Solutions of Exam problems (January 10, 2017)

1. An affine map can be written as \( A : x \mapsto Lx + b \). We have

\[
    b = A \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad L = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}.
\]

Hence

\[
    A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

An affine map preserves affine combinations. The point \((1,1)^T\) can be written as \((1,0)^T + (0,1)^T - (0,0)^T\), which is such a combination. Hence

\[
    A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} + A \begin{pmatrix} 0 \\ 1 \end{pmatrix} - A \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

An affine map takes parallel lines to parallel lines, hence a square will be taken to a parallelogram by a generic affine map. In fact, all parallelograms are affinely equivalent.

2. Convexity of sets.

- \( \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) is an intersection of closed half-spaces and hence convex;
- \( \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1, \ z \geq 0 \} \) is the intersection of the Euclidean unit ball with a half-space and is hence convex;
- \( \{ (x,y) \in \mathbb{R}^2 \mid y \geq x^2 - x^2 + 1 \} \) is the epigraph of a non-convex function and hence not convex;
- \( \{ (x,y) \in \mathbb{R} \times \mathbb{R}^n \mid x \leq \|y\|_2 \} \) is isomorphic to the epigraph of minus the 2-norm and hence not convex;
- \( \{ (x,y) \in \mathbb{R}^2 \mid x > 0, \ 2x - 3 \leq y \leq \log x \} \) is the intersection of the epigraph of a convex function and the hypograph of a concave function and hence convex.

3. Dual cone of \( K = \{ (x,y,z) \in \mathbb{R}^3 \mid z \geq 0, \ (x,y) \in [0,z]^2 \} \).

First solution: The cone \( K \) is polyhedral and equal to the convex conic hull of the vectors \((0,0,1)^T, (1,0,1)^T, (0,1,1)^T, (1,1,1)^T\). A vector \( u \in \mathbb{R}^3 \) is in \( K^* \) if and only if it has a nonnegative scalar product with each of these four extremal elements. Thus

\[
    K^* = \{ (a,b,c)^T \mid c \geq 0, \ a + c \geq 0, \ b + c \geq 0, \ a + b + c \geq 0 \}.
\]

Second solution: The cone \( K \) is polyhedral with four facets having inward pointing normal vectors \((1,0,0)^T, (0,1,0)^T, (1,1,0)^T, (0,1,2)^T\). Thus \( K^* \) is the convex conic hull of these vectors, \( K^* = \{ (\alpha - \gamma, \beta - \delta, \gamma + \delta)^T \mid \alpha, \beta, \gamma, \delta \geq 0 \} \).

4. Fenchel dual.

4.1. We have \( p = f'(x) = -\frac{1}{x} \). If \( x \) runs through \( \mathbb{R}_{++} \), \( p \) runs through \( \mathbb{R}_{--} \). For \( p < 0 \) we get \( x = \frac{1}{\sqrt{-p}} \) and

\[
    f^*(p) = \frac{p}{\sqrt{-p}} - \sqrt{-p} = -2\sqrt{-p}.
\]

For \( p = 0 \) we have \( f^*(0) = \sup_{x>0}(-x^{-1}) = 0 \), for \( p > 0 \) we get \( f^*(p) = +\infty \).

4.2. The function \( f \) is of the form \( f(x, y) = c + Q(x, y) \) with \( Q \) a quadratic form. Let \( A = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \) be the matrix of \( Q \) and denote the adjoint variables by \( p, q \). We get

\[
    \begin{pmatrix} p \\ q \end{pmatrix} = 2A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}A^{-1} \begin{pmatrix} p \\ q \end{pmatrix},
\]
\[ f^*(p, q) = \frac{1}{2} (\begin{pmatrix} p \\ q \end{pmatrix})^T A^{-1} \begin{pmatrix} p \\ q \end{pmatrix} - c - \frac{1}{4} (\begin{pmatrix} p \\ q \end{pmatrix})^T A^{-1} A A^{-1} (\begin{pmatrix} p \\ q \end{pmatrix}) = -c + \frac{1}{4} (\begin{pmatrix} p \\ q \end{pmatrix})^T A^{-1} (\begin{pmatrix} p \\ q \end{pmatrix}). \]

We have \( A^{-1} = \frac{4}{3} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \). Hence

\[ f^*(p, q) = -1 + \frac{1}{3} (p^2 + pq + q^2). \]

4.3. We have

\[ f(x) = \begin{cases} e^x, & x \geq 0, \\ e^{-x}, & x \leq 0, \end{cases} \quad f'(x) = \begin{cases} e^x, & x > 0, \\ -e^{-x}, & x < 0. \end{cases} \]

Hence every \( p \) with \(|p| > 1\) is a gradient of \( f \) at \( x = \text{sgn} \log |p| \) and every \( p \in [-1, 1] \) is a subgradient of \( f \) at \( x = 0 \). We then get

\[ f^*(p) = \begin{cases} p \log p - p, & p > 1, \\ -1, & p \in [-1, 1], \\ -p \log(-p) + p, & p < -1. \end{cases} \]

5. Let \( e, c \) be the amount of shelves of the expensive and the cheap kind, respectively, to be produced in a single day. Let \( m, b \) be the amount of massive wood and of chipboard, respectively, to be processed in a single day. We have as constraints that these amounts are nonnegative, the equation linking the amount of material to the amount of produced units, and the limiting capacities. The profit to be maximized is the price of the units minus the price of the material. Thus we get the LP

\[
\begin{align*}
\max \quad & \begin{pmatrix} 20 \\ 50 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} c \\ e \\ b \\ m \end{pmatrix} = 0,
\begin{pmatrix} 8 & 2 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c \\ e \\ b \\ m \end{pmatrix} = 0,
\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c \\ e \\ b \\ m \end{pmatrix} = 0,
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ e \\ b \\ m \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 30 \\ 320 \end{pmatrix}.
\end{align*}
\]