Using divergence-free and curl-free wavelets for the simulation of turbulent flows

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We present a numerical method based on divergence-free and curl-free wavelets to solve the incompressible Navier-Stokes equations. We introduce a new scheme which uses anisotropic divergence-free wavelets for the decomposition of the velocity, and which only needs Fast Wavelet Transform algorithms. Numerical results finally show the feasibility of the method.

1 Introduction

The prediction of fully-developed turbulent flows represents an extremely challenging field of research in scientific computing. The Direct Numerical Simulation (DNS) of turbulence requires the integration in time of the full nonlinear Navier-Stokes equations: in order to compute accurately all the scales of the turbulent flow, the discretizations in space and time must be of very small size, leading to a huge number of degrees of freedom. In that context, wavelet bases provide a decomposition of the solution allowing to represent the intermittent spatial structure of turbulent flows with only few degrees of freedom. For the incompressible Stokes problem, an interesting approach first considered by K. Urban [8], was to use the divergence-free wavelet bases originally designed by P.G. Lemarié-Rieusset [6].
In that paper we present a new scheme for solving the incompressible Navier-Stokes equations based on divergence-free and curl-free vector wavelets. The method is based on the Helmholtz decomposition of the nonlinear term in the wavelet domain. Finally we present numerical tests in 2D to validate the approach.

2 A new numerical scheme for Navier-Stokes equations

We consider the incompressible Navier-Stokes equations, written in velocity-pressure formulation (without forcing term) in the whole space ($d = 2, 3$):

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = 0\ ,\ t\in[0,T],\ \mathbf{x}\in\mathbb{R}^d \\ \nabla\cdot\mathbf{u} = 0 \end{array} \right. \tag{1}$$

with initial data $\mathbf{u}_0$.We summarize below the different steps for the construction of the numerical scheme.

(i) Leray formulation of Navier-Stokes equations: equation (1) is first projected onto the space of divergence-free vector functions. Let $\mathbb{P}$ be the Leray projector onto this space, (1) rewrites:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbb{P}\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \nu\Delta\mathbf{u} = 0 \tag{2}$$

The pressure is eliminated, but it can be simply recovered through the Helmholtz decomposition of the nonlinear term: the Helmholtz decomposition of the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ consists in splitting $(\mathbf{u}\cdot\nabla)\mathbf{u}$ into a divergence-free part, and a gradient function, which writes:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbb{P}\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \nabla p$$

(ii) Decomposition of the velocity onto divergence-free vector wavelets:
In order to fulfill the incompressible condition, the velocity $\mathbf{u}$ is searched as a sum of divergence-free wavelets:

$$u(t,\mathbf{x}) = \sum_{\mathbf{j},\mathbf{k}} d_{\mathbf{j},\mathbf{k}}(t)\ \Psi^{div}_{\mathbf{j},\mathbf{k}}(\mathbf{x})$$

Such a basis exists, since the pioneer work of Lemarié [6]. In [3], we have constructed anisotropic divergence-free wavelets, and derived fast decomposition algorithms (in O(N) operations where N is the number of grid points).

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(iii) Decomposition of the pressure gradient onto curl-free wavelets: Curl-free wavelets have been defined by simply taking the gradient of a differentiable wavelet basis in $\mathbb{R}^d$ [9]. Fast algorithms associated to anisotropic curl-free wavelets $\Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}}$ have been implemented in [3]. The gradient of the pressure will be searched as a finite sum:

$$\nabla p(t,\mathbf{x}) = \sum_{\mathbf{j},\mathbf{k}} d^{\text{curl}}_{\mathbf{j},\mathbf{k}}(t)\ \Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}}(\mathbf{x})$$

Since the curl-free wavelets are wavelet gradients, this formula provides readily the pressure at grid points, by a (fast) inverse transform applied to the coefficients $d^{\text{curl}}_{\mathbf{j},\mathbf{k}}$ in the appropriate wavelet basis.

(iv) Computation of the wavelet coefficients of the velocity and of the pressure: In order to solve (2), we introduce a semi-implicit finite-difference discretization in time, for instance (in practice we will use a second order scheme):

$$(I - \nu\delta t\ \Delta)\mathbf{u}^{n+1} = \mathbf{u}^n + \delta t\ \mathbb{P}\left[(\mathbf{u^n}\cdot\nabla)\mathbf{u^n}\right] \tag{3}$$

where $\mathbf{u}^n(x) \approx \mathbf{u}(n\delta t, x) \approx \sum_{\mathbf{j},\mathbf{k}} d^n_{\mathbf{j},\mathbf{k}}\ \Psi^{div}_{\mathbf{j},\mathbf{k}}(\mathbf{x})$. Then at each time-step n the computation of the $d^{n+1}_{\mathbf{j},\mathbf{k}}$ requires:

1) the Helmholtz decomposition of the nonlinear term: the grid values of $(\mathbf{u^n}\cdot\nabla)\mathbf{u^n}$ are recovered by an inverse (div-free) wavelet transform from the $d^n_{\mathbf{j},\mathbf{k}}$, followed by pointwise products in physical space. The Helmholtz decomposition in the wavelet domain corresponds to $(L^2(\mathbb{R}^d))^d = span\{\Psi^{\text{div}}_{\mathbf{j},\mathbf{k}}\} \oplus span\{\Psi^{\text{curl}}_{\mathbf{j},\mathbf{k}}\}$. A method for computing the divergence-free wavelet coefficients and the curl-free wavelet coefficients of any vector function has been presented in [2, 4]. This method is here applied for computing the div-free wavelet coefficients of $\mathbb{P}\left[(\mathbf{u^n}\cdot\nabla)\mathbf{u^n}\right]$, and the curl-free wavelet coefficients of ∇p^{n+1}.

2) a resolution of the discrete Heat equation in the wavelet domain: this second point is achieved by mean of an iterative scheme developed in [1], based on divergence-free wavelet coefficients of both sides, and which only uses Fast Wavelet Transforms.

3 Numerical results

We present below a numerical experiment in 2D with periodic boundary conditions to test the ability of our approach. It consists in a three vortex interaction often used in the literature to test new numerical methods [7]. This experiment is precisely described in [5]: figure 1 shows the time evolution of the vorticity field, reconstructed from the wavelet coefficients of the velocity field. In this experiment, only the "active" wavelet coefficients of the velocity (which means coefficients larger in absolute value than a given threshold) have been used in the computations. Each time-step requires in practice a few number of fast wavelet transforms to solve the system (3). The obtained result is very close to the result of a pseudo-spectral code with the same resolution (256^2), but with a few number of active modes in the solution (less than 0.6%)!

a) b) c) d)

Fig. 1 "Merging of three vortices" with the wavelet code 256^2, adding a threshold on the coefficients (depending on the scale). a) t=0, b) t=10, c) t=20, d) t=40

Current works are on the way to extend the method in three dimensions (there is no theoretical obstacle), and for physical boundary conditions, which will require the construction of divergence-free wavelets in bounded square domains.

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