Conic programs

In this section we consider a large class of convex optimization problems, the conic programs. Actually, every convex optimization problem can be formulated as a conic program, so this is merely a form into which a convex problem can be cast. However, this form is more convenient for certain problems and less convenient for others.

**Definition 2.1.** A regular, or proper convex cone is a closed convex cone with non-empty interior which does not contain a line.

A closed convex cone is hence regular if it is not degenerated one way or another.

**Definition 2.2.** A conic program over a regular convex cone $K$ is an optimization problem of the form

$$\inf_{x \in K} c^T x : Ax = b.$$ 

Here $x \in K$ is the conic constraint, $Ax = b$ is a linear constraint, and the objective function is linear. The feasible set of a conic program is hence an intersection of the cone with an affine subspace. As the intersection of convex sets it is convex. The objective function is convex too, and hence a conic program formally belongs to the class of convex problems. The complexity of a conic program is encoded in the cone $K$, and whether the conic program is efficiently solvable depends on which descriptions of the cone $K$ are available.

*Example:* Let $C^n$ be the cone of real symmetric $n \times n$ matrices $A$ such that $x^T Ax \geq 0$ for every $x \in \mathbb{R}^n_+$. This cone is called the copositive cone. Its difference with the positive semi-definite cone is that we require $A$ to be nonnegative only on nonnegative vectors. The cone $C^n$ is regular, but to decide whether a given point is not in $C^n$ is NP-complete (Murty, Cabadi 1987). Efficient methods for solving generic programs over the copositive cone are hence unlikely to exist.

Since every closed convex set can be written as an intersection of an affine subspace with a regular convex cone, and minimizing a convex function is equivalent to minimizing a linear function over the epigraph of the original function, every convex problem can actually be rewritten as a conic program.

We may write a conic program also in the form

$$\min_x c^T x : Ax + b \in K.$$ 

Here the decision variable $x$ directly parameterizes the affine hull of the feasible set.

We shall now introduce the class of regular convex cones which is mostly used in practice and for which the theory of conic programming is well developed.

Homogeneous cones

**Definition 2.3.** Let $K \subset V$ be a closed convex cone in some vector space. An automorphism of $K$ is an automorphism $A$ of $V$ such that $A[K] = K$.

The automorphisms of $K$ form a group, the automorphism group Aut $K$ of the cone. The automorphism group always contains the 1-parametric subgroup of homotheties, i.e., maps $g_\lambda : v \mapsto \lambda v$ which multiply every vector by a positive constant $\lambda$.

**Definition 2.4.** A regular convex cone $K$ is called homogeneous if its automorphism group acts transitively on the interior of $K$, i.e., if for every $x, y \in K^o$ there exists an automorphism $A \in \text{Aut} K$ such that $Ax = y$.

In a homogeneous cone every interior point is hence equivalent to any other interior point, and every interior point can be considered as the center of the cone. The homogeneous cones can be put into 1-to-1 correspondence with algebraic structures, so-called $T$-algebras (Vinberg 1963). A more explicit classification is also available (Kaneyuki, Tsuji 1974).
Duality

**Definition 2.5.** Let $V$ be a vector space. The set of all linear functionals on $V$ is also a vector space and is called the *dual space*, denoted by $V^*$.

For finite-dimensional vector spaces, we may canonically identify $V$ with the dual $(V^*)^*$ to the dual space. Namely, $x \in V$ is identified with the linear functional $p \mapsto \langle p, x \rangle$ on $V^*$.

*Remark:* For infinite-dimensional complete normed vector spaces (Banach spaces), "linear functional" is replaced by "continuous linear functional", and $V$ is in general only a subset of $(V^*)^*$.

**Definition 2.6.** Let $K \subset V$ be a convex cone. Then the dual cone $K^*$ is the set of points $p \in V^*$ such that $\langle p, x \rangle \geq 0$ for all $x \in K$.

The dual cone hence consists of all linear functionals on $V$ which take on only nonnegative values on $K$. It is not hard to see that $K^*$ is a closed convex cone.

**Lemma 2.7.** Let $K$ be a closed convex cone. Then $(K^*)^* = K$.

*Proof.* For every $x \in K$ and $p \in K^*$ we have $\langle p, x \rangle \geq 0$. Hence $K \subset (K^*)^*$.

Suppose for the sake of contradiction that there exists $y \in (K^*)^*$ such that $y \notin K$. Then $y$ has a positive distance from $K$, since $K$ is closed, and can be strongly separated from $K$. Therefore there exists a linear functional $p \in V^*$ such that $\langle p, y \rangle < 0$ and $\langle p, x \rangle \geq 0$ for all $x \in K$. From the second condition it follows that $p \in K^*$. But this is in contradiction with $y \in (K^*)^*$. Thus $(K^*)^* \subset K$. $\square$

**Lemma 2.8.** Let $K$ be a closed convex cone. Then $K$ has non-empty interior if and only if $K^*$ does not contain a line, and $K$ does not contain a line if and only if $K^*$ has non-empty interior.

*Proof.* Let $K$ have empty interior. Then $K$ is contained in a proper linear subspace, and there exists a non-zero linear functional $p \in V^*$ such that $\langle p, x \rangle = 0$ for all $x \in K$. Then $K^*$ contains the line $\{\alpha p \mid \alpha \in \mathbb{R}\}$.

On the other hand, let $p \in V^*$ be a non-zero functional such that the line $\{\alpha p \mid \alpha \in \mathbb{R}\}$ is contained in $K^*$. Then $\langle p, x \rangle \geq 0$ and $\langle -p, x \rangle \geq 0$, and hence $\langle p, x \rangle = 0$ for all $x \in K$. Hence $K$ is contained in a proper linear subspace and has empty interior.

The second assertion is proved in a similar way. $\square$

Therefore the dual cone $K^*$ of a closed convex cone $K$ is regular if and only if $K$ itself is regular.

We now consider how $K^*$ behaves with respect to linear isomorphisms. Let $A : V \rightarrow V$ be a linear isomorphism of a real vector space. We look for an isomorphism $B : V^* \rightarrow V^*$ of the dual space such that $\langle By, Ax \rangle = \langle y, x \rangle$ for all $x \in V$, $y \in V^*$. Clearly such an isomorphism $B$ exists, is unique, and is given by the inverse of the adjoint map $A^T$.

**Lemma 2.9.** Let $K \subset V$ be a convex cone, and let $K' = A[K]$ be its image under a linear isomorphism $A$ of $V$. Then $(K')^* = (A^*)^{-1}[K^*]$.

*Proof.* We have

$$(K')^* = \{y \in V^* \mid \langle y, x \rangle \geq 0 \forall x \in K'\} = \{y \in V^* \mid \langle Ax, y \rangle \geq 0 \forall x \in K\} = \{y \in V^* \mid \langle A^*y, x \rangle \geq 0 \forall x \in K\} = \{y \in V^* \mid A^*y \in K^*\} = (A^*)^{-1}[K^*].$$

*Note:* That if a basis $\{e_1, \ldots, e_n\}$ is given in $V$, $V^*$ is equipped with the dual basis $\{e^1, \ldots, e^n\}$ (i.e., $(e_i, e_j) = \delta_{ij}$ with $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$ being the Kronecker symbol), and the linear isomorphism $A$ is expressed as a real $n \times n$ matrix in this basis, then $A^*$ expressed in the dual basis will have coordinate matrix $A^T$.

Lemma 2.9 allows to compute the dual cone of a cone $K$ if the dual of an isomorphic cone is already known. The following result is helpful for computing the dual cone if such information is not available.

**Lemma 2.10.** Let $K \subset V$ be a regular convex cone. A non-zero point $y \in V^*$ lies on the boundary of $K^*$ if and only if the kernel $H$ of $y$ is a supporting hyperplane to $K$ at some non-zero boundary point $x \in \partial K$ such that $K$ lies in the closed half-space with respect to $H$ where $y$ assumes nonnegative values.
Proof. First note that $y$ is by definition a linear functional on $V$. If $y \neq 0$, then its kernel $H = \{x \in V \mid \langle x, y \rangle = 0\}$ is indeed a linear subspace of $V$ of codimension 1.

Let $y \in \partial K^* \setminus \{0\}$. The linear functional $y$ is then non-constant on $V$, and in particular on $K$. Then $\langle x, y \rangle > 0$ for all $x$ in the interior of $K$, otherwise there would exist $x \in K$ such that $\langle x, y \rangle < 0$. Hence the interior of $K$ lies in the open half-space delineated by the hyperplane $H$ where $y$ assumes positive values.

Since $y \in \partial K^*$, there exists a sequence of points $y_k \in V \setminus K^*$ which converges to $y$. To every such $y_k \notin K^*$ we find a point $x_k \in K$ such that $\langle x_k, y_k \rangle < 0$. Since $x_k \neq 0$, we may normalize it such that $||x_k|| = 1$ in some fixed Euclidean norm on $V$. Then there exists a subsequence of $\{x_k\}$ which converges to some point $x \in K$ of unit norm, because the intersection of the unit sphere with $K$ is compact. By continuity we have $\langle x, y \rangle \leq 0$ and hence $\langle x, y \rangle = 0$, because $x \in K$ and $y \in K^*$. Therefore $x \in H$, and $H$ separates $\{x\}$ properly from $K$. This proves one direction.

Let now $x \in \partial K$ be non-zero, and let $H$ be a hyperplane separating $\{x\}$ from $K$ properly. Then $K$ lies in one of the closed half-spaces delineated by $H$. Let $y \in V^*$ be non-zero such that $H$ is the kernel of $y$ and such that $y$ assumes nonnegative values on $K$. Then $y \in K^*$ by definition. However, $\langle x, y \rangle = 0$, because $x \in H$, and $x$ is not constant as a linear functional on $V^*$. Therefore $y \notin \text{int}K^*$, which shows the other direction.

In order to find the boundary of $K^*$, we hence have to compute all hyperplanes which support $K$ at non-zero boundary points. If the boundary $\partial K \setminus \{0\}$ is smooth, then at every non-zero boundary point of $K$ there exists exactly one such hyperplane, namely the tangent plane to $\partial K$. In this case $\partial K^* \setminus \{0\}$ is comprised of the normals to all tangent planes to $\partial K \setminus \{0\}$.

**Symmetric cones**

Let the vector space $V$ be equipped with a Euclidean scalar product $\langle \cdot, \cdot \rangle$. This allows to identify the dual vector space $V^*$ with $V$ itself. The linear functional $p \in V^*$ is identified with the element $y \in V$ such that $(p, x) = (y, x)$. Here on the left we have the dual pairing, and on the right the scalar product.

**Definition 2.11.** Let $K \subset V$ be a closed convex cone, with the ambient vector space $V$ equipped with a scalar product. Then $K$ is called **self-dual** if $K^* = K$ under the identification of $V$ with $V^*$ by the scalar product.

If no scalar product is defined a priori, then $K$ is called **self-dual** if there exists a scalar product on $V$ such that $K^* = K$ under the identification of $V$ with $V^*$ generated by this scalar product.

**Definition 2.12.** A closed convex cone $K$ is called **symmetric** if it is both homogeneous and self-dual.

Symmetric cones possess a rich algebraic structure and are closely linked to a class of non-associative algebras.

**Definition 2.13.** A **real algebra** $A$ is a real vector space equipped with a bilinear multiplication $\bullet : A \times A \to A$. It is called **commutative** if $a \bullet b = b \bullet a$ for all $a, b \in A$, and **associative** if $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all $a, b, c \in A$.

**Definition 2.14.** A real algebra $A$ is called a **Jordan algebra** if it is commutative and satisfies the Jordan identity $(x \bullet x) \bullet (y \bullet y) = x \bullet ((x \bullet x) \bullet y)$ for all $x, y \in A$.

A Jordan algebra is called **Euclidean** if $\sum_{k=1}^m x_k \bullet x_k = 0$ implies $x_1 = \cdots = x_m = 0$ for all $x_1, \ldots, x_m \in A$.

Then the symmetric cones are exactly the cones of squares of the Euclidean Jordan algebras.

**Theorem 2.15.** Let $K \subset V$ be a symmetric cone. Then there exists a Euclidean Jordan algebra structure with multiplication $\bullet$ on $V$ such that $K = \{x \bullet x \mid x \in V\}$.

On the other hand, for every Euclidean Jordan algebra $A$ the set $\{x \bullet x \mid x \in A\}$ is a symmetric cone.

The symmetric cones have been fully classified.

**Theorem 2.16.** Every symmetric cone $K$ is a direct product of a finite number of symmetric cones $K_1, \ldots, K_m$, each of which is either a member of one of the following families, indexed by a natural number $n \geq 1$:
• Lorentz cone $L_n = \{ x \in \mathbb{R}^n \mid x_0 \geq \sqrt{\sum_{j=1}^{n-1} x_j^2} \}$,

• real symmetric positive semi-definite matrix cone $S^n_+ = \{ A = A^T \mid x^T A x \geq 0 \ \forall \ x \in \mathbb{R}^n \}$,

• complex Hermitian positive semi-definite matrix cone $\mathcal{H}^n_+ = \{ A = A^* \mid x^* A x \geq 0 \ \forall \ x \in \mathbb{C}^n \}$,

• quaternionic Hermitian positive semi-definite matrix cone $\mathcal{Q}^n_+ = \{ A = A^* \mid x^* A x \geq 0 \ \forall \ x \in \mathbb{H}^n \}$;

or the exceptional 27-dimensional Albert cone.

Every cone which was listed above is symmetric and cannot be further decomposed in a non-trivial manner (i.e., it is irreducible).

For $n = 1$ all families yield the 1-dimensional cone $\mathbb{R}_+$, and the matrix cones for $n = 2$ are all isomorphic to a Lorentz cone. Apart from these exceptions, the irreducible cones listed above are mutually non-isomorphic. Note that the orthant $\mathbb{R}^n_+$ is a direct product of $n$ copies of the 1-dimensional cone and is hence also symmetric.

The Jordan multiplication for the Lorentz cone is given by
\[
\begin{pmatrix} x_0 \\ \tilde{x} \end{pmatrix} \bullet \begin{pmatrix} y_0 \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x_0 y_0 + \tilde{x}^T \tilde{y} \\ y_0 \tilde{x} + x_0 \tilde{y} \end{pmatrix}.
\]

The Jordan multiplication for the matrix cones is given by
\[
A \bullet B = \frac{AB + BA}{2},
\]
where the product on the right-hand side is the ordinary matrix multiplication.

The Jordan multiplication for the orthant $\mathbb{R}^n_+$ is hence the element-wise multiplication of real vectors in $\mathbb{R}^n$.  

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