

# MULTICOMPONENT SIGNAL DENOISING WITH SYNCHROSQUEEZING

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## ABSTRACT

In this paper, we develop a new technique based on the synchrosqueezing method to denoise multicomponent signals. The approach proposed is based on a two step strategy: a mode detection step followed by a reconstruction one. The emphasis is put on the robustness of the detection step in a noisy context, a key issue in the implementation of the method. Numerical applications show the improvement in terms of multicomponent signal denoising brought about by the proposed method over the translation-invariant wavelet thresholding.

**Index Terms**— Synchrosqueezing, Wavelets, Time-Frequency, Denoising

## 1. INTRODUCTION

A popular class of techniques to carry out the retrieval of the components of a multicomponent signal are either based on time-frequency or time-scale signal representations. In such frameworks, many different approaches have been proposed, such as spectrogram reassignment techniques [1], reconstruction based on  $l_1$  minimization of the ambiguity function associated with the Wigner-Ville distribution [2], synchrosqueezing using the wavelet or the short-time Fourier transforms [3][4] or Fourier ridges [5]. Apart the notable exception of the approach of Carmona *et al.* [5], they are not designed for component retrieval in a noisy context. In [5], an algorithm based on a Markov Chain Monte Carlo was proposed, the derivation of which was quite involved.

In the present paper, we introduce a variant of the technique proposed in [6], based on the synchrosqueezing method (SM) introduced by Daubechies *et al.* [4], which enables component retrieval in a noisy context. The focus is on additive Gaussian white noise and exploiting the properties of the wavelet decomposition for that type of noise to set the parameters of the method. We show that this new algorithm for component retrieval provides a denoising technique that outperforms the translation-invariant wavelet thresholding method on the signals studied.

## 2. SYNCHROSQUEEZING BASICS

Let us define a multicomponent signal  $f$  by:

$$f(t) = \sum_{k=1}^K A_k(t) \cos(2\pi\phi_k(t)) = \sum_{k=1}^K f_k(t). \quad (1)$$

The principle of the SM is to retrieve the modes  $f_k$  by appropriately reallocating the wavelet transform of  $f$ . For the sake of consistency, we recall some theoretical results stated in [4] which were established for  $f$  defined as a superposition of *intrinsic-mode-type* functions (IMT):

**Definition 1** A continuous function  $f : \mathbb{R} \rightarrow \mathbb{R} \in L^\infty(\mathbb{R})$  is said to be of *intrinsic-mode-type* (IMT) with accuracy  $\epsilon$  if  $f(t) = A(t) \cos(2\pi\phi(t))$  with  $A$  and  $\phi$  satisfying the following properties:

$$\begin{aligned} & A \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \phi \in C^2(\mathbb{R}) \\ & \inf_{t \in \mathbb{R}} \phi'(t) > 0, \sup_{t \in \mathbb{R}} \phi'(t) < \infty, \sup_{t \in \mathbb{R}} |\phi''(t)| < \infty \\ & A(t) > 0, |A'(t)|, |\phi''(t)| \leq \epsilon |\phi'(t)|, \forall t \in \mathbb{R}. \end{aligned}$$

**Definition 2** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is said to be a superposition of well-separated IMTs with accuracy  $\epsilon$  (or  $\epsilon$ -IMT) and separation  $d$  (the set of which is denoted by  $\mathcal{A}_{\epsilon,d}$  in the sequel), if there exists a finite  $K$  such that  $f(t) = \sum_{k=1}^K f_k(t)$  where all the  $f_k$  are  $\epsilon$ -IMTs satisfying  $\phi'_k(t) > \phi'_{k-1}(t)$  and  $|\phi'_k(t) - \phi'_{k-1}(t)| \geq d(\phi'_{k-1}(t) + \phi'_k(t))$ .

Note that  $d$  will be called in the sequel the *separation condition*. Let us denote by  $\hat{f} := \int_{\mathbb{R}} f(x) e^{-2i\pi\xi x} dx$  the Fourier transform of  $f \in L^1(\mathbb{R})$  and  $W_f(a, t) := \int_{\mathbb{R}} f(x) \frac{1}{a} \overline{\Psi(\frac{x-t}{a})} dx$  its continuous wavelet transform ( $\Psi$  being in  $L^2(\mathbb{R})$  and satisfying the condition  $\int_0^{+\infty} \frac{|\hat{\Psi}(\xi)|^2}{\xi} d\xi < +\infty$ ). Recalling the description  $\bar{\omega}(a, t) = \frac{\partial_t W_f(a, t)}{2i\pi W_f(a, t)}$  of the frequency representation of  $f$ , the following theorem defines the SM from a theoretical point of view [4]:

**Theorem 1** Let  $f$  be a function in  $\mathcal{A}_{\epsilon,d}$  and set  $\tilde{\epsilon} = \epsilon^{1/3}$ . Select a function  $h$  in  $C_c^\infty(\mathbb{R})$  with  $\int h(t) dt = 1$  such that  $\frac{1}{\alpha} h(\frac{\cdot}{\alpha}) \rightarrow \delta$ , the Dirac distribution when  $\alpha \rightarrow 0$  and a

wavelet  $\Psi$  in the Schwartz class such that its Fourier transform  $\hat{\Psi}$  is supported in  $[\xi_\Psi - \Delta, \xi_\Psi + \Delta]$ , with  $\Delta < \xi_\Psi d / (1 + d)$ . Consider the function obtained by synchrosqueezing  $W_f$ , with threshold  $\tilde{\epsilon}$  and accuracy  $\delta$ , i.e.

$$S_{f,\tilde{\epsilon}}^\delta(t, \omega) = \int_{A_{\tilde{\epsilon},f}(b)} W_f(a, t) \frac{1}{\alpha} h\left(\frac{\omega - \bar{\omega}(a, t)}{\alpha}\right) \frac{da}{a},$$

where  $A_{\tilde{\epsilon},f} = \{a \in \mathbb{R}^+; |W_f(a, t)| > \tilde{\epsilon}\}$ . Then provided  $\epsilon$  is sufficiently small, the following condition holds:

- $|W_f(a, t)| > \tilde{\epsilon}$  only when for some  $1 \leq k \leq K$ ,  $(a, t) \in Z_k := \{(a, t), |a\phi'_k(t) - \xi_\Psi| < \Delta\}$ .
- For each  $k$  and pair  $(a, t) \in Z_k$ , for which  $|W_f(a, t)| > \tilde{\epsilon}$ , we have  $|\bar{\omega}(a, t) - \phi'_k(t)| \leq \tilde{\epsilon}$
- For each  $k$  there exists a constant  $C$  such that for any  $t$  in  $\mathbb{R}$ ,

$$\left| \lim_{\delta \rightarrow 0} \frac{2}{C_\Psi} \mathcal{R}e \left[ \int_{B_{\tilde{\epsilon},k}} S_{f,\tilde{\epsilon}}^\delta(t, \omega) d\omega \right] - A_k(t) \cos(2\pi\phi_k(t)) \right| \leq C\tilde{\epsilon},$$

where  $C_\Psi = \int_0^\infty \frac{\hat{\Psi}(\xi)}{\xi} d\xi$  and  $B_{\tilde{\epsilon},k} := \{\omega, |\omega - \phi'_k(t)| < \tilde{\epsilon}\}$ .

To summarize, the basic idea exploited by the SM to obtain the  $k^{\text{th}}$  mode is to sum up the wavelet transform in the frequency domain by considering that  $\bar{\omega}(a, t)$  is a good approximation of  $\phi'_k(t)$  for  $(a, t)$  in  $Z_k$ . Some limitations of modes reconstruction with synchrosqueezing are that  $\phi'_k(t)$  is a priori unknown and that the accuracy of the method requires a small modulation parameter  $\epsilon$ . Another one is related to the fact that the wavelet must be analytic, i.e.  $\hat{\Psi}(\xi) = 0$  when  $\xi < 0$ , and determined depending on  $\epsilon$  and  $d$  (for more details on the wavelet choice see [6]). An important issue is about the potential presence of noise which both makes the estimation of  $\phi'_k(t)$  harder and the wavelet coefficients overtake  $\tilde{\epsilon}$ . Despite these drawbacks, we show in what follows that some ideas contained in the SM can be used to derive a components retrieval algorithm particularly efficient in a noisy context.

### 3. PRACTICAL IMPLEMENTATION

The algorithm for components retrieval we introduce next is a variant of that proposed in [6], putting the emphasis on the treatment of noisy signals. Our approach is based on two distinct steps: a mode detection step followed by a mode reconstruction one. The detection step is based on the determination of the number of modes for which the following quantity is introduced:

$$C_f(\gamma, t) = \#(\{j, |W_f(a_j, t)| < \gamma \text{ and } |W_f(a_{j+1}, t)| > \gamma\}), \quad (2)$$

where  $\#X$  is the cardinality of  $X$  and where  $a_j = 2^{j/n_v} \Delta t$ ,  $\Delta t$  being the time span and  $j = 0, \dots, L n_v - 1$ ,  $L$  giving

the number of scales per octave. The natural expectation is that, given  $t$  and when  $\gamma$  is appropriately chosen, each mode be associated with a single  $j$  as defined in (2). Therefore, we consider a particular class of analytic wavelets for signal analysis which are those admitting a unique peak frequency. Then, the quantity

$$N_f(\Gamma) = \text{round}(\text{median}_{t, \gamma \in \Gamma(t)} C_f(\gamma, t)), \quad (3)$$

defines the average number of modes over all times  $t$  and  $\gamma$  in  $\Gamma(t)$ . We will come back later on the determination of  $\Gamma(t)$  in a noisy context. Having computed the above estimation of the number of modes, an appropriate wavelet threshold is determined as follows:

$$\bar{\gamma}(t) = \text{median}_{\gamma \in \Gamma(t)} \{\gamma, C_f(\gamma, t) = N_f(\Gamma)\}. \quad (4)$$

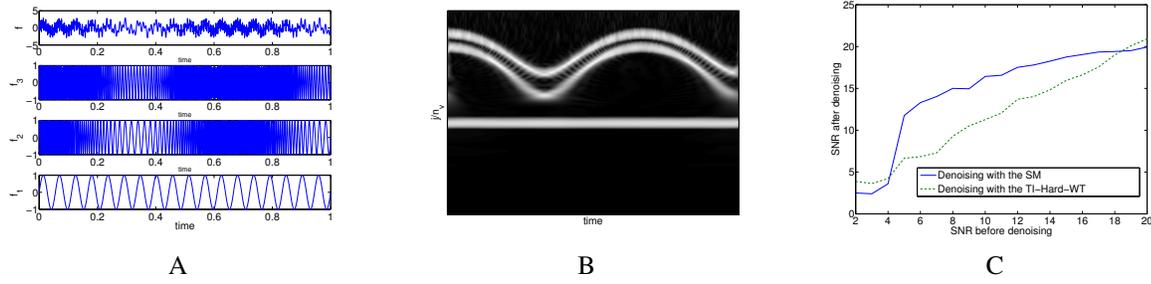
The existence of this quantity requires that for some  $\gamma$  in  $\Gamma(t)$  we have  $C_f(\gamma, t) = N_f(\Gamma)$ . The corresponding  $t$  define a subset  $T_0$  of  $T$ , the signal duration (as in applications  $T_0 = T$  we will not further expand on that). In contrast to [6], the parameter  $\bar{\gamma}$  depends on  $t$  and the mean is replaced by the median to obtain more robust estimates. For each  $t \in T_0$ , we have a set of  $N_f$  intervals defined for  $j_k(t)$ ,  $1 \leq k \leq N_f(\Gamma)$ , in  $C_f(\bar{\gamma}(t), t)$  by:

$$I_{j_k(t)} = [j_k(t), \tilde{j}_k(t)] \text{ s.t. } \forall j_k(t) \leq j \leq \tilde{j}_k(t) \\ |W_f(a_j, t)| > \bar{\gamma}(t) \text{ and } |W_f(a_{\tilde{j}_k(t)+1}, t)| < \bar{\gamma}(t). \quad (5)$$

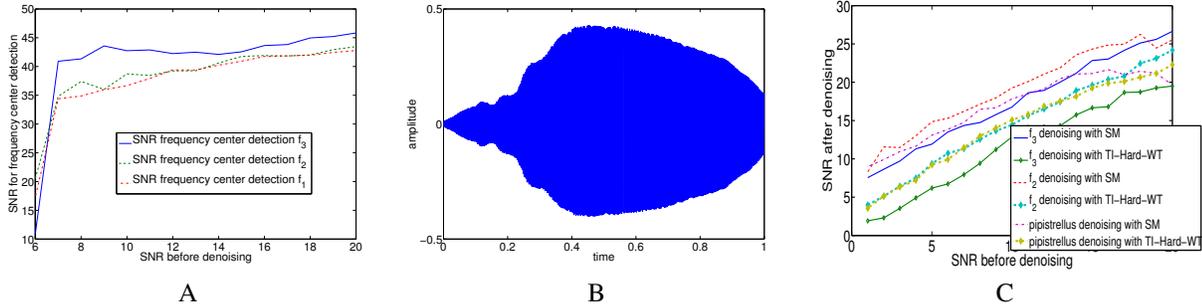
Then, one can associate with each mode  $k$ , the set of intervals  $(I_{j_k(t)})_{t \in T_0}$ , which we use to define the  $k^{\text{th}}$  mode reconstruction formula. Indeed, given  $t$ , the interval  $I_{j_k(t)}$  gives us an insight into what scales to use for the reconstruction of that mode. However, we do not directly use it. Rather, we remark that if the Fourier transform of the wavelet is symmetric with respect to its peak frequency and if it has a single peak frequency, the wavelet representation also has a single peak frequency and is almost symmetric with respect to the scale associated with this peak. Consequently, a good estimation of  $\phi'_k(t)$  is obtained by considering  $\hat{\phi}'_k(t) := \frac{\xi_\Psi}{m_k(t)}$  where  $m_k(t)$  is the midpart of the interval  $[a_{j_k(t)}, a_{\tilde{j}_k(t)}]$  and  $\xi_\Psi$  is the frequency center of the wavelet. Bearing in mind the definition of the set  $Z_k$  and assuming  $\hat{\Psi}$  is supported inside  $[\xi_\Psi - \Delta, \xi_\Psi + \Delta]$ , the scales of interest for the mode  $k$  at time  $t$  are  $[\frac{\xi_\Psi - \Delta}{\phi'_k(t)}, \frac{\xi_\Psi + \Delta}{\phi'_k(t)}]$ , which we approximate by  $J_{k,q} := [\frac{\xi_\Psi - \Delta}{\phi'_k(q\Delta t)}, \frac{\xi_\Psi + \Delta}{\phi'_k(q\Delta t)}]$  for  $t = q\Delta t \in T_0$ . Using the Morlet reconstruction formula considering only the scales of interest and taking into account that the scales  $a_j$  are on an exponential basis, we obtain the following approximation:

$$f_k(q\Delta t) \approx \frac{2}{C_\Psi} \mathcal{R}e \left[ \sum_{a_j \in J_{k,q}} W_f(a_j, q\Delta t) \frac{\log(2)}{n_v} \right]. \quad (6)$$

Note that when the threshold  $\bar{\gamma}(t)$  is high, the scales associated with  $J_{k,q}$  are much more relevant than those associated



**Fig. 1.** A: The analyzed signal  $f$  and its components  $f_1, f_2$  and  $f_3$ , B: The wavelet representation of the signal  $s$  ( $\sigma = 0.1$ ,  $\mu = 0.4$ ), C: Denoising results applying the proposed algorithm or the TI-Hard-WT to  $f$



**Fig. 2.** A: SNR associated with the frequency center detection, B : Pipistrellus sound C: Denoising results applying the proposed algorithm or the TI-Hard-WT to  $f_3, f_2$  or the pipistrellus sound ( $\sigma = 0.2$ ,  $\mu = 0.4$ )

with  $I_{jk}(q\Delta t)$ . We explain in the next section how to determine the parameter  $\Gamma(t)$  in a noisy context and we then illustrate the algorithm performance on multicomponent noisy signals.

## 4. PARAMETERS SELECTION AND ILLUSTRATIVE EXAMPLES

### 4.1. Determination of $\Gamma(t)$

In what follows, we assume that the separation condition  $d$  (see Definition 1) is known, giving an upper bound for the wavelet support (since we have  $\Delta < d$  see [6]). As already mentioned, the wavelet used must be analytic and have a single peak frequency. The wavelet we use in practice is the *bump wavelet* defined by  $\hat{\Psi}(\xi) = e^{-\frac{1}{1-(\frac{\xi}{\sigma}-\mu)^2}} \chi_{[\mu-\sigma, \mu+\sigma]}$ . Then, we need to discuss the choice for the set  $\Gamma(t)$ . Assuming the parameter  $\epsilon$  is sufficiently small, for any given pair  $(a, t)$  satisfying  $\mu - \Delta < a\phi'_k(t) < \mu + \Delta$ , the following approximation

$$W_f(a, t) \approx \frac{1}{2} A_k(t) e^{2i\pi\phi_k(t)} \overline{\hat{\Psi}(a\phi'_k(t))} \quad (7)$$

holds, meaning that in a noise-free context the local extrema of  $g : a \rightarrow |W_f(a, t)|$  give a good approximation

of  $\frac{1}{2}|A_k(t)|$  for the different  $k$ . Clearly, in that context, the upper bound  $\Gamma(t)_{\max}$  of  $\Gamma(t)$  should be taken equal to  $\min\{g(a), s.t. g'(a) = 0\}$  because otherwise at least one of the mode would not be detected. Nevertheless, the meaning of the extrema of  $g'$  is much less obvious for noisy signals and we can only say is that  $\Gamma(t)_{\max} \leq \max_a |W_f(a, t)|$ . This bound can however be improved by remarking that one wants to detect at least one mode, so that the upper bound for  $\Gamma(t)$  is set to  $\Gamma(t)_{\max} = \min(\max_a |W_f(a, t)|, \operatorname{argmin}_{\gamma} \{C_f(\gamma, t) = 1\})$ . The net effect of considering that upper bound is to limit the range for the search for  $\bar{\gamma}(t)$  when one of the modes have a significantly larger amplitude. Finally, we need to define a value for the lower bound for  $\Gamma(t)$ . Assume that the standard deviation of the noise  $\sigma$  is known, then we deduce from (7) that only the wavelet coefficients larger than  $\frac{\sigma}{2}$  should be taken into account. To estimate the standard deviation of the noise, we use a robust estimator  $\bar{\sigma}$  of  $\sigma$  based on the median component of the finest detail coefficients of an orthogonal wavelet decomposition [7] (in our simulations, we have used the symmlet with 4 vanishing moments, changing the number of the vanishing moments does not change the estimate a great deal). The lower bound for  $\Gamma(t)$  is finally set to  $\Gamma(t)_{\min} = \frac{\bar{\sigma}}{2}$ . Note that the noise level will also impact the computation of  $\Gamma(t)_{\max}$ .

## 4.2. Illustrative Examples

We now show that the proposed reconstruction formula (6) defines a denoising algorithm that outperforms most wavelet denoising techniques. To do so, instead of comparing our method to a set of wavelet techniques, evoking [8] we only compare our algorithm to the best performing wavelet technique, i.e. the translation invariant hard wavelet thresholding technique which we denote by TI-Hard-WT.

Our modes retrieval algorithm works as follows: it determines some frequency bands of interest that are associated with the modes and then integrates the wavelet transform over these bands. The computation of these bands is essentially based on the detection of their frequency center  $\phi'_k(t)$  which must be robust to noise. When the frequency bands are correctly estimated, the quality of the reconstruction then depends on how perturbed the coefficients are inside the bands.

As a first illustration, we consider the multicomponent signal of Figure 1 A (its wavelet representation being displayed in Figure 1 B) to which we add a Gaussian white noise so as to obtain a prescribe SNR. Then, we apply the proposed algorithm to these noisy signals (considering the bump wavelet with  $\mu = 0.4$  and  $\sigma = 0.1$  in each case,  $d$  being equal to 0.2) to obtain the so-called *denoised signal* by summing up all the reconstructed modes. By computing the SNR after denoising we are able to compare the proposed algorithm to the TI-Hard-WT (see Figure 1 C, the wavelet used for wavelet denoising being the symmetlet 4). We notice that the denoising procedure is significantly better provided the true number of modes is computed. Indeed, when the SNR becomes lower than 4, the number of detected modes is no longer correct resulting in poor denoising performance. To understand why the proposed method behaves so well, we investigate the quality of the estimation of  $\phi'_k(t)$  by computing its SNR (assuming the reference estimation is the noise-free one) depending on the noise level. The results shown in Figure 2 A for the modes  $f_1, f_2$  and  $f_3$ , show that the detection of the frequency centers is very stable when the noise level increases and thus, so are the frequency bands used in the reconstruction formula.

When one considers the denoising of a single mode, the restriction on the noise level in relation with the correct determination of the number of modes disappears and so does the constraint on the wavelet choice depending on the separation condition  $d$ . In such a case, the comparison of the denoising performance of the proposed method with the TI-Hard-WT is even more favorable to the former. As an illustration, we again consider the denoising of the signal  $f_2$  and  $f_3$  along with pipistrellus echolocation sound of Figure 2 B. In any of these cases, we notice that the proposed method significantly outperforms the TI-Hard-WT.

## 5. CONCLUSION

The algorithm for modes retrieval introduced in this paper uses the ideas of the synchrosqueezing and appears to be particularly efficient to denoise multicomponent signals. A limitation of the method is that it requires the knowledge of the separation condition therefore future work should involve the seek for some automatic procedure to determine this parameter. Nevertheless, this method can already be used without any restriction to denoise monocomponent signals.

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