The Analytic Stockwell Transform and its Zeros

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Analytic signal

Definition

For a signal $f(x) \in \mathbf{L}^2(\mathbb{R})$ the corresponding analytic signal $f^+(x)$ is defined as follows:

$$f^+(x) = 2\mathcal{F}^{-1}\left(1_{\mathbb{R}_+}\mathcal{F}f\right)(x), \forall x \in \mathbb{R}$$

where $\mathcal{F}$ is the Fourier operator:

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi it\xi}dt, \quad \xi \in \mathbb{R}$$

Theorem (Paley-Wiener)

Suppose f and $\hat{f}$ have moderate decrease. Then $\hat{f}(\xi) = 0$ for all $\xi < 0$ if and only if f can be extended to a continuous and bounded function in the closed upper half-plane $\{z = x + iy : y \geq 0\}$ with f holomorphic in the interior.

Analytic Time-Frequency Transform

Let $V_g f(x,\xi)$ and $W_\psi f(x,y)$ be a time-frequency (e.g. STFT rwith window g) and time-scale (e.g. wavelet with mother wavelet ψ) representations for a signal $f(x) \in \mathbf{L}^2(\mathbb{R})$:

$$V_g f(x,\xi) = M_g^f(x,\xi) e^{i\Phi_g^f(x,\xi)}$$

$$W_\psi f(x,y) = M_\psi^f(x,y) e^{i\Phi_\psi^f(x,y)}$$

- Transforms generated by analytic functions: analytic wavelets whose Fourier transform vanishes at negative frequencies $\widehat{\psi}(\xi) = 0$ for $\xi < 0$.
- Transforms that map to the space of analytic functions: a wavelet transform using an analytic wavelet at a fixed scale y_0 also can be extended to an analytic function on the upper half-plane [Holighaus et al. 19].

Analytic STFT and wavelet transforms [Ascensi et al. 09]

Theorem

Consider the model space of a Gabor atom $g \in L^2(\mathbb{R})$ *and* $z = x + i\xi$ *a point in the complex plane:*

$$H_g = \left\{ F(z) = V_g f(x,\xi) = \int_{\mathbb{R}} f(t)\overline{g(t-x)} e^{-2\pi i t\xi} dt, f \in L^2(\mathbb{R}) \right\}$$

Then this space is a space of holomorphic functions, modulo a multiplication by a weight, if and only if g *is a time-frequency translation of the Gaussian function.*

Characterization of analytic wavelets [Holighaus et al. 19]

Theorem

Let $\psi \in \mathbf{L}^2(\mathbb{R})$ *with* $\widehat{\psi}(\xi) = 0$ *for* $\xi < 0$. *There exist constants* $a \in \mathbb{R}, b \in \mathbb{R}^+$ *and a* $\mathcal{C}^\infty$ *function* $I : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$ *with* $I(x, y) \neq 0$ *such that*

$$\begin{aligned} h : \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} &\to \mathbb{C} \\ x + iy &\mapsto I(x, y) W_\psi f(x - aby, by) \end{aligned}$$

is analytic for all $f \in \mathbf{L}^2(\mathbb{R})$, *if and only if*

$$\widehat{\psi}(\xi) = c\xi^{\frac{\alpha-1}{2}} e^{-2\pi\gamma\xi} e^{i\beta \log \xi}$$

Phase retrieval

Direct connection between the phase and magnitude (phase-magnitude relationship)

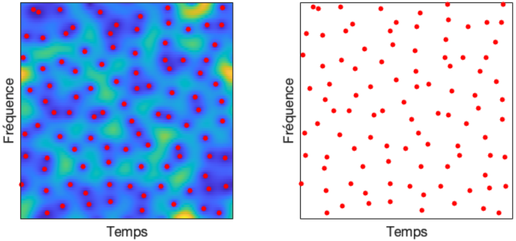
- For the STFT with gaussian window $g(t) = \lambda^{-1/2}\pi^{-1/4}e^{-t^2/(2\lambda^2)}$ [Auger 12]

$$\frac{\partial}{\partial x}\phi_g^f(x,\xi) = \lambda^{-2}\frac{\partial}{\partial \xi}\log\left(M_g^f(x,\xi)\right) + \frac{\xi}{2}$$
$$\frac{\partial}{\partial \xi}\phi_g^f(x,\xi) = -\lambda^{2}\frac{\partial}{\partial x}\log\left(M_g^f(x,\xi)\right) - \frac{x}{2}$$

- For the wavelet transform with $\hat{\psi}(\xi) = \xi^{\frac{\alpha-1}{2}}e^{-2\pi\gamma\xi}e^{i\beta\log\xi}$ [Holighaus et al. 19]

$$\frac{\partial}{\partial x}\phi_\psi^f(x,y) = -\frac{\partial}{\partial y}\log\left(M_\psi^f(x,y)\right) + \frac{\alpha}{2y}$$
$$\frac{\partial}{\partial y}\phi_\psi^f(x,y) = \frac{\partial}{\partial x}\log\left(M_\psi^f(x,y)\right) - \frac{\beta}{y}$$

Bridge between TF methods and GAFs



Let $z = x + i\xi \in \mathbb{C}$ and a is WGN. The STFT of white noise $V_g a(z)$ coincides with planar GAF [Bardenet 20]:

$$V_g a(z) = \sqrt{\pi} e^{i\pi x\xi} e^{-\frac{\pi}{2}|z|^2} \sum_{k=0}^{\infty} \langle a, h_k \rangle \frac{\pi^{k/2} z^k}{\sqrt{k!}}$$

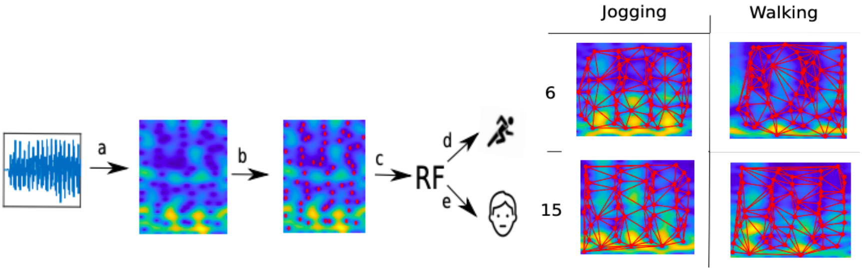
$$\mathbf{GAF}_{\mathbb{C}}^{(\ell)}(z) := \sum_{k=0}^{\infty} a_k \frac{1}{\sqrt{k!}} \left(\frac{z}{\ell}\right)^k$$

Correspondences TF-GAFs [Bardenet 21]

$\mathcal{H}$	Transformation	Polynôme	GAF	Théorème
$L^2(\mathbb{R},\mathbb{C})$	$\frac{\mathrm{e}^{-z^2/2}}{\pi^{1/4}} \int_{\mathbb{R}} \overline{f(x)} \mathrm{e}^{\sqrt{2}xz-x^2/2}\, \mathrm{d}x$	Hermite	$\mathbb{C}$	Th. 2.1
$\ell^2(\mathbb{N},\mathbb{C})$	$\sum_{x\in\mathbb{N}} \overline{f(x)} \frac{z^x}{\sqrt{x!}}$	Charlier	$\mathbb{C}$	Th. 2.2
$H^2(\mathbb{R})$	$\frac{1}{(1-z)^{2\beta+1}} \int_{\mathbf{R}_+} \overline{\hat{f}(x)} x^\beta \mathrm{e}^{-\frac{x}{2}\frac{1+z}{1-z}}\, \mathrm{d}x$	Laguerre	$\mathbb{H}$	Th. 2.3
$\ell^2(\mathbb{N},\mathbb{C})$	$\sum_{x\in\mathbb{N}} \overline{f(x)} \sqrt{\frac{\Gamma(x+\alpha+1)}{x!}} z^x$	Meixner	$\mathbb{H}$	Th. 2.4
$\mathbb{C}^{N+1}$	$\sum_{x=0}^{N} \overline{f(x)} \sqrt{\binom{N}{x}} z^x$	Krawtchouk	$\mathbb{S}$	Th. 2.5

Filtering and feature extraction based on the TF zeros

Use the distribution of zeros in the time-frequency plane to filter non stationary signals [Flandrin 15, Bardenet 20] and extract features from acceleration signals [Rouge 22].

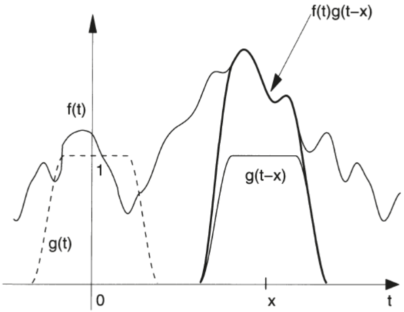


Short-Time Fourier Transform (STFT)

Let $f \in L^2(\mathbb{R})$ be a signal and $g \in L^2(\mathbb{R}, \mathbb{C})$ the analysis window function. The STFT of f respect to g, denoted $V_g f$ is defined as :

$$V_g f(x, \xi) = \langle f, \mathbf{M}_\xi \mathbf{T}_x g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i t \xi} dt \qquad \text{pour } x, \xi \in \mathbb{R}.$$

with $\mathbf{M}_\xi f(t) = e^{2\pi i \xi t} f(t)$ et $\mathbf{T}_x f(t) = f(t-x)$.



Continuous Wavelet Transform (CWT)

Let a mother wavelet function $\psi \in \mathbf{L}^2(\mathbb{R})$, the continuous wavelet transform of f, denoted $W_\psi f$ is given as:

$$W_\psi f(x,y) = \langle f, \mathbf{T}_x \mathbf{D}_y \psi \rangle = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(t) \overline{\psi\left(\frac{t-x}{y}\right)} dt$$

The admissibility constant C_ψ of a wavelet ψ is given as:

$$C_\psi = \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{\xi} d\xi$$

and for the wavelet ψ to be admissible it is necessary that $C_\psi < \infty$.

An hybrid version: The Stockwell Transform

Let $f \in \mathbf{L}^2(\mathbb{R})$ be a signal. The ST with respect to the window $g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$ with $\sigma = 1/|\xi|$, denoted $S_g f$ can be given as [Stockwell 96]:

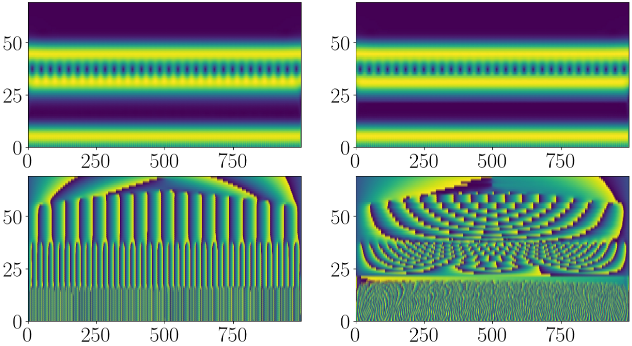
$$S_g f(x,\xi) = \frac{|\xi|}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-(t-x)^2\xi^2/2} e^{-2\pi i t\xi} dt, \quad x \in \mathbb{R}, \xi \in \mathbb{R}^*$$

An alternative formulation with respect to Fourier transform of f deduced by rewriting $S_g f(x,\xi)$ as a convolution product, can be given as:

$$S_g f(x,\xi) = \int_{\mathbb{R}} \widehat{f}(\nu + \xi) e^{-2\pi^2\nu^2/\xi^2} e^{2\pi i x \nu} d\nu$$

Phase information

For ST we have an absolute referenced phase information: the oscillatory kernel $e^{-2\pi i t\xi}$ remains stationary while translating the time localizing envelope (Gaussian window).



The Generalized Stockwell Transform (GST)

Definition

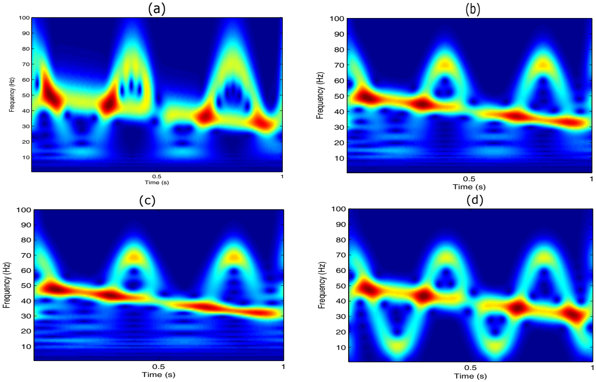
Let $\varphi \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ be an arbitrary window such that $\int_{\mathbb{R}} \varphi(t)dt = 1$ and whose width is adjusted by an arbitrary function $\sigma(\xi)$. Then the generalized Stockwell transform of f, denoted $S_{\varphi}^{\sigma} f$ can be written as:

$$S_{\varphi}^{\sigma} f(x, \xi) = \langle f, \mathbf{M}_{\xi} \mathbf{T}_x \mathbf{D}_{\sigma(\xi)} \varphi \rangle = \frac{1}{\sigma(\xi)} \int_{\mathbb{R}} f(t) \overline{\varphi\left(\frac{t-x}{\sigma(\xi)}\right)} e^{-2\pi i t\xi} dt$$

By choosing $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ and $\sigma(\xi) = 1/|\xi|$, we retrieve the classical ST. For more "flexibility" we can introduce more parameters on $\sigma(\xi)$, e.g. [Moukadem et al. 15]:

$$\sigma(\xi) = \frac{m\xi^p + k}{\xi^r}, \quad m, p, k, r \in \mathbb{R}$$

Examples



The Generalized Stockwell Transform (GST)

Proposition

Let $\hat{f}$ the Fourier transform of a signal $f \in \mathbf{L}^2(\mathbb{R})$, the generalized Stockwell Transform can be then formulated as follows:

$$S_\varphi^\sigma f(x,\xi) = \int_{\mathbb{R}} \widehat{f}(\nu+\xi)\overline{\widehat{\varphi}(\sigma(\xi)\nu)} e^{2\pi i x\nu} d\nu, \quad x \in \mathbb{R}, \xi \in \mathbb{R}^*$$

Proof.

The GST can be written as a convolution product as follows:

$$S_\varphi^\sigma f(x,\xi) = \mathbf{M}_\xi f * \mathbf{D}_{\sigma(\xi)} \overline{\tilde{\varphi}}(t)$$

where $\tilde{\varphi}(t) = \overline{\varphi(-t)}$. By applying the Fourier transform in both sides we obtain:

$$\mathcal{F}\left\{S_\varphi^\sigma f(x,\xi)\right\} = \widehat{f}(\nu+\xi)\overline{\widehat{\varphi}(\nu\sigma(\xi))}$$

Therefore, $S_\varphi^\sigma f$ can be obtained by applying the inverse Fourier transform $\mathcal{F}^{-1}$:

Relation with the CWT

Proposition

The GST S_φ^σ with a generalized window $\varphi \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ adjusted by an arbitrary function $\sigma(\xi)$, can be written as a Wavelet transform as follows:

$$S_\varphi^\sigma f(x,\xi) = \frac{e^{-i2\pi\xi x}}{\sqrt{\sigma(\xi)}} W_\psi f(x, \sigma(\xi))$$

with a mother wavelet $\psi(t)$, can be expressed as a function of the generalized window $\varphi(t)$:

$$\psi(t) = \varphi(t) e^{i2\pi\xi\sigma(\xi)t}$$

and satisfying the admissibility condition $C_\psi = \int_{\mathbb{R}^+} \frac{|\widehat{\psi}(\xi)|^2}{\xi} d\xi < \infty$, which can be written in terms of C_φ:

$$C_\varphi = \int_{\mathbb{R}^+} \frac{|\widehat{\varphi}(\xi - \xi\sigma(\xi)|^2}{\xi} d\xi < \infty$$

Relation with the CWT

Proof.

To establish the link between the WT and the generalized ST, we set the mother wavelet as $\psi(t) = \varphi(t)e^{i2\pi\xi\sigma(\xi)t}$ and $y = \sigma(\xi)$. Therefore, $W_\psi f(x, y)$ can be written as:

$$
\begin{aligned}
W_\psi f(x, \sigma(\xi)) &= \frac{1}{\sqrt{\sigma(\xi)}} \int_{\mathbb{R}} f(t)\overline{\varphi\left(\frac{t-x}{\sigma(\xi)}\right)} e^{-i2\pi\xi\sigma(\xi)\left(\frac{t-x}{\sigma(\xi)}\right)} dt \\
&= \frac{e^{i2\pi\xi x}}{\sqrt{\sigma(\xi)}} \int_{\mathbb{R}} f(t)\overline{\varphi\left(\frac{t-x}{\sigma(\xi)}\right)} e^{-i2\pi\xi t} dt \\
&= \sqrt{\sigma(\xi)} e^{i2\pi\xi x} S^\sigma_\varphi(x, \xi)
\end{aligned}
$$

□

GST from a wavelet point of view

Corollary

Let $\varphi \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ with $\widehat{\varphi}(\nu) = 0$ for $\nu < 0$ and $\sigma(\xi)$ a function which controls the width of φ. There exist a $\mathcal{C}^\infty$ function $I : \mathbb{R}\times \mathbb{R}^+ \to \mathbb{C}$ with $I(x,\xi) \neq 0$ such that

$$h : \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \to \mathbb{C}$$
$$x + iy \mapsto I(x,y) S_\varphi^y f(x,\xi)$$

is analytic for all $f \in \mathbf{L}^2(\mathbb{R})$, if and only if

$$\hat{\varphi}(\nu) = \begin{cases} c(\nu + \xi\sigma(\xi))^{\frac{\alpha-1}{2}} e^{-2\pi\gamma(\nu+\xi\sigma(\xi))} e^{i\beta \log(\nu+\xi\sigma(\xi))} & \nu \in \mathbb{R}^+ \\ 0 & \textit{otherwise} \end{cases} \qquad (1)$$

In the case of $\sigma(\xi) = 1/\xi$, we have $\hat{\varphi}(\nu) = c(\nu+1)^{\frac{\alpha-1}{2}} e^{-2\pi\gamma(\nu+1)} e^{i\beta \log(\nu+1)}$.

The partial derivatives of $S^{\sigma}_{\varphi}(x, \xi)$

Proposition

Let a signal $f(t) \in \mathbf{L}^2(\mathbb{R})$, an arbitrary window $\varphi(t) \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$, a function $\sigma(\xi)$ such that $\int_{\mathbb{R}} \varphi(t)dt = 1$, then for all x and $\xi \in \mathbb{R}$ the partial derivatives of the generalized Stockwell transform of f, denoted $S^{\sigma}_{\varphi}(x, \xi)$ can be given as follows:

$$\frac{\partial}{\partial x} S^{\sigma}_{\varphi} f(x, \xi) = \frac{-1}{\sigma(\xi)} S^{\sigma}_{\varphi'} f(x, \xi) \tag{2}$$

and

$$\frac{\partial}{\partial \xi} S^{\sigma}_{\varphi} f(x, \xi) = \frac{-\sigma'(\xi)}{\sigma(\xi)} S^{\sigma}_{(\mathbf{T}\varphi)'} f(x, \xi) - i2\pi \left(S^{\sigma}_{(\mathbf{T}\varphi)} f(x, \xi) + x S^{\sigma}_{\varphi} f(x, \xi) \right) \tag{3}$$

where $\mathbf{T}$ denotes the time-weighting operator $\mathbf{T} f(t) = t f(t)$.

Time-Frequency interpretation

Theorem

Let $\varphi \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ with $\widehat{\varphi}(\nu) = 0$ for $\nu < 0$ and $\sigma(\xi)$ a function which controls the width of φ. There exist a $\mathcal{C}^\infty$ function $I : \mathbb{R}\times \mathbb{R}^+ \to \mathbb{C}$ with $I(x,\xi) \neq 0$ such that

$$\begin{aligned} h : \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} &\to \mathbb{C} \\ x + i\xi &\mapsto I(x,\xi) S_\varphi^\sigma f(x,\xi) \end{aligned}$$

is analytic for all $f \in \mathbf{L}^2(\mathbb{R})$, if and only if

$$\hat{\varphi}(\nu) = c e^{i \frac{\sigma(\xi)}{\sigma'(\xi)} \overline{g} \ln(\sigma'(\xi)\nu + \sigma(\xi))} e^{\frac{2\pi}{\sigma'(\xi)}\nu} e^{-2\pi \frac{\sigma(\xi)}{(\sigma'(\xi))^2} \ln(\sigma'(\xi)\nu + \sigma(\xi))}$$

for all $\nu > 0$ and $g(x,\xi) = \frac{\frac{\partial}{\partial x} I(x,\xi) + 2\pi x I(x,\xi) + i \frac{\partial}{\partial \xi} I(x,\xi)}{I(x,\xi)}$

Elements of proof

The analyticity of h is equivalent to the satisfaction of the Cauchy-Riemann equations $\frac{\partial}{\partial x} h = -i \frac{\partial}{\partial \xi} h$. This will lead to the following differential equation:

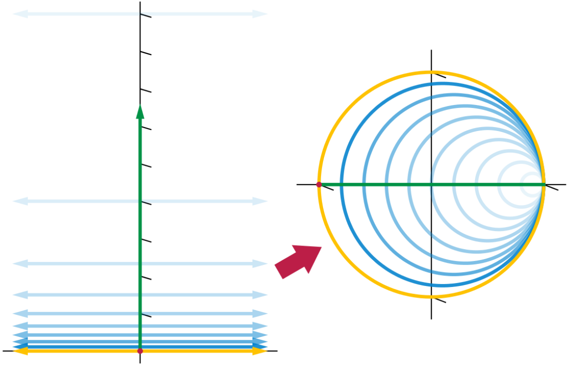
$$\left(\overline{g} - \frac{i2\pi\nu}{\sigma(\xi)} \right) \hat{\varphi}(\nu) - i \left(\frac{\sigma'(\xi)\nu}{\sigma(\xi)} - 1 \right) (\hat{\varphi}(\nu))' = 0$$

which gives the given solution $\hat{\varphi}(\nu)$. For the case $\sigma(\xi) = 1/\xi$, it can be written as:

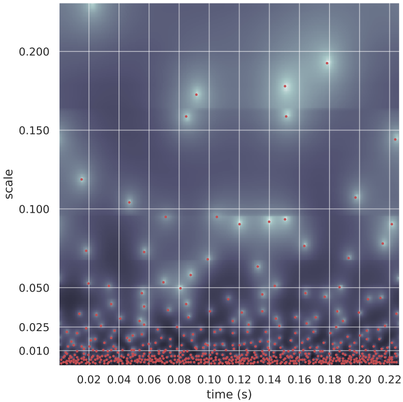
$$\hat{\varphi}(\nu) = c(\nu + \xi)^{\left(\mathrm{Im}(g)\xi + 2\pi\xi^3\right)} e^{i\xi\,\mathrm{Re}(g)\log(\nu+\xi)} e^{-2\pi\xi^2\nu}$$

Poincaré Disk Model for Hyperbolic geometry

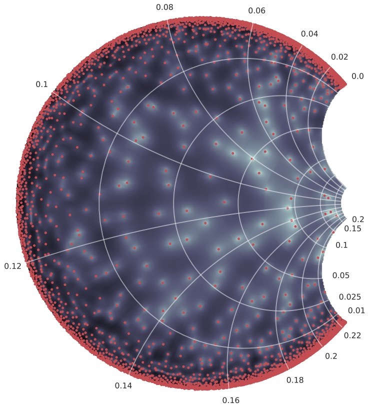
Conformal map between half plane model and Poincaré disk model (the Cayley transform)



The zeros of the AST of white noise

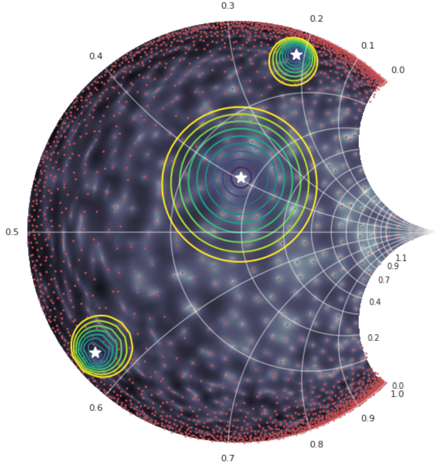


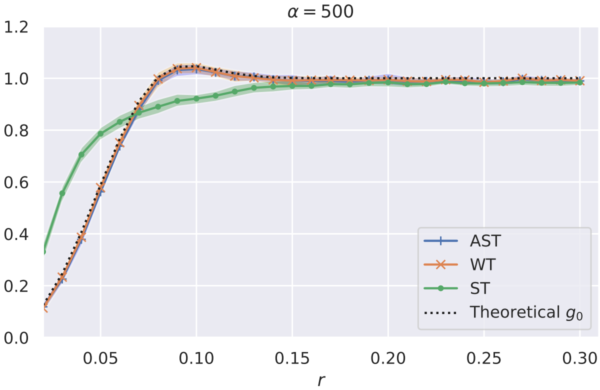
(c) Time-scale representation of the AST and its zeros set.



(d) Poincaré projection of the AST and its zeros set

Estimation of the pair correlation function





Summary

- The ST is an hybrid version between STFT and CWT.
- Generalized ST allows us to define an analytical ST under certain conditions in time-scale and time-frequency planes.
- It seems that analytic ST, like analytic WT, coincides with hyperbolic GAF (confirmed empirically).

Perspectives

- Application to filtering, feature extraction and phase retrieval (comparisons with existing methods based on STFT and WT).
- Extension of the work carried out in [Courbot et al. 22] "Sparse off-the-grid computation of the zeros of STFT" to the zeros of the ST.