamma_1(t)) \Longrightarrow$ Spectrum estimation of $\mathscr{S}_{X_1}$

$y_2(t) = \alpha_2(t) x_2(\gamma_2(t)) \Longrightarrow$ Deformation estimations of α_2 and γ_2

Cross synthesis :
$y_{1\to 2}(t) = \tilde{\alpha}_2(t)\tilde{x}_1(\tilde{\gamma}_2(t))$

		Deformation		
		Song	Wind	Formula 1
Spectrum	Song			
	Wind			
	Formula 1			

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IMPORTANT DATES

Submission of Special Session proposals	Jan. 14, 2024
Notification of acceptance of Special Sessions	Jan. 30, 2024
Submission of Tutorial proposals	Mar. 1, 2024
Full paper submission	Mar. 3, 2024
Paper acceptance notification	May 22, 2024
Camera-ready paper deadline	Jun. 1, 2024
3-Minute Thesis contest	Jun. 15, 2024

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