10 Applications

In this lecture we consider several applications of second-order cone programming and semi-definite programming. More details can be found in [2].

10.1 Inscribed ellipsoid of maximal volume

Consider a polytope $P = \{x \mid Ax \le b\} \subset \mathbb{R}^n$ with non-empty interior. We want to find the ellipsoid E inscribed in the polytope which has the maximal volume among all such ellipsoids. Note that the polytope is given by a set of linear inequalities. Hence the inclusion $E \subset P$ is equivalent to the condition that all points of E satisfy these linear conditions. We shall describe the ellipsoid by an affine image of a Euclidean ball, more precisely by the set

$$E = \{x = Cu + c \mid \|u\|_2 \le 1\},$$

defined by an $n \times n$ square matrix C, and a vector $c \in \mathbb{R}^n$ defining the off-set. Without loss of generality we may assume that the matrix C is positive definite, retaining only the symmetric factor of the polar decomposition.

This form is convenient because we can directly insert the vector x in the linear inequalities defining the polytope. We have $E \subset P$ if and only if

$$Ax = A(Cu + c) \le b \qquad \forall\, u: \ \|u\|_2 \le 1.$$

This can be rewritten as

$$ACu \le b - Ac \qquad \Leftrightarrow \qquad \langle (AC)_i, u\rangle \le (b - Ac)_i \quad \forall\, i, \quad \forall\, u: \ \|u\|_2 \le 1.$$

Note that here A, b are given, and C, c are the decision variables of the problem. Each of the scalar inequalities finally becomes

$$\|(AC)_i\|_2 \le (b - Ac)_i,$$

which is a second-order cone condition.

The cost function is given by the volume of the ellipsoid. It does not depend on the off-set, but only on the determinant of the matrix C, to which it is proportional.

In order to represent the cost function $\det C$ in the framework of a semi-definite program, we need a semi-definite description of the hypo-graph of this function.

Lemma 10.1. *Let $X \succeq 0$ be a positive semi-definite matrix. Then there exists a lower triangular matrix with diagonal η such that $\prod_{i=1}^n \eta_i = \det X$ and*

$$\begin{pmatrix} X & \Delta \\ \Delta^T & \operatorname{diag}\eta \end{pmatrix} \succeq 0.$$

Proof. Indeed, let $X = LL^T$ be the Cholesky decomposition of X. Let η be the element-wise square of the diagonal of L. Then

$$\begin{pmatrix} X & L\operatorname{diag}(\sqrt{\eta}) \\ \operatorname{diag}(\sqrt{\eta}) \cdot L^T & \operatorname{diag}\eta \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & \operatorname{diag}(\sqrt{\eta}) \\ \operatorname{diag}(\sqrt{\eta}) & \operatorname{diag}\eta \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix}^T \succeq 0.$$

Moreover, $\det X = (\det L)^2 = \left(\prod_{i=1}^n L_{ii}\right)^2 = \prod_{i=1}^n \eta_i$. □

On the other hand, if there exists a lower-triangular matrix Δ with nonnegative diagonal η, such that the LMI in the lemma holds, then $\det X \ge \prod_{i=1}^n \eta_i$. Indeed, for every $\epsilon > 0$ we have that by Schur complement

$$X \succeq \Delta \cdot (\operatorname{diag}\eta + \epsilon I)^{-1} \cdot \Delta^T.$$

Hence

$$\det X \ge (\det \Delta)^2 \cdot \prod_{i=1}^n \frac{1}{\eta_i + \epsilon} = \prod_{i=1}^n \frac{\eta_i^2}{\eta_i + \epsilon} = \prod_{i=1}^n \eta_i \cdot \frac{\eta_i}{\eta_i + \epsilon}.$$

If η is not strictly positive, then the claim holds. If $\eta > 0$, then $\lim_{\epsilon\to 0} \frac{\eta_i}{\eta_i+\epsilon} = 1$ for all i, and the inequality is verified too.

We now need a semi-definite representation of the hypo-graph of the product of n nonnegative numbers. However, such a representation cannot exist, because the product is not concave. However, we can still represent the hypo-graph of the geometric mean, which is a monotone function of the product. Recall that $0 \le t \le \sqrt{t_1 t_2}$ for nonnegative t_1, t_2 if and only if

$$t \ge 0, \qquad \begin{pmatrix} t_1 & t \\ t & t_2 \end{pmatrix} \succeq 0.$$

Complementing the n values $\eta_1, \dots, \eta_n$ by ones to $2^{\lceil \log_2 n \rceil}$ values, we may use the semi-definite representation of the hypo-graph of the geometric mean recursively to obtain a representation of the hypo-graph of the mean of the $2^{\lceil \log_2 n \rceil}$ values.

10.2 Circumscribed ellipsoid of minimal volume

Now we consider the in some sense dual problem, namely to find the ellipsoid of minimal volume which contains a given polytope. We shall describe the polytope as the convex hull of its extreme points,

$$P = \mathrm{conv}\{x_1, \dots, x_m\}.$$

We assume that the polytope has a non-empty interior, otherwise the problem does not make sense.

We look for an ellipsoid

$$E = \{x \mid (x - D^{-1}d)^T D(x - D^{-1}d) \le 1\}$$

containing P. Here D is a shape matrix and $D^{-1}d$ is an off-set. We have chosen this representation, different from the one in Section 10.1, because minimizing the volume of E is equivalent to maximizing the determinant of D. In Section 10.1 we have already seen how to incorporate this cost function into a semi-definite program.

Note that although the off-set is not directly a design variable, if we have found d, D, then we can easily recover the off-set by computing the corresponding matrix-vector product. The condition $P \subset E$ is satisfied if and only if every extreme point of P is in E, i.e.,

$$(x_i - D^{-1}d)^T D(x_i - D^{-1}d) = x_i^T D x_i - 2x_i^T d + d^T D^{-1} d \le 1 \qquad \forall\, i = 1, \dots, m.$$

Now the epi-graph of the scalar expression $d^T D^{-1} d$ can be easily modelled by Schur complements. We have $s \ge d^T D^{-1} d$ with $D \succeq 0$ if and only if

$$\begin{pmatrix} s & d^T \\ d & D \end{pmatrix} \succeq 0.$$

On the other hand, the condition $x_i^T D x_i - 2x_i^T d + s \le 1$ is just a scalar inequality which is jointly linear in d, D.

10.3 Truss topology design

A truss is a system of K bars, interconnected at n nodes, which serves to carry loads. The loads act by external forces at some nodes, to which the truss reacts by deformation. The deformation in turn generates reaction forces which compensate the load. The sum of all forces at every node is zero in equilibrium position, and the forces applied by the loads are redistributed on those nodes which are anchored in the ground or in walls, which compensate the load by a reaction force in turn. As a consequence, the sum of all *internal* forces is known at the nodes where the loads apply, unknown at the nodes which are anchored, and zero at all other nodes.

A particularity of a truss is that the deformation consists solely in stretching and squeezing the bars. Thus the forces applied by a bar at the nodes between which it is located are parallel to the direction of the bar and compensate each other.

The deformation of a bar leads to the build-up of potential energy in the bar. The less the total energy generated in the truss is, the *stiffer* it is said to be. The goal of truss topology design is to minimize the deformation energy generated by application of a given load.

Let us consider a single bar, let its index be k, of length $l = \|v_i - v_j\|$ and cross-section σ, located between two nodes i, j with coordinates v_i, v_j. Let the displacements of the node locations be given by x_i, x_j. The length of the deformed bar will up to first order in the displacements be given by

$$\|v_i + x_i - v_j - x_j\| \approx \sqrt{\|v_i - v_j\|^2 + 2\langle v_i - v_j, x_i - x_j\rangle} \approx l\left(1 + \frac{\langle v_i - v_j, x_i - x_j\rangle}{\|v_i - v_j\|^2}\right).$$

The change in length is hence given by $\delta l = \frac{\langle v_i - v_j, x_i - x_j\rangle}{l}$.

The mass m_k of bar k is given by $\rho \cdot l \cdot \sigma$, where ρ is the density of the material. The force generated by a deformation resulting in a change of length $l \mapsto l + \delta l$ is determined by the Young modulus c. It is proportional to the cross-section, hence the mass divided by the length, and to the relative change in length of the bar. If δl is negative, then the force at x_i is in the direction of $x_i - x_j$, i.e., outward, while for $\delta l < 0$ it is directed inward, in the direction $x_j - x_i$. Hence we obtain

$$f_{ik} = -f_{jk} = -c \cdot \frac{m_k}{\|v_i - v_j\|} \cdot \left\langle \frac{v_i - v_j}{\|v_i - v_j\|}, x_i - x_j \right\rangle \cdot \frac{1}{\|v_i - v_j\|} \cdot \frac{v_i - v_j}{\|v_i - v_j\|}$$

for the force exerted by bar k on node i. Here the first factor is the Young modulus, the second one the cross-section multiplied by the density, the third one the change in length, the fourth one the inverse length (to obtain the relative change in length), and the last factor is the direction of the force.

We assume that the deformations are small enough to stay in the linear regime of elastic deformations. Note also that the positions v_i of the nodes are determined by the design of the truss, while the displacements x_i are generated by the applied force.

For each bar k, let us define a vector $b_k = (b_{ik})_{i=1,\dots,n} \in \mathbb{R}^{3n}$, where $b_{ik} = -b_{jk} = \sqrt{c}\frac{v_i - v_j}{\|v_i - v_j\|^2}$ if k connects the nodes i, j, and $b_{lk} = 0$ for $l \not\in \{i, j\}$. Let us also collect the displacements into a large vector $x = (x_i)_{i=1,\dots,n} \in \mathbb{R}^{3n}$. The the force generated by bar k on the ensemble of nodes can be written as

$$f_k = -m_k b_k b_k^T x.$$

The philosophy of truss topology design posits that the v_i are *known*, while the masses m_k of the nodes are the design variables. The equality $m_k = 0$ then simply means that bar k is absent in the construction. By filling the space with a dense grid of potential node locations and optimizing over all bars that connect reasonably distant nodes, we hence obtain a topology without using the node positions explicitly as design variables. Let us formulate the truss topology design problem.

The external force is given by

$$f = -\sum_k f_k = \sum_k m_k b_k b_k^T x = Ax, \tag{1}$$

with A an appropriately defined positive semi-definite matrix of size $3n$. It is part of the problem data for the nodes $i \in L$ at which the loads act, is unknown at the nodes $i \in J$ attached to the support, and zero for every $i \in I = \{1, \dots, n\} \setminus (L \cup J)$.

Finally, the energy stored in the truss after deformation is given by the work $\langle f, x\rangle$ of the force against the displacement.

Let us analyze the problem. We have linear constraints on the components of f, namely

$$f_i = g_i, \quad i \in L; \qquad f_i = 0, \quad i \in I,$$

where g_i is the known external force acting on node i. Further, we have the constraint (1), which contains the design variables f, m, x in a non-linear fashion.

However, we can exclude the variables x by Schur complement, minimizing an additional variable t under the semi-definite constraint

$$\begin{pmatrix} t & -f^T \\ -f & \sum_k m_k b_k b_k^T \end{pmatrix} \succeq 0.$$

Clearly we can make t arbitrarily small by increasing the masses m_k of the bars. Therefore it is reasonable to add a constraint on the masses, e.g., that the total mass $\sum_k m_k$ equals some maximal material budget.

For more details see [1].

10.4 Lyapunov function design

Consider the linear homogeneous dynamical system

$$\dot{x}(t) = A(t)x(t). \tag{2}$$

The coefficient matrix $A(t)$ is time-variant, but at any moment it belongs to some known set $\mathcal{U}$.

Definition 10.2. We call a strictly convex quadratic function $L(x) = x^T Xx$, $X \succ 0$, a *Lyapunov function* for system (2) if the condition

$$\frac{d}{dt}L = x(t)^T(A(t)^T X + XA(t))x(t) \leq -sL(x(t))$$

holds for all t and for some fixed constant $s > 0$.

The existence of a Lyapunov function guarantees global stability of the system, because the value of the function $L(x(t))$ decreases exponentially to zero together with the norm of $x(t)$.

In order for the condition to hold no matter how $A(t)$ behaves, it is sufficient to demand that

$$A^T X + XA \preceq -sX \quad \forall\ A \in \mathcal{U}. \tag{3}$$

In order to solve the design problem we have to impose conditions on the shape of the uncertainty set $\mathcal{U}$. Suppose that $\mathcal{U} = \text{conv}\{A_1, \dots, A_m\}$ is a polytope, defined by its vertices A_i. Then (3) holds for all $A \in \mathcal{U}$ if and only if it holds for all $A = A_1, \dots, A_m$.

On the other hand, if X is any positive definite solution to the problem, and the matrix inequality is linear and homogeneous in X, we may scale X by a positive factor to achieve $X \succeq I$. Thus the strict matrix inequality $X \succ 0$ can be transformed into a non-strict one.

Since we demand only the existence of a positive constant s, and all norms in a finite-dimensional space are equivalent, condition (3) can also be written in the form

$$X \succeq I, \quad A^T X + XA \preceq -sI \quad \forall\ A = A_i,\ i = 1, \dots, m.$$

We may check above condition by solving the semi-definite program

$$\min s:\ X \succeq I,\ sI - A_i^T X - XA_i \succeq 0\ \forall\ i.$$

If the optimal value of the SDP is strictly negative, the matrix X is a Lyapunov function for system (2).

10.5 Control design

Consider the controlled dynamical system

$$\dot{x} = Ax + Bu,$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, and A, B are known coefficient matrices.

The goal is to design a *feedback control*, i.e., a rule according to which the control u is chosen as a function of the state x. We shall consider the problem of *linear* stabilizing control design, i.e., we look for a coefficient matrix K such that the system with control $u = Kx$ is globally stable.

According to the previous section, it is sufficient to find a K such that the system possesses a Lyapunov function. Note that $\dot{x} = Ax + B(Kx) = (A + BK)x$, and the system matrix depends linearly on the design variable K. We look for a Lyapunov function of the form $L(x) = x^T Xx$, $X \succ 0$, satisfying the condition

$$\frac{d}{dt}L = x^T((A + BK)^T X + X(A + BK))x \leq -sL = -sx^T Xx$$

for some $s > 0$. We hence look for a matrix $X \succ 0$ and a constant $s > 0$ satisfying the matrix inequality

$$(A + BK)^T X + X(A + BK) \preceq -sX.$$

Multiplying from the left and the right by the inverse of X, we obtain the equivalent condition

$$X^{-1}(A^T + K^T B^T) + (A + BK)X^{-1} \preceq -sX^{-1}, \quad s > 0, \quad X \succ 0.$$

Note that if X and the product KX^{-1} are known, then K can easily be recovered. Hence we may define new design variables $Y = X^{-1}$, $Z = KX^{-1}$, and the condition becomes

$$(AY + BZ)^T + (AY + BZ) \preceq -sY, \quad s > 0, \quad Y \succ 0.$$

As in the previous section, we may replace the strict condition $Y \succ 0$ by the non-strict one $Y \succeq I$, and the right-hand side by a multiple of the identity. We obtain the semi-definite program

$$\min s : \ sI - (AY + BZ) - (AY + BZ)^T \succeq 0, \ Y \succeq I.$$

If the value of this program is strictly negative, then $X = Y^{-1}$, $K = ZX$ yield the sought Lyapunov function and stabilizing feedback law.

References

[1] Aharon Ben-Tal and Arkadi Nemirovski. Robust truss topology design via semidefinite programming. *SIAM J. Optim.*, 7(4):991–1016, 1997.

[2] Aharon Ben-Tal and Arkadi Nemirovski. Lectures on modern convex optimization - analysis, algorithms, engineering applications. https://www2.isye.gatech.edu/ nemirovs/LMCO_LN.pdf, 2019.