Generalized Focal Surfaces : A New Method for Surface Interrogation

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Abstract

The generation of smooth surfaces from a mesh of three-dimensional data points is an important problem in geometric modeling. Apart from the pure construction of these curves and surfaces, the analysis of their quality is equally important in the design and manufacturing process. Generalized focal surfaces are presented here as a new surface interrogation tool.

1 Introduction

The geometric modeling of free-form curves and surfaces is of central importance for sophisticated CAD/ CAM systems. Apart from the pure construction of these curves and surfaces, the analysis of their quality is equally important in the design and manufacturing process. It is for example very important to test the convexity of a surface, to pinpoint inflection points, to visualize flat points and to visualize technical smoothness of surfaces. The purpose of this paper is to introduce generalized focal surfaces as a new tool for surface interrogation. A critical survey on other surface interrogation methods is given in [Hagen et al,'90].

2 Focal surfaces

Focal surfaces are known in the field of line congruences. Line congruences have been introduced in the field of visualization by Hagen and Pottmann (see [Hagen et al,'91]). They can be used to visualize the pressure and heat distribution on an airplane, temperature, rainfall, ozone over the earth's surface, etc. However, the setting in this paper is different. Here, focal surfaces are used as a surface interrogation tool to analyze the "quality" of the surface before further processing of the surface, for example in a NC-milling operation.

We represent surfaces parametrically as vector valued functions $X(u,w)$ and $N(u,w)$ is the unit normal vector of the surface.
Given a set of unit vectors $E(u,w)$, a *line congruence* is defined in parameter form :

$$C(u,w,z) = X(u,w) + zE(u,w)\ . \qquad (2.1)$$

For each pair (u,w), (2.1) is a line of the congruence, the so called *generator* of the line congruence C. z is the parameter of its points indicating the signed distance of the corresponding point on X. On each generator of C, there are two special (real, complex, identical) points, called *focal points.* They are the osculating points of the generators with the so called Gratlinien der Kongruenztorsen. Therefore the *focal surface* is the locus of the focal points. Generally, there are two parts of the focal surface.

If $E(u,w) = N(u,w)$, then $C = C_N$ is a *normal congruence.* The parametric representation of the focal surfaces of C_N is given by

$$F_i(u,w) = X(u,w) + \kappa_i^{-1}(u,w)N(u,w)\ \ ;\ \ i = 1,2 \qquad (2.2)$$

where κ_1, κ_2 are the principal curvatures. For more details about differential geometry see the appendix.

Fig. 1: Focal points

Considering fundamental facts from differential geometry, it is obvious that the centers of curvature of the normal section curves at a particular point on a surface fill out a certain segment of the normal vector at this point. The extremities of these segments are the centers of curvature of two principal directions. These two points are called the *focal points* of this particular normal. This terminology is justified by the fact that a line congruence can be considered as the set of lines touching two surfaces, the focal surfaces of the line congruence. The points of contact between a line of the congruence and the two focal surfaces are the focal points of this line. It turns out that the focal points of a normal congruence are the centers of curvature of the two principal directions (see [J. Hoschek,'71]).

The sphere is the only surface for which the two sheets of the focal surface degenerate into a point and the Dupin cyclides are the only surfaces whose focal surfaces degenerate into curves.

3 Generalized focal surfaces

We introduce a generalization of the "classical" focal surface concept to achieve a new curve and surface interrogation tool.

For a planar parametric curve $X = X(t)$ one chooses: (see [K.-P. Beier,'87])

$$F(t) = X(t) + a\, f(\kappa(t))\, N(t) \quad \text{with} \quad a \in I\!R \qquad (3.1)$$

as the "*variable curvature offset*" (VCO), where f is a scalar factor depending on the curvature $\kappa = \kappa(t)$ (some examples are $f = \kappa$, $f = \kappa^2$ or $f = 1/\kappa$).

Using $F(t) = X(t) + a\,\kappa(t)\,N(t)$ as VCO (with the same parametrization as $X(t)$), we show, that these curves visualize certain properties of the original curve X. Such properties are

(1) inflection points

(2) G^2, G^3 discontinuity

(3) curvature behaviour.

An inflection point of the curve $X(t)$ is marked by an intersection of F and X at this parameter (see Fig. 3).

C^2 (respectively C^1) continuity of two curve segments are visible in the curvature offsets by C^1 (respectively C^0) discontinuity at the corresponding point.
In Fig. 3 and Fig. 4 one can see the advantage of this method versus curvature plots and orthotomics: the VCO pinpoints the unwanted situations. In the case of inflection points, the intersection is the pinpoint; the connecting normal marks the discontinuity.

The **Generalized focal surfaces** have the following form:

$$F(u,w) = X(u,w) + a\,f(\kappa_1,\kappa_2)\,N(u,w) \quad \text{with} \quad a \in I\!R \tag{3.2}$$

where the scalar function f now depends on the the principal curvatures $\kappa_1 = \kappa_1(u,w)$, $\kappa_2 = \kappa_2(u,w)$ of X. The real number a is used as a scale factor. If the curvatures are very small you need a very large number a to distinct the two surfaces $X(u,w)$ and $F(u,w)$ on the screen. Variation of this factor can also improve the visibility of several properties of the focal surface, for ex. one can get intersections clearer.

For different applications we use different functions $f(\kappa_1,\kappa_2)$:

3.1 Convexity test

$$f = \kappa_1 \cdot \kappa_2 \tag{3.3}$$

It is important to know whether a certain region of the surface is convex, non-convex or whether it contains flat points ($\kappa_1 = \kappa_2 = 0$).

The convexity of a surface is given by its Gaussian curvature $K = \kappa_1 \cdot \kappa_2$. If K is positive all over the surface, it is called convex. Therefore, one is interested in the change of sign in the Gaussian curvature, which is the corresponding property to the undesired inflection points of curves. The generalized focal surface can visualize the change of sign in the Gaussian curvature with the factor $f = \kappa_1 \cdot \kappa_2$: $X(u,w)$ and $F(u,w)$ intersect at points of vanishing Gaussian curvature.

In Fig. 5 one can see the original surface $X(u,w)$ in red, which has a line of vanishing Gaussian curvature near the four corners.

Fig. 5: Convexity test of a bicubic surface

3.2 Visualization of flat points

$$f = \kappa_1^2 + \kappa_2^2 \tag{3.4}$$

A surface has a *flat point* at the parameter (u,w) if the principal curvatures are vanishing there identically, i.e. $\kappa_1(u,w) = \kappa_2(u,w) = 0$. Such flat points or flat regions resp. are also undesired points with regard to the manufacture of those surfaces. A convex surface with flat points looks to have dents.

To test a surface for flat regions, the VCO factor $f(u,w) = \kappa_1^2(u,w) + \kappa_2^2(u,w)$ is useful, as the next focal analysis shows: A flat point exists, wherever both surfaces touch see Fig. 6.

Fig. 6: Visualization of a flat point

3.3 Continuity of the surface

The factor $f(u,w) = \kappa_1^2(u,w) + \kappa_2^2(u,w)$ can also be used to visualize the continuity of the surface.
The order of differentiation of the generalized focal surface decreases by 2. This means that a surface $X(u,w)$ of class C^3 has a generalized focal surface of class C^1. On the other hand if the focal surface is C^0 but not C^1, so the surface X is not G^3. And a surface with a non continuous focal surface could be maximal of class C^1 (i.e. not G^2).

The test surface Fig. 7 consisting of 16 bicubic patches has only C^1 continuity between the patches. This is visualized with its generalized focal surface which is not continuous between these patches (see Fig. 8).

Fig. 7: test surface

Fig. 8: G^2-discontinuity of the test surface

In Fig. 9, we see a generalized focal surface of a C^2 continuous surface (4 patches), which is only C^0 there.

Fig. 9: G^3-discontinuous surface with gen. focal surface

3.4 Visualization of the technical smoothness of a surface

$$f = \frac{\kappa_1^2 + \kappa_2^2}{\kappa_1 + \kappa_2} \qquad (3.5)$$

In the last couple of years, successful algorithms were developed to design and construct technically smooth curves and surfaces (see [Hagen - Santarelli,'92]).

Technical smoothness means that the data can be immediately transfered to the milling process. In this production chain, one part is still missing, the visualization of technical smoothness as a quality test before the milling process. We close this gap with the next result, which proves the fact that a technically smooth surface has a generalized focal surface in the sense of $f = \frac{\kappa_1^2+\kappa_2^2}{\kappa_1+\kappa_2}$ with a minimal distance of the surface areas.

Theorem :

Supposed that $X(u,w)$ is a surface with $\kappa_1 \neq -\kappa_2$ and its generalized focal surface is given by

$$F(u,w) = X(u,w) + a\frac{\kappa_1^2 + \kappa_2^2}{\kappa_1 + \kappa_2} N(u,w) \; . \qquad (3.6)$$

If $X(u,w)$ is technically smooth in the sense of $\int_s (\kappa_1^2 + \kappa_2^2) ds \rightarrow min$ then

$$|A(F) - A(X)| \; \rightarrow \; min$$

$A(F)$ and $A(X)$ is the surface area of $F(u,w)$ and $X(u,w)$.

Proof :

To prove this theorem we have to state more precisely some conditions on X and F :

(a) $X = X(u,w)$ is a C^2-mapping of an open subset $U \subset I\!R^2$ into $I\!R^3$. X is a regular parametrization of a surface, which is equivalent to say that the determinant g of the first fundamental matrix has to be non zero : $|g| > 0$ for all $(u,w) \in \bar{U}$, where $\bar{U}$ denotes the closure of the set U.

(b) $X(u,w)$ with $\kappa_1 \neq -\kappa_2$ in the theorem means that there is no change of sign in the mean curvature M on X : It exists a real number $\alpha > 0$ with $|\kappa_1 + \kappa_2| > \alpha$ for all $(u,w) \in U$

(c) Here the generalized focal surface

$$F(u,w) = X(u,w) + \varepsilon f(u,w) N(u,w)$$

with $f(u,w) = \frac{\kappa_1^2+\kappa_2^2}{\kappa_1+\kappa_2}$ is a neighbour surface who converges to X when $\varepsilon \rightarrow 0$.

In a first step we prove, that f is bounded because we need this property in the second step: A function $f : U \rightarrow I\!R$ is bounded if there exists a real number m, so that $|f(u,w)| \leq m$ for all (u,w). Therefore our function f is bounded if we can find an upper bound for the nominator $\kappa_1^2 + \kappa_2^2$ and a lower bound for its denominator $\kappa_1 + \kappa_2$. Considering condition (b), there exists such a lower bound, namely α. Considering condition (a) we have $|g| > 0$ for all $(u,w) \in \bar{U}$ and $g \in C^1(\bar{U})$ because $X \in C^2(\bar{U})$. A function which is continuous over a compact set takes its absolute minimum in this set. It follows that there exists a minimum $\beta > 0$ of g such that $|g| \geq \beta$ for all $(u,w) \in \bar{U}$. This result indicates directly that the principal curvatures κ_1 and κ_2 of the surface X are bounded : There exists $\gamma > 0$: $|\kappa_1(u,w)|$, $|\kappa_2(u,w)| \leq \gamma$ for all $(u,w) \in \bar{U}$. It follows $|f(u,w)| \leq \frac{2\gamma^2}{\alpha} < \infty$.

In the next step we want to determine the surface area of the generalized focal surface F. The surface area $A(X)$ is defined by $A(X) = \iint_U \sqrt{g}\, dudw$. So we have to calculate

$$A(F) = \iint_U \sqrt{g^F}\, dudw \tag{3.7}$$

$g^F = g_{11}^F g_{22}^F - (g_{12}^F)^2$ is the determinant of the first fundamental form of F. Partial derivation of F gives us

$$\begin{aligned} F_u(u,w) &= x_u(u,w) + \varepsilon f_u N(u,w) + \varepsilon f N_u(u,w) \\ F_w(u,w) &= x_w(u,w) + \varepsilon f_w N(u,w) + \varepsilon f N_w(u,w)\ . \end{aligned}$$

F_u denotes $\frac{\delta F}{\delta u}$. The components of the first fundamental form are defined as follows : $g_{11}^F =< F_u, F_u >$, $g_{12}^F =< F_u, F_w >$, $g_{22}^F =< F_w, F_w >$, where $<\ ,\ >$ denotes the scalar product. Because ε is a arbitrarily small number and f is bounded we can leave out all the terms which have ε in second order. Making some elementary calculations we get

$$\begin{aligned} g_{11}^F &= g_{11} - 2\varepsilon h_{11}\ , \\ g_{12}^F &= g_{12} - 2\varepsilon h_{12}\ , \\ g_{22}^F &= g_{22} - 2\varepsilon h_{22}\ . \end{aligned}$$

h_{ij} are the components of the second fundamental form of X. Further calculations give

$$\sqrt{g^F} = \sqrt{g}\,(1 - 2\varepsilon f M).$$

M denotes the mean curvature of the origin surface X. After application of the surface area formula (3.7) one gets

$$A(F) = A(X) - 2\varepsilon \iint_U \sqrt{g}(\kappa_1^2 + \kappa_2^2) dudw \tag{3.8}$$

Because X is a surface with $\int_s(\kappa_1^2 + \kappa_2^2)ds \rightarrow min$ one gets $\|A(F) - A(X)\| \ \rightarrow\ min$ □

Conclusion :

The difference of the surface areas of the surface $X(u,w)$ and the generalized focal surface $F(u,w)$ in the sense of $f := \frac{\kappa_1^2+\kappa_2^2}{\kappa_1+\kappa_2}$ is a measure for the technical smoothness of the surface $X(u,w)$.

Now at the end we give a focal analysis of an practical example. Industrial data of a part of a hair dryer are used. There are two composed surfaces consisting of several patches, see Fig. 10 at the bottom. For each surface you can see the generalized focal surfaces with $f = \kappa_1^2 + \kappa_2^2$. It is visible that the two surfaces (red) are not C^2-continuous, because the focal surfaces aren't continuous. In addition this focal analysis shows the curvature behaviour of these surfaces.

Fig. 10: Focal analysis of a hair dryer

4 References

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5 Appendix: Fundamentals of differential geometry

A parametrized C^r-surface is a C^r-differential map $X : U \rightarrow E^3$ of an open domain $U \in R^2$ into the Euclidean space E^3 , where $X_1 := \frac{\partial x}{\partial u}$ and $X_2 := \frac{\partial x}{\partial w}$ are linearly independent.
The two-dimensional linear subspace $T_P X$ of E^3 generated by the span $\{X_1, X_2\}$ is called the tangent space of X at P. The unit normal field N is given by:

$$N := \frac{[X_1, X_2]}{\|[X_1, X_2]\|} \qquad (A.1)$$

the moving frame $\{X_1, X_2, N\}$ is the Gaussian frame. The Gaussian frame is in general not an orthogonal frame. Every tangential vector field y along the surface $X : U \rightarrow E^3$ can be represented in the form:

$$y(u, w) = \Delta u(u, w) \cdot X_1(u, w) + \Delta w(u, w) \cdot X_2(u, w) \qquad (A.2)$$

The bilinear form on $T_P X$ induced by the inner product of E^3 by restriction is called the first fundamental form of the surface. The matrix representation of the first fundamental form I_P with respect to the basis $\{X_1, X_2\}$ of $T_P X$ is given by:

$$\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \begin{vmatrix} < X_1, X_1 > & < X_1, X_2 > \\ < X_2, X_1 > & < X_2, X_2 > \end{vmatrix} \qquad (A.3)$$

The first fundamental form I_P is symmetric, positive definite and geometric invariant. Geometrically the first fundamental form allows measurements on the surface (length of curves, angles of tangent vectors, areas of regions) without referring back to the space E^3, in which the surface lies.
The linear map $L : T_P X \longrightarrow T_P X$ defined by $L := -dN \circ dX^{-1}$ is called the Weingarten map. The bilinear form II_P defined by $II_P(A, B) := < L(A), B >$ for each $A, B \in T_P X$ is called the second fundamental form of the surface. The matrix representation of II_P with respect to the basis $\{X_1, X_2\}$ of $T_P X$ is given by:

$$h_{ij} = < -N_i, X_j > = < N, X_{ij} > \quad i, j = 1, 2 \qquad (A.4)$$

The Weingarten map L is self-adjoint, the eigenvalues k_1, k_2 are therefore real and the corresponding eigenvectors are orthogonal. The eigenvalues k_1, k_2 are the principal curvatures of the surface.

$$K := k_1 \cdot k_2 = \det(L) = \frac{\det(II)}{\det(I)} \qquad (A.5)$$

is called the Gaussian curvature and

$$H := \text{trace}(L) = \frac{1}{2}(k_1 + k_2) \qquad (A.6)$$

is called the mean curvature. For curves on surfaces the geometric interpretations of the second fundamental form follows. Let $A := \Delta u \cdot X_1 + \Delta w \cdot X_2$ be a tangent vector with $\|A\| = 1$. If we intersect the surface with the plane given by N and A, we get an intersection curve y with the following properties:

$$\dot{y}(s) = A \quad \text{and} \quad e_2 = \pm N \qquad (A.7)$$

where e_2 is the principal normal vector of the space curve y. The implicit function theorem implies the existence of this normal section curve. To calculate the extreme values of the curvature of a normal section curve (the normal section curvature) we can use the method of Lagrange multipliers because we are looking for the extreme values of the normal section curvature k_N with the condition $\|\dot{y}(s)\| = 1$. As the result of these considerations we obtain:

> Unless the normal section curvature is the same for all directions there are two perpendicular directions A_1 and A_2 in which k_N attains its absolute maximum and its absolute minimum values. These directions are the principal directions with the corresponding normal section curvatures k_1 and k_2.

For $A = A_1 \cos\varphi + A_2 \sin\varphi$ we get Euler's formula:

$$k_N = k_1 \cos\varphi + k_2 \sin\varphi \qquad (A.8)$$

If the principal directions are taken as coordinate axes, Euler's formula implies the so-called Dupin indicatrix:

$$k_1(u)^2 + k_2(u)^2 = \pm 1 \qquad (A.9)$$

We use the Dupin indicatrices as a tool to visualize curvature situations on surfaces. The Dupin indicatrices at elliptic points ($k > 0$) are ellipses, at hyperbolic points ($k < 0$) pairs of hyperbolas, and at parabolic points ($k = 0$) pairs of parallel lines. Planar points ($k_1 = k_2 = 0$) are degenerated parabolic cases.

Fig. 5: Convexity test of a bicubic surface

Fig. 6: Visualization of a flat point

Fig. 7: Test surface

Fig. 8: G^2-discontinuity of the test surface

Fig. 9: G^3-discontinuous surface with generalized focal surface

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