A partial differential equation characterizing determinants of symmetric cones

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1. Symmetric cones
   - Geometric characterization
   - Algebraic characterization

2. Jordan algebras
   - Exponential and logarithm
   - Trace forms and determinant

3. The partial differential equation
   - Hessian metrics
   - The PDE
   - Connection with Jordan algebras
Outline

1. **Symmetric cones**
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A PDE characterizing determinants of symmetric cones
Regular convex cones

Definition

A regular convex cone $K \subset \mathbb{R}^n$ is a closed convex cone having nonempty interior and containing no lines.

Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathbb{R}^n$

$$K^* = \{ p \in \mathbb{R}^n | \langle x, p \rangle \geq 0 \quad \forall x \in K \}$$

is called the dual cone
Symmetric cones

Definition
A regular convex cone $K \subset \mathbb{R}^n$ is called self-dual if there exists a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that $K = K^*$. 

Definition
A regular convex cone $K \subset \mathbb{R}^n$ is called homogeneous if the automorphism group Aut($K$) acts transitively on $K^o$. 

Definition
A self-dual, homogeneous regular convex cone is called symmetric.
Jordan algebras

an algebra $A$ is a vector space $V$ equipped with a bilinear operation $\bullet : V \times V \rightarrow V$

**Definition**

An algebra $J$ is a Jordan algebra if

- $x \bullet y = y \bullet x$ for all $x, y \in J$ (commutativity)
- $x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y)$ for all $x, y \in J$ (Jordan identity)

where $x^2 = x \bullet x$.

**Definition**

A Jordan algebra is formally real or Euclidean if $\sum_{k=1}^{m} x_k^2 = 0$ implies $x_k = 0$ for all $k, m$. 
Examples

let $Q$ be a real symmetric matrix and $e \in \mathbb{R}^n$ such that $e^T Q e = 1$

the **quadratic factor** $J_n(Q)$ is the space $\mathbb{R}^n$ equipped with the multiplication

$$x \bullet y = e^T Q x \cdot y + e^T Q y \cdot x - x^T Q y \cdot e$$

let $\mathcal{H}$ be an algebra of Hermitian matrices over a real coordinate algebra $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$

then the corresponding **Hermitian Jordan algebra** is the vector space underlying $\mathcal{H}$ equipped with the multiplication

$$A \bullet B = \frac{AB + BA}{2}$$
Examples

let $Q$ be a real symmetric matrix and $e \in \mathbb{R}^n$ such that $e^T Q e = 1$

the quadratic factor $\mathcal{J}_n(Q)$ is the space $\mathbb{R}^n$ equipped with the multiplication

$$x \bullet y = e^T Q x \cdot y + e^T Q y \cdot x - x^T Q y \cdot e$$

let $\mathcal{H}$ be an algebra of Hermitian matrices over a real coordinate algebra $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$

then the corresponding Hermitian Jordan algebra is the vector space underlying $\mathcal{H}$ equipped with the multiplication

$$A \bullet B = \frac{AB + BA}{2}$$
Theorem (Jordan, von Neumann, Wigner 1934)

*Every Euclidean Jordan algebra is a direct product of a finite number of Jordan algebras of the following types:*

- **quadratic factor with matrix** $Q$ of signature $+ - \cdots -$
- real symmetric matrices
- complex Hermitian matrices
- quaternionic Hermitian matrices
- octonionic Hermitian $3 \times 3$ matrices
Symmetric cones
Jordan algebras
The partial differential equation
Geometric characterization
Algebraic characterization

Classification of symmetric cones

Theorem (Vinberg, 1960; Koecher, 1962)

The symmetric cones are exactly the cones of squares of Euclidean Jordan algebras, \( K = \{ x^2 \mid x \in J \} \).

Every symmetric cone can be hence represented as a direct product of a finite number of the following irreducible symmetric cones:

- Lorentz (or second order) cone
  \[ L_n = \left\{ (x_0, \ldots, x_{n-1}) \mid x_0 \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2} \right\} \]

- matrix cones \( S_+(n), H_+(n), Q_+(n) \) of real, complex, or quaternionic hermitian positive semi-definite matrices

- Albert cone \( O_+(3) \) of octonionic hermitian positive semi-definite \( 3 \times 3 \) matrices
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Definition

A Jordan algebra is called **unital** if it possesses a unit element $e$, satisfying $u \cdot e = u$ for all $u \in J$.

Definition

A Jordan algebra is called **simple** if it is not nil and has no non-trivial ideal.

Theorem (Jordan, von Neumann, Wigner 1934)

*Euclidean Jordan algebras are unital and decompose in a unique way into a direct product of simple Jordan algebras.*
define recursively $u^{m+1} = u \cdot u^m$
with $u^0 = e$, define the exponential map

$$\exp(u) = \sum_{k=0}^{\infty} \frac{u^k}{k!}$$

**Theorem (Köcher)**

Let $J$ be a Euclidean Jordan algebra and $K$ its cone of squares. Then the exponential map is injective and its image is the interior of $K$,

$$\exp[J] = K^o.$$
let $J$ be a Euclidean Jordan algebra with cone of squares $K$
then we can define the logarithm

$$\log : K^o \rightarrow J$$

as the inverse of the exponential map
Definition

Let $J$ be a Jordan algebra. A symmetric bilinear form $\gamma$ on $J$ is called **trace form** if $\gamma(u, v \bullet w) = \gamma(u \bullet v, w)$ for all $u, v, w \in J$. 
for every $u$ in a unital Jordan algebra there exists $m$ such that

- $u^0, u^1, \ldots, u^{m-1}$ are linearly independent
- $u^m = \sigma_1 u^{m-1} - \sigma_2 u^{m-2} + \cdots - (-1)^m \sigma_m u^0$

$p_u(\lambda) = \lambda^m - \sigma_1 \lambda^{m-1} + \cdots + (-1)^m \sigma_m$ is the minimum polynomial of $u$

**Theorem (Jacobson, 1963)**

There exists a unique minimal polynomial $p(\lambda) = \lambda^m - \sigma_1(u) \lambda^{m-1} + \cdots + (-1)^m \sigma_m(u)$, the generic minimum polynomial, such that $p_u | p$ for all $u$. The coefficient $\sigma_k(u)$ is homogeneous of degree $k$ in $u$. The coefficient $t(u) = \sigma_1(u)$ is called generic trace and the coefficient $n(u) = \sigma_m(u)$ the generic norm.
Theorem (Jacobson)

Let $J$ be a unital Jordan algebra. The symmetric bilinear form

$$
\tau(u, v) = t(u \cdot v)
$$

is a trace form, called the \textit{generic bilinear trace form}.

for Euclidean Jordan algebras with cone of squares $K$ we have

$$
\log n(x) = t(\log x) = \tau(e, \log x)
$$

for all $x \in K^o$
Euclidean Jordan algebras

Theorem (Köcher)

Let $J$ be a unital real Jordan algebra. Then the following conditions are equivalent.

- $J$ is Euclidean
- there exists a positive definite trace form $\gamma$ on $J$.

if $J$ is a simple Euclidean Jordan algebra, then any non-degenerate trace form $\gamma$ on $J$ is proportional to the generic bilinear trace form $\tau$

hence $\gamma(e, \log x)$ is proportional to $\log n(x)$
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Notation for derivatives

let $F : U \to \mathbb{R}$ be a smooth function on $U \subset \mathbb{A}^n$, where $\mathbb{A}^n$ is the $n$-dimensional affine real space

we note $\frac{\partial F}{\partial x^\alpha} = F,\alpha$, $\frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} = F,\alpha\beta$ etc.

note $F,\alpha\beta$ for the inverse of the Hessian
Hessian metrics

**Definition**

Let \( U \subset \mathbb{A}^n \) be a domain equipped with a pseudo-metric \( g \). Then \( g \) is called **Hessian** if there locally exists a smooth function \( F \) such that \( g = F'' \). The function \( F \) is called **Hessian potential**.

The geodesics of a pseudo-metric obey the equation

\[
\ddot{x}^\alpha + \sum_{\beta \gamma} \Gamma^\alpha_{\beta \gamma} \dot{x}^\beta \dot{x}^\gamma
\]

with \( \Gamma^\alpha_{\beta \gamma} \) the Christoffel symbols

for a **Hessian metric** we have

\[
\Gamma^\alpha_{\beta \gamma} = \frac{1}{2} \sum_{\delta} F_{,\alpha \delta} F_{,\beta \gamma \delta}
\]
the third derivative $F'''$ of the Hessian potential is parallel with respect to the Hessian metric $F''$ if

$$\frac{\partial}{\partial x^\delta} F,_{\alpha\beta\gamma} + \sum_{\eta} \left( \Gamma^{\eta}_{\alpha\delta} F,_{\beta\gamma\eta} + \Gamma^{\eta}_{\beta\delta} F,_{\alpha\gamma\eta} + \Gamma^{\eta}_{\gamma\delta} F,_{\alpha\beta\eta} \right) = 0$$

in short notation $\hat{D} D^3 F = 0$, with $D$ the flat connection of $\mathbb{A}^n$ and $\hat{D}$ the Levi-Civita connection of the Hessian metric.

$$F,_{\alpha\beta\gamma\delta} = \frac{1}{2} \sum_{\rho\sigma} F,^{\rho\sigma} \left( F,_{\alpha\beta\rho} F,_{\gamma\delta\sigma} + F,_{\alpha\gamma\rho} F,_{\beta\delta\sigma} + F,_{\alpha\delta\rho} F,_{\beta\gamma\sigma} \right)$$
Integrability condition

differentiating with respect to $x^\eta$ and substituting the fourth order derivatives by the right-hand side, we get

$$F_{\alpha \beta \gamma \delta \eta} = \frac{1}{4} \sum_{\rho, \sigma, \mu, \nu} F_{\rho \sigma} F_{\mu \nu} \left( F_{\beta \eta \nu} F_{\alpha \rho \mu} F_{\gamma \delta \sigma} + F_{\alpha \eta \mu} F_{\rho \beta \nu} F_{\gamma \delta \sigma} \right. \left. + F_{\gamma \eta \nu} F_{\alpha \rho \mu} F_{\beta \delta \sigma} + F_{\alpha \eta \mu} F_{\rho \gamma \nu} F_{\beta \delta \sigma} + F_{\beta \eta \nu} F_{\gamma \rho \mu} F_{\alpha \delta \sigma} \right. \left. + F_{\gamma \eta \nu} F_{\alpha \rho \mu} F_{\beta \delta \sigma} + F_{\alpha \eta \mu} F_{\rho \gamma \nu} F_{\beta \delta \sigma} + F_{\beta \eta \nu} F_{\gamma \rho \mu} F_{\alpha \delta \sigma} \right)$$

anti-commuting $\delta, \eta$ gives the integrability condition

$$F_{\rho \sigma} F_{\mu \nu} \left( F_{\beta \eta \nu} F_{\delta \rho \mu} F_{\alpha \gamma \sigma} + F_{\alpha \eta \mu} F_{\rho \delta \nu} F_{\beta \gamma \sigma} + F_{\gamma \eta \mu} F_{\rho \delta \nu} F_{\alpha \beta \sigma} \right. \left. - F_{\beta \delta \nu} F_{\eta \rho \mu} F_{\alpha \gamma \sigma} - F_{\alpha \delta \mu} F_{\rho \eta \nu} F_{\beta \gamma \sigma} - F_{\gamma \delta \mu} F_{\rho \eta \nu} F_{\alpha \beta \sigma} \right) = 0.$$
let $K_{\alpha}^{\beta\gamma} = -\Gamma_{\beta\gamma}^{\alpha} = -\frac{1}{2} \sum_{\delta} F,_{\alpha\delta} F,_{\beta\gamma\delta}$, then $K_{\beta\gamma}^{\alpha} = K_{\gamma\beta}^{\alpha}$

contracting the integrability condition with $F,_{\eta\zeta}$, we get

$$\sum_{\mu,\rho} \left( K_{\alpha\mu}^{\zeta} K_{\delta\rho}^{\mu} K_{\beta\gamma}^{\rho} + K_{\beta\mu}^{\zeta} K_{\delta\rho}^{\mu} K_{\alpha\gamma}^{\rho} + K_{\gamma\mu}^{\zeta} K_{\delta\rho}^{\mu} K_{\alpha\beta}^{\rho} - K_{\mu\alpha\delta}^{\zeta} K_{\rho\mu}^{\zeta} K_{\beta\gamma}^{\rho} - K_{\beta\delta}^{\zeta} K_{\rho\mu}^{\zeta} K_{\alpha\gamma}^{\rho} - K_{\gamma\delta}^{\zeta} K_{\rho\mu}^{\zeta} K_{\alpha\beta}^{\rho} \right) = 0$$

this is satisfied if and only if

$$\sum_{\alpha,\beta,\gamma,\delta,\mu,\rho} \left( K_{\alpha\mu}^{\zeta} K_{\delta\rho}^{\mu} K_{\beta\gamma}^{\rho} u^{\alpha} u^{\beta} u^{\gamma} v^{\delta} - K_{\alpha\delta}^{\mu} K_{\rho\mu}^{\zeta} K_{\beta\gamma}^{\rho} u^{\alpha} u^{\beta} u^{\gamma} v^{\delta} \right) = 0$$

for all tangent vectors $u, v$
Jordan algebra defined by $F$

choose a point $e \in U$ and define a multiplication on $T_e U$ by

$$u \cdot v = K(u, v),$$

$$(u \cdot v)^\alpha = \sum_{\beta, \gamma} K_{\beta \gamma}^\alpha u^\beta v^\gamma$$

then $T_e U$ becomes a commutative algebra $J$

the integrability condition becomes

$$K(K(K(u, u), v), u) = K(K(u, v), K(u, u))$$

or

$$(u^2 \cdot v) \cdot u = (u \cdot v) \cdot u^2$$

hence $J$ is a Jordan algebra
the pseudo-metric $g = F''(e)$ satisfies

$$g(u \cdot v, w) = \sum_{\beta, \gamma, \delta, \rho} F,_{\beta \gamma} K^\beta_{\delta \rho} u^\delta v^\rho w^\gamma$$

$$= -\frac{1}{2} \sum_{\beta, \gamma, \delta, \rho, \sigma} F,_{\beta \gamma} F,_{\delta \rho \sigma} F,_{\sigma \beta} u^\delta v^\rho w^\gamma = -\frac{1}{2} \sum_{\gamma, \delta, \rho} F,_{\delta \rho \gamma} u^\delta v^\rho w^\gamma$$

$$= -\frac{1}{2} \sum_{\beta, \gamma, \delta, \rho, \sigma} F,_{\beta \delta} u^\delta F,_{\rho \gamma \sigma} F,_{\sigma \beta} v^\rho w^\gamma$$

$$= \sum_{\beta, \gamma, \delta, \rho} F,_{\delta \beta} u^\delta K^\beta_{\rho \gamma} v^\rho w^\gamma = g(u, v \cdot w).$$

hence $g$ is a trace form
Let $F : U \to \mathbb{R}$ be a solution of the equation $\hat{D}D^3 F = 0$. Let $e \in U$ and let $J$ be the algebra defined on $T_e U$ by the structure coefficients $K_{\beta\gamma}^\alpha = -\frac{1}{2} \sum \delta F,_{\alpha\delta} F,_{\beta\gamma\delta}$ at $e$.

Then $J$ is a Jordan algebra, and the Hessian metric $g = F''(e)$ is a non-degenerate trace form on $J$.

- If $F$ is convex and log-homogeneous, then $J$ is Euclidean.
- If in addition $J$ is simple, then $g$ is proportional to the generic bilinear trace $\tau$. 

Theorem (H., 2012)
Logarithmically homogeneous functions

Definition

Let $U \subset \mathbb{R}^n$ be an open conic set. A logarithmically homogeneous function on $U$ is a smooth function $F : U \rightarrow \mathbb{R}$ such that

$$F(\alpha x) = -\nu \log \alpha + F(x)$$

for all $\alpha > 0$, $x \in U$. The scalar $\nu$ is called the homogeneity parameter.
Let $J$ be a Euclidean Jordan algebra and $K$ its cone of squares. Let $\gamma$ be a non-degenerate trace form on $J$. Then $F : K^0 \to \mathbb{R}$ defined by

$$F(x) = -\gamma(e, \log x)$$

is a solution of the equation $\hat{D}D^3 F = 0$ such that $F''(e) = \gamma$ and, under identification of $T_eK^0$ and $J$, the multiplication in $J$ is given by $K_{\beta\gamma}^\alpha = -\frac{1}{2} \sum_{\delta} F_{,\alpha\delta}^\gamma F_{,\beta\gamma\delta}$ at $e$. 
Theorem (H., 2012)

Let $K = K_1 \times \cdots \times K_m$ be a symmetric cone and $K_1, \ldots, K_m$ its irreducible factors. Then for every set of non-zero reals $\alpha_1, \ldots, \alpha_m$, the function $F : K^o \to \mathbb{R}$ given by

$$F(A_1, \ldots, A_m) = - \sum_{k=1}^{m} \alpha_k \log n(A_k)$$

is log-homogeneous and satisfies the equation $\hat{D}D^3 F = 0$. The function $F$ is convex if and only if $\alpha_k > 0$ for all $k$. 
Theorem (H., 2012)

Let $U \subset \mathbb{A}^n$ be a subset of affine space and let $F : U \to \mathbb{R}$ be a log-homogeneous convex solution of the equation $\hat{D}D^3F = 0$. Then there exists a symmetric cone $K = K_1 \times \cdots \times K_m \subset \mathbb{A}^n$, positive reals $\alpha_1, \ldots, \alpha_m$, and a constant $c$ such that $F$ can be extended to a solution $\tilde{F} : K^o \to \mathbb{R}$ given by

$$\tilde{F}(A_1, \ldots, A_m) = - \sum_{k=1}^{m} \alpha_k \log n(A_k) + c.$$
when dropping convexity assumption, generalization beyond Euclidean Jordan algebras possible:


Thank you