Point processes in TF representations: on the zeros of some time-frequency transforms of white noise

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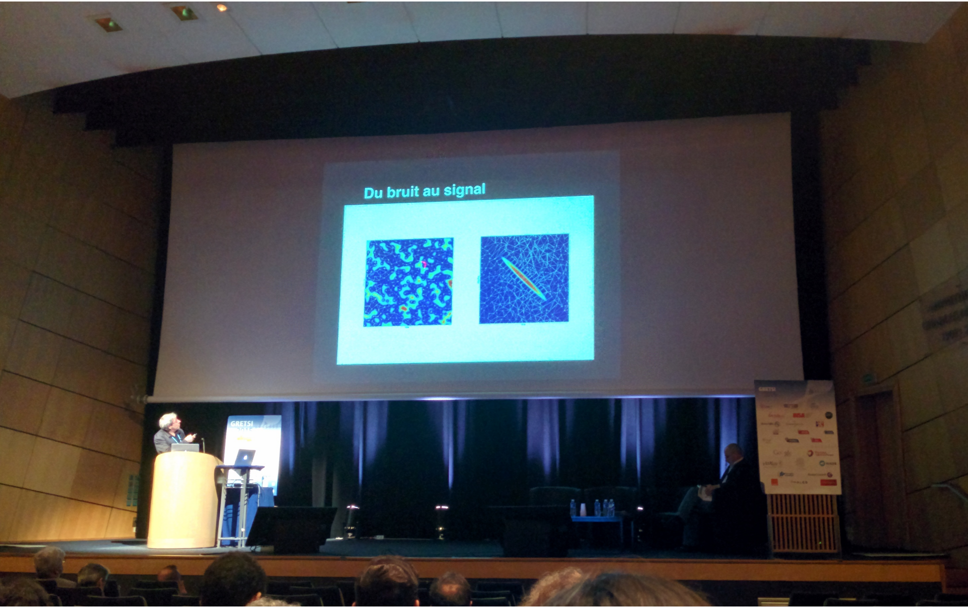
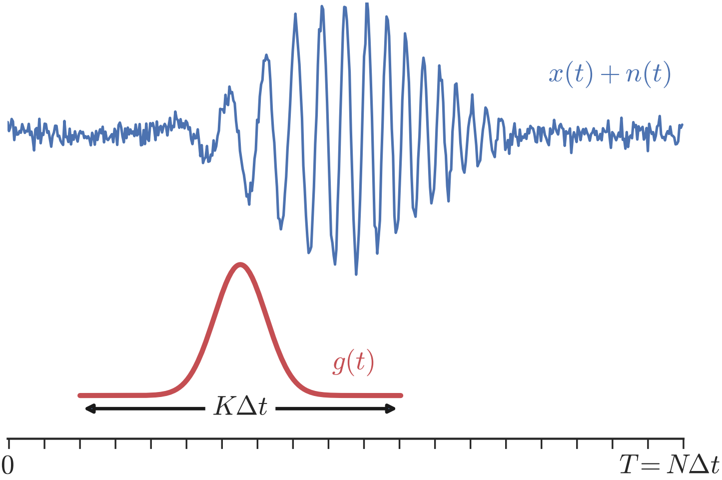
Joint work with R. Bardenet (Lille), J. Flamant (Nancy), P. Flandrin (Lyon)

Gretsi 2015



The short-time Fourier transform

- $V_g f(u,v) = \int f(t)\overline{g(t-u)}e^{-2i\pi tv}dt = \langle f, M_v T_u g\rangle$



The effect of signal onto the distribution of zeros [Flandrin 2015]

Patrick showed these two videos

- One of a spectrogram.
- One of its zeros

where he

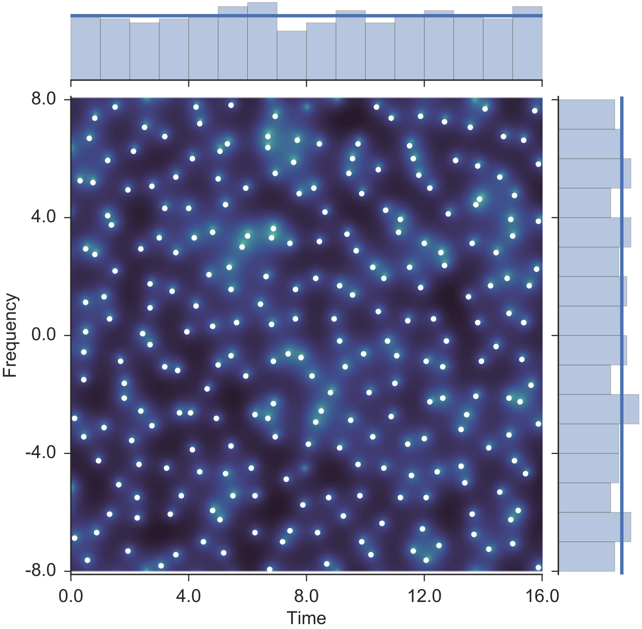
- fixed a synthetic signal $s[i], i = 1, \dots, T$ with three components,
- sampled $n[i] \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ i.i.d.,
- showed the discretized spectrogram of $\alpha s + n$, with α increasing from zero to some large value.

This talk

- a probabilistic model for a Gaussian window g,
- the distribution of the zeros of the STFT of white noise.

Bardenet, Flamant, C., On the zeros of the spectrogram of white noise, ACHA, 2020.

The spectrogram of complex white Gaussian noise



The overarching idea

- We know [Gröchenig 2001] an ON basis (f_k) of L^2 such that the STFT
$$V_g f_k(u,-v) \propto \frac{(u+\mathrm{i}v)^k}{\sqrt{k!}}.$$
- Let $(\xi_n)_{n\geq 1}$ be i.i.d. unit complex Gaussians. We know
  - that $\sum \xi_k \frac{z^k}{\sqrt{k!}}$ defines a random entire function,
  - and we know the distribution of its zeros !
- **Research program**
  - Justify we can take $\sum \xi_k f_k$ to be 'the' white noise.
  - Show that it is reasonable to think
$$\underbrace{V_g\left(\sum \xi_k f_k\right)(u,-v)}_{spectro.of WGN} \propto \underbrace{\sum \xi_k \frac{(u+iv)^k}{\sqrt{k!}}}_{GAF}$$

    and that their zeros are identical.

From (t, f) to $z = u + iv$: the Bargmann transform

Let $f \in L^2(\mathbb{R})$, $u, v \in \mathbb{R}$, $a > 0$, and $z = au + i\frac{v}{a}$, then

$$V_{\gamma_a}(f)(u, -v) \quad \propto \quad e^{-i\pi uv} e^{-\frac{\pi}{2}|z|^2} B\left(f(\cdot/a)\right)(z), \qquad (1)$$

where the Bargmann transform B is defined by

$$Bf(z) = 2^{1/4} \int f(t) e^{2\pi tz - \pi t^2 - \frac{\pi}{2} z^2} dt.$$

$\Rightarrow$ the zeros of the spectrogram $u, v \mapsto |V_{\gamma_a}(f)(u, v)|^2$ of f are the zeros of the Bargmann transform of $s \mapsto f(s/a)$ up to a symmetry w.r.t. the real axis.

A special basis of L^2 with simple Bargmann transform

- Let $H_k(x)$ be the k-th Hermite polynomial, satisfying

$$\int_{\mathbb{R}} H_k(x)H_\ell(x)\mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}2^k k!\,\delta_{k\ell}.$$

- The following functions form an orthonormal basis of L^2

$$f_k(x) := \frac{1}{\sqrt{\sqrt{\pi}2^k k!}} H_k(x)\mathrm{e}^{-x^2/2}.$$

- They satisfy

$$V_g(f_k)(u,-v) \propto \frac{(u+\mathrm{i}v)^k}{\sqrt{k!}}.$$

Random point processes : DPP, Ginibre,...

- **Poisson process** : independent random points, uniform density ;
- **More generally** : k-point correlation function...

$$\rho^{(k)}(x_1,\dots,x_k)dx_1\dots dx_k = \mathbb{P}\left(\begin{array}{c}\text{There are at least } k \text{ points, one in each of the}\\ \text{infinitesimal balls } B(x_i,dx_i), i=1,\dots,k\end{array}\right),$$

e.g. $\qquad g(x,y) = \frac{\rho^{(2)}(x,y)}{\rho^{(1)}(x)\rho^{(1)}(y)},$

- **DPP : Determinantal Point Processes**

$$\rho^{(k)}(x_1,\dots,x_k) = \det\left[\kappa(x_i,x_j)\right]_{1\le i,j\le k}$$

- **Ginibre ensemble** (a classical DPP) :

$$\kappa^{\mathrm{Gin}}(z,w) = e^{-\frac{\pi}{2}|z|^2}e^{\pi z\bar{w}}e^{-\frac{\pi}{2}|w|^2},$$

its pair correlation is

$$g_0^{\mathrm{Gin}}(r) = 1 - e^{-\pi r^2}.$$

Zeros of Gaussian analytic functions

Point process on $\mathbb{C}$ = zeros of random analytic functions

The simplest random analytic functions have Gaussian coefficients : GAF.

Let (ξ_n) be a sequence of i.i.d. complex unit Gaussians, then with probability one

$$\sum \xi_n \frac{z^n}{\sqrt{n!}}$$

converges uniformly on compact subsets of $\mathbb{C}$.

The limit almost surely defines an entire function.

White Gaussian noise(s)

The Bochner-Minlos theorem [Holden et al., 2010] states that there exists a unique probability measure μ_1 on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ (tempered dist.) such that

$$\forall \varphi \in \mathcal{S}, \quad \mathbb{E}_{\mu_1} e^{i\langle \cdot, \varphi \rangle} = e^{-\frac{1}{2}\|\varphi\|_2^2}.$$

We call this measure **Gaussian white noise**, and $(\mathcal{S}', B(\mathcal{S}'), \mu_1)$ the white noise probability space.

Characterizing the zeros of the spectrogram of WGN

$\xi = \mathrm{WGN}(\mu_1)$

Let $u, v \in \mathbb{R}^2$, and write $z = u + iv \in \mathbb{C}$. Then

$$\langle \xi, M_v T_u \gamma \rangle = \sqrt{\pi} e^{i\pi uv} e^{-\frac{\pi}{2}|z|^2} \sum_{k=0}^{\infty} \langle \xi, h_k \rangle \frac{\pi^{k/2} z^k}{\sqrt{k!}}$$

where (h_k) denote the orthonormal Hermite functions, and convergence is in $L^2(\mu_1)$.

The random series

$$\sum_{k=0}^{\infty} \langle \xi, h_k \rangle \frac{\pi^{k/2} z^k}{\sqrt{k!}}$$

μ_1-almost surely defines an entire function.

$\Rightarrow$ identity of the zeros

The planar Gaussian analytic function

Proposition (HannayHough et al '09, Nishry'10)

The planar GAF satisfies the following properties :

1. *The distribution of its zeros is invariant to rotations and translations in the complex plane. In particular, it is a stationary point process.*
2. *Its correlation functions are known. In particular, the intensity in constant equal to* 1*.*
3. *The hole probability*

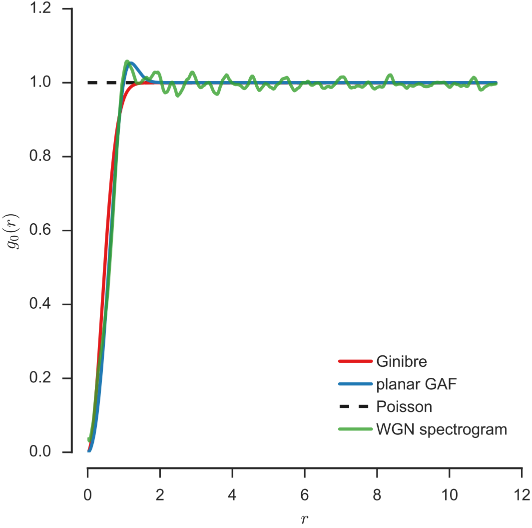
$$p_r = \mathbb{P}(\textit{no points in the disk centered at } 0 \textit{ and with radius } r)$$

scales as

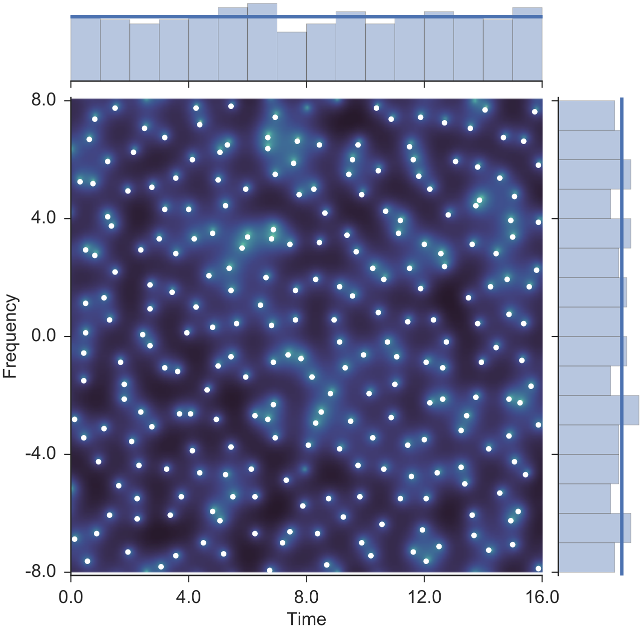
$$r^{-4} \log p_r \to -3e^2/4$$

as $r \to +\infty$*.*

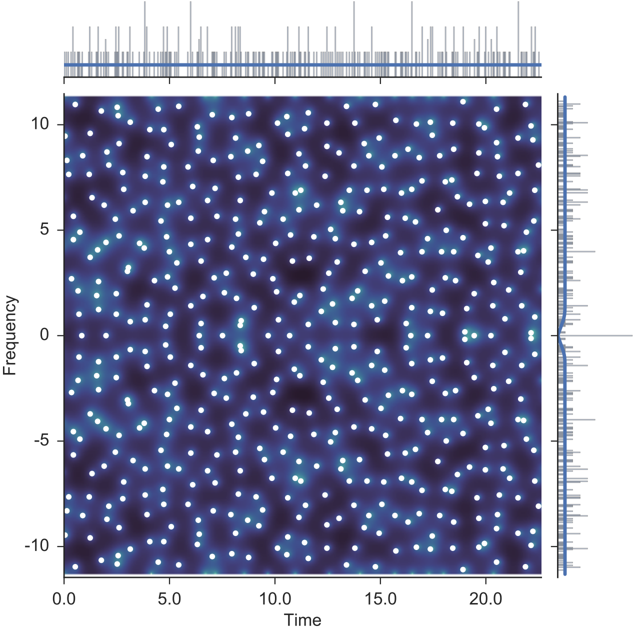
The two-point correlation function



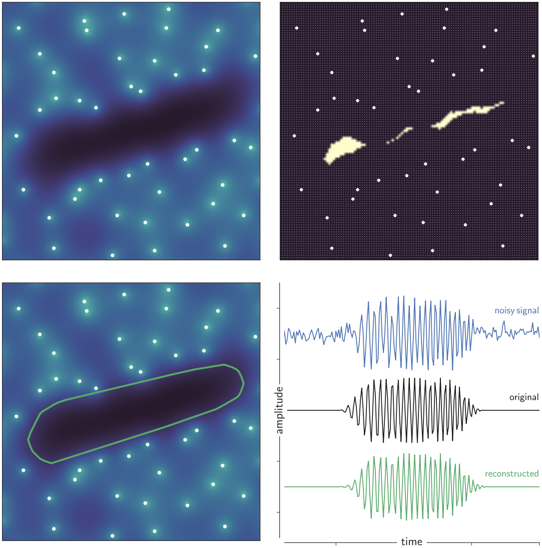
The spectrogram of complex white Gaussian noise



The spectrogram of real white Gaussian noise

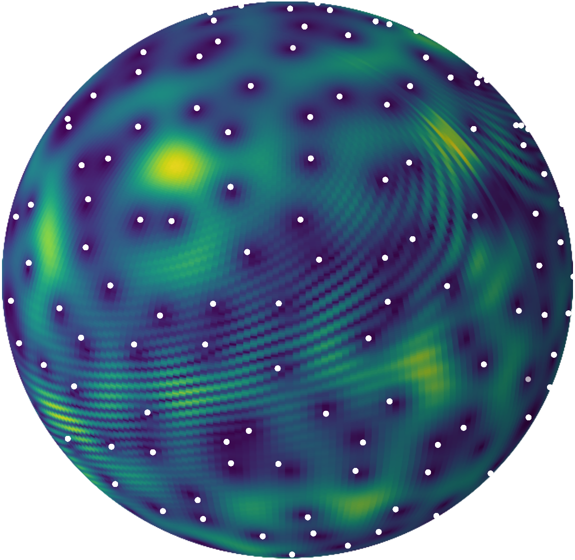


Reconstruction



What more can we get ?

Zeros of the spherical GAF



Open questions

- how efficient can we be with zeros ?
- can we turn the FFT into an exact sampler ?
- can we characterize the zeros in the presence of signal ?
- when are the zeros of a GAF determinantal ?
- can we go beyond complex analysis ?