Large scale reduction principle

M.Clausel–F.Roueff–M.S.Taqqu–C. Tudor
Memory parameter of a time series

- $X = \{X_t\}_{t \in \mathbb{Z}}$ : centered stationary time series with unit variance and spectral density $f$.

- $d_X$ memory parameter of $X$ (Hurvich et al. 1995) if

  \[
  f(\lambda) \underset{\lambda=0}{\sim} |\lambda|^{-2d_X}.
  \]

- $X$ : long memory process if $0 < d_X < 1/2$, short memory process if $d_X = 0$, negative memory process if $d_X < 0$.

- Extension to the case where $\Delta^K X$ stationary for $K \geq 1$ considering the generalized spectral density of $X$

  \[
  f(\lambda) = |1 - e^{-i\lambda}|^{-2K} f_{\Delta^K X}(\lambda).
  \]
Examples

- FARIMA model: $\Delta^d X_\ell = \xi_\ell$ with $\Delta^d$ fractional differentiation operator of order $d \in (-1/2, 1/2)$ and $(\xi_t)$ iid $\mathcal{N}(0, 1)$. Stationary time series with memory parameter $d_X = d$.

- $\{B_H(k)\}_{k \in \mathbb{Z}}$ discretized version of usual FBM $\{B_H(t)\}_{t \in \mathbb{R}}$ with Hurst index $H \in (0, 1)$. Memory parameter $d_{B_H} = H + 1/2$. 

M. Clausel–F. Roueff–M.S. Taqqu–C. Tudor

Large scale reduction principle
Main goals

- Estimation of the memory parameter of a non linear time series of the form $G(X)$, $X$ Gaussian time series.
- Statistical properties and asymptotical behavior of the estimator.
- Application to hypothesis testing.
A wavelet based estimator

**Wavelet bases**

- Compactly supported MRA defined from $\varphi, \psi \in L^2(\mathbb{R})$ compactly supported.
- $\psi$ : function admitting $M$ vanishing moments.
- Wavelet coefficients of $F \in L^2(\mathbb{R})$

$$W_{j,k}^{(F)} = \int_{\mathbb{R}} F(t) \psi_{j,k}(t) dt, \text{ with } \psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j} t - k).$$

- Wavelet expansion of $F$ in $L^2(\mathbb{R})$ : $F = \sum_{(j,k) \in \mathbb{Z}^2} W_{j,k}^{(F)} \psi_{j,k}$.
- Case $X$ time series $x(t) = \sum_\ell X_\ell \varphi(t - \ell)$ and

$$W_{j,k}^{(X)} = \int_{\mathbb{R}} x(t) \psi_{j,k}(t) dt = \sum_\ell h_{j,2j,k-\ell} X_\ell = (h_j, \cdot \star X)_{2j,k}$$

with $h_j(m) = \int_{\mathbb{R}} \phi(t + m) \psi_{j,0}(t) dt$. 

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**Large scale reduction principle**
A wavelet based estimator
The example of FBM (Wornell et al. 1992, Bardet 2002)

- FBM case \( \{B_H(t)\} \) with Hurst index \( H \), variance of wavelet coefficients related to \( d_{B_H} = H + 1/2 \).
- \( H \)–self–similarity
  \[
  \mathbb{E}[|W_{j,k}^{B_H}|^2] = C 2^{2j(H+1/2)} = C 2^{2jd_H}
  \]
- Gaussian or linear time series \( X \)
  \[
  \mathbb{E}[|W_{j,k}^X|^2] \sim C(f^*(0), d)2^{2jd_X} \quad \text{as} \quad j \to \infty.
  \]

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Large scale reduction principle
A wavelet based estimator
The estimator of Abry–Veicht (1998)

- \( X_1, \ldots, X_N \) sample of the time series \( X \) with memory parameter \( d_X \).
- **Empirical variance** of the wavelet coefficients at scale \( j \)
  \[
  \hat{\sigma}_{N,j} = \frac{1}{n} \sum_{k=0}^{n-1} \left( W_{j,k}(X) \right)^2,
  \]
  with \( n \sim N2^{-j} \) number of coefficients available at scale \( j \).
- Expected result \( \hat{\sigma}_{N,j} \sim \mathbb{E}[|W_{j,k}|^2] \sim C(f^*(0), \psi)2^{2jd_X} \) as \( N, j \to \infty \)
A wavelet based estimator
The estimator of Abry–Veicht (1998)

- Wavelet estimator

\[ \hat{d}_{N,j}(X) = \sum_{i=0}^{p} w_i \log \hat{\sigma}^2_{N,j+i} \]

with \( w_0, \ldots, w_p \) s.t. \( \sum_{i=0}^{p} w_i = 0 \) and \( \sum_{i=0}^{p} iw_i = 1/(2 \log 2) \).

- Gaussian/linear case: \( \hat{\sigma}_{N,j} \) and \( \hat{d}_{N,j} \) both satisfying a CLT (Moulines–Roueff–Taqqu (2007), Roueff–Taqqu (2009)). This means that under mild assumptions

\[ (N2^{-j})^{1/2}(\hat{d}_{N,j}(X) - d_X) \]

admits a Gaussian limit \( U_1 \) which can be given explicitly.
A wavelet based estimator
Beyond linear case?

- Extension to the general non linear case using the Abry–Veicht estimator.
Statistical properties of the wavelet–based estimator

Preliminary results

- $X$ Gaussian centered stationary time series with memory parameter $d_X$, $Y = G(X)$ with $G$ non linear function.
- Memory parameter of $Y$?
- Depends on $d_X$ and on the Hermite expansion of $G$

$$G = \sum_q c_q H_q,$$

where $\sum_q c_q^2 / q! < +\infty$, $H_q$ $q$–th Hermite polynomial.
- Hermite rank of $G$ $q_0 = \min\{q, c_q \neq 0\}$.
- Memory parameter of $Y$: $\delta(q_0) = d_X q_0 - (q_0 - 1)/2$ (Dalla et al. 2006).
We apply wavelet–based estimation to $Y = G(X)$.

**Theorem (Clausel et al., 2015) General case $Y = G(X)$**

Let $(j_N)$ increasing sequence s.t. $\lim_{N \to \infty} N2^{-j_N} = \infty$. Suppose that $M \geq K + \delta(q_0)$. Then, as $N \to \infty$,

$$\hat{d}_{N,j_N}(Y) \xrightarrow{(P)} d_Y = \delta(q_0).$$
Asymptotical properties of the wavelet–based estimator

Some questions

- Consistency not sufficient in view of statistical applications as hypothesis testing.
- Convergence rate and asymptotical behavior of the estimator?
- **Reduction principle true** for the wavelet coefficients (Clausel et al., 2012). For \( Y = G(X) \) with
  \[
  G = c_{q_0} H_{q_0}/q_0! + \cdots
  \]
  \[
  \mathcal{W}_{j,k}^{(Y)} \approx \mathcal{W}_{j,k}^{(c_{q_0} H_{q_0}(X)/q_0!)}
  \]
  Does \( \hat{d}_{N,j}(Y) \) satisfy the **reduction principle**? If such the case, for \( Y = G(X) \) with
  \[
  G = c_{q_0} H_{q_0}/q_0! + \cdots
  \]
  then \( \hat{d}_{N,j}(Y) \) behaves as \( \hat{d}_{N,j}(c_{q_0} H_{q_0}(X)/q_0!) \) as \( N, j \to \infty \)
- Behavior of \( \hat{d}_{N,j}(c_{q_0} H_{q_0}(X)/q_0!) \)?
The special case $G = c_{q_0} H_{q_0}(X)/q_0!$, $q_0 \geq 2$

**Theorem (Clausel et al., 2014) Case $Y = H_{q_0}(X)/q_0!$, $q_0 \geq 2$**

Assume $M \geq K + \delta(q_0)$. As $N \to \infty$, if $j = j(N)$ is such that $j \to \infty$ and $N2^{-j} \to \infty$, then

$$
\hat{d}_{N,j}(Y) = d_Y + (N2^{-j})^{2d_X - 1} O_P(1) + O \left(2^{-\tilde{\beta}j}\right).
$$

where $\tilde{\beta}$ is related to the smoothness at 0 of $f^*(\lambda) = |\lambda|^{2d_X} f(\lambda)$. Moreover the $O_P$-term converges in distribution to a Rosenblatt variable $U_2$. 

**M.Clausel–F.Roueff–M.S.Taqqu–C. Tudor**

Large scale reduction principle
Asymptotical properties of the wavelet–based estimator

Hint of the proof (1)

- Harmonizable representation of $X$
  \[ X_\ell = \int_{-\pi}^{\pi} e^{i\lambda \ell} f^{1/2}(\lambda) d\hat{W}(\lambda), \]

- $Y_\ell = c_{q_0} H_{q_0}(X_\ell)/q_0!$ multiple stochastic integral of order $q_0$.

- $W_{j,k}^{(Y)}$ linear in $Y$ : multiple stochastic integral of order $q_0$.

- Centered empirical variance ? Need to estimate
  \[ \frac{1}{n} \sum_{k=0}^{n-1} [W_{j,k}^{(Y)}]^2 - \mathbb{E} \left[ \frac{1}{n} \sum_{k=0}^{n-1} [W_{j,k}^{(Y)}]^2 \right] \]
Asymptotical properties of the wavelet–based estimator

Hint of the proof (2)

- Product formula for multiple stochastic integrals applied to the multiple stochastic integral $W_{j,k}^{(Y)}$ → decomposition into Wiener chaos of the centered empirical variance

$$\hat{\sigma}_{N,j} - \mathbb{E}[\hat{\sigma}_{N,j}] = \sum_{q=1}^{q_0} I_{N,j}^{(2q)}$$

with $I_{2q}$ multiple integrals of order $2q$.

- Dominating term $I_{N,j}^{(2)}$: Rosenblatt variable whose asymptotic variance is known → asymptotical behavior of the empirical variance and the estimator of the memory parameter using the delta method.
End of the story?

- We know the asymptotic limit of the estimator if $G = H_{q_0}$ in the two cases $q_0 = 1$ (Gaussian) and $q_0 \geq 2$ (Rosenblatt).
- If $G = c_{q_0} H_{q_0} / q_0! + \cdots$ and reduction principle true

$$d_{N,j}(Y) \approx d_{N,j}(c_{q_0} H_{q_0}(X) / q_0!) . . . .$$

Unfortunately not so simple (Abry et al. 2011)!!

- The reduction principle may not hold....
Asymptotical properties of the wavelet–based estimator

A counterexample for the reduction principle

- Case $Y = G(X)$ with $G = H_{q_0} + H_{q_0+1}$, $q_0 \geq 2$.
- $W_{j,k}^{(Y)} = W_{j,k}^{(q_0)} + W_{j,k}^{(q_0+1)}$ with $W_{j,k}^{(q)}$ in the $q$–th Wiener chaos for $q = q_0, q_0 + 1$.
- Product formula

$$[W_{j,k}^{(Y)}]^2 = [W_{j,k}^{(q_0)}]^2 + [W_{j,k}^{(q_0+1)}]^2 + 2W_{j,k}^{(q_0)} W_{j,k}^{(q_0+1)}$$

- If reduction principle \textit{true}

$$\frac{1}{n} \sum_{k=0}^{n-1} [W_{j,k}^{(Y)}]^2 \approx \frac{1}{n} \sum_{k=0}^{n-1} [W_{j,k}^{(q_0)}]^2$$
Statistical properties of the wavelet–based estimator
A counterexample for the reduction principle

- If $N \ll 2^{2j}$, the sum
  \[
  \frac{1}{n} \left( \sum_{k=0}^{n-1} [W_{j,k}^{(q_0)}]^2 \right)
  \]
  is dominating in the empirical variance and the reduction principle holds.

- Unfortunately, if $2^{2j} \ll N$, the sum
  \[
  \frac{1}{n} \left( \sum_{k=0}^{n-1} W_{j,k}^{(q_0)} W_{j,k}^{(q_0+1)} \right)
  \]
  is dominating and the reduction principle does not hold!!

- Extension to the case $G = H_{q_0} + H_{q_\ell_0} + H_{q_\ell_0+1}$. The reduction principle holds or not depending whether $N \ll 2^{j(\nu+1)}$ or $2^{j(\nu+1)} \ll N$ with $\nu = 2q_\ell_0 + 1 - 2q_0$. 

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Large scale reduction principle
Statistical properties of the wavelet–based estimator

A counterexample for the reduction principle

- General case (Clausel et al. 2013). Maybe complicated: limit may be Gaussian, Rosenblatt or Hermite and the rate of convergence of the estimator can be different!
- Depends on the value of \( \nu \) such that \( N \sim 2^j(\nu+1) \) with respect to some critical indices depending on the whole function \( G \)....
- Sufficient conditions for the reduction principle to hold?
Assume $M \geq K + \delta(q_0)$. There exists some critical index $\nu_c$ which can be defined explicitly and depends only on $G, d_X$, such that if $N \ll 2^j(\nu_c+1)$, the reduction principle holds as $j, N \to \infty$.

In some cases this critical index is very simple.

- $G$ even: $\nu_c = \infty$.
- $q_0 \geq 2$ and there is two LRD terms in the Hermite expansion of $G$

$$\nu_c = 1 + 2(q_{\ell_0} - 2q_0)$$
Statistical properties of the wavelet–based estimator
Application to hypothesis testing

- Definition of a statistical test procedure which applies to a general $G$.
- Let $d_0^*$ : possible value for the true unknown memory parameter $d_Y$ of $Y$.
- Hypotheses

\[ H_0 : d_Y = d_0^* \quad \text{against} \quad H_1 : d_Y \in (0, \bar{K} + 1/2) \setminus \{d_0^*\}. \]
\( \alpha \in (0, 1) \) be a significance level.

Statistical test

\[
\delta_s = \begin{cases} 
1 & \text{if } |\hat{d}_0 - d_0^*| > s_N(\alpha), \\
0 & \text{otherwise.} 
\end{cases}
\] (1)

where \( s_N(\alpha) \) is the \((1 - \alpha/2)\) quantile of \( U_1/(N2^{-j})^{1/2} \) or \( U_1/(N2^{-j})^{1/2} \) depending on the Hermite rank of \( G \).
The constant $\zeta$ is a constant depending on the behavior of $f(\lambda)|\lambda|^{2d}$ at $\lambda = 0$.

**Theorem**

Let $j = (j_{N})$ s.t. $N2^{-j} \to \infty$ holds, $M \geq K + \delta(q_{0})$. Suppose moreover that, as $N \to \infty$,

$$N2^{-j} \ll 2^{j\nu^{*}_c},$$

and that

$$2^{-\zeta j} \ll u_{N}^{-1},$$

with $u_{N} = (N2^{-j})^{1/2}$ if $q_{0} = 1$, $u_{N} = (N2^{-j})^{1-2d_{X}}$ otherwise. Then, the test $\delta_{s}$ is a consistent test with asymptotic significance level $\alpha$. 


• M. Clausel, F. Roueff, M.S. Taqqu, C. Tudor, Large scale behavior of wavelet coefficients of non–linear subordinated processes with long memory. ACHA (2012).


• M. Clausel, F. Roueff, M.S. Taqqu, Large scale reduction principle and application to hypothesis testing. EJS (2015)
The critical exponent is

\[
\nu_c = \begin{cases} 
\infty, & \text{if } \mathcal{L} = \{0\} \text{ or if } q_0 = 1, \ d \leq 1/4 \text{ and } I_0 = \emptyset, \\
\frac{d+1/2-2\delta_+ (q_{\ell_0})}{d}, & \text{if } q_0 = 1, \ d \leq 1/4 \text{ and } I_0 \neq \emptyset, \\
\frac{1-2\delta_+ (q_1-1)}{2d-1/2}, & \text{if } q_0 = 1, \ d > 1/4, \ 1 \in \mathcal{L} \text{ and } J_d = \emptyset, \\
\min \left( \frac{1-2\delta_+ (q_1-1)}{2d-1/2}, \frac{2d+1/2-2\delta_+ (q_{\ell_r})-\delta(r+1)}{\delta(r+1)} : r \in \mathcal{I}_r \right), & \text{if } q_0 = 1, \ d > 1/4 \text{ and } J_d \neq \emptyset, \\
\infty, & \text{if } q_0 \geq 2 \text{ and } I_0 = \emptyset, \\
1 + \frac{4(\delta(q_0)-\delta_+ (q_{\ell_0}))}{1-2d}, & \text{if } q_0 \geq 2 \text{ and } I_0 \neq \emptyset.
\end{cases}
\]