Hyperbolic wavelet transform: an efficient tool for multifractal analysis of anisotropic fields

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Abstract

Global and local regularities of functions are analyzed in anisotropic function spaces, under the common framework supplied by hyperbolic wavelet bases. Local and directional regularity features are characterized by means of global quantities derived from the coefficients of hyperbolic wavelet decompositions. A multifractal analysis is introduced, that jointly accounts for scale invariance and anisotropy, and its properties are investigated.

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1 Introduction

Natural images often display various forms of anisotropy. For a wide range of applications, anisotropy has been quantified through regularity characteristics and features that strongly differ when measured in different directions. This is, for instance, the case in medical imaging (osteoarthritis, muscular tissues, mammographies,...), cf. e.g. [13, 14], hydrology [42], fracture surfaces analysis [20], ... For such images, a key issue consists first in describing the anisotropy of the texture, and then in defining regularity anisotropy parameters that can be measured via numerical procedures and further involved into e.g., classification schemes. This requires the design of a mathematical framework that allows to define and estimate these parameters. Such a program can be split into several questions, some of them having already been either solved or, at least, partially addressed. We start by briefly sketching this program.

Important examples of anisotropic self-similar fields driven by two parameters (an anisotropy matrix and a self-similarity index) have been introduced and studied by H. Biermé, M. Meerschaert and H. Scheffler, in [13], as a relevant model to describe osteoporosis.

In [19], two of the authors (M. Clausel and B. Vedel) addressed the question of defining in a proper way the concept of anisotropy of an image in relation to its global regularity. In

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particular they showed that the two parameters characterizing the above mentioned random fields can be recovered without a priori knowledge of the characteristics of the model, by studying the global smoothness properties of the process. This result indicated that some of the properties characterizing anisotropy are revealed by the regularity of the sample paths when analyzed with functional spaces well-adapted to anisotropy: Anisotropic Besov spaces, which extend (isotropic) Besov spaces (see Section 2.1 for a definition); this first result put into light the key role that such spaces might play in the analysis of random anisotropic textures. For example, in two variables, such spaces present the following anisotropy property: They are invariant after a rescaling of \( \lambda^\alpha \) in the \( x \) axis and \( \lambda^{2-\alpha} \) in the \( y \) axis, where \( \alpha \) is the anisotropy parameter.

Note that this point of view follows, in the isotropic setting, ideas that are common in multifractal analysis, where global function space regularity yields pertinent parameters for classification, see [1] and references therein. This also explains the choice of Besov spaces instead of Sobolev spaces, which (on top of a simpler wavelet characterization) allows to use Lebesgue integrability indices \( p \in (0,1) \), therefore yielding a larger range of classification indices.

Note that, in the isotropic setting, wavelet bases supply an efficient tool for measuring smoothness in a large range of functional spaces (cf. [37] for details). For an a priori given anisotropy parameter, anisotropic wavelets such as curvelets, bandelets, contourlets, shearlets, ridgelets, or wedgelets have been introduced (see e.g. [32] for a thorough review of these representation systems and a comparison of their properties for image processing). Natural criteria of choice between these alternatives are, on the mathematical side, that these variations on isotropic wavelets supply bases for the corresponding anisotropic spaces, and, on the applied side, that practically tractable procedures can be devised and implemented to permit the characterization of real-world data according to these function spaces.

On the mathematical side, many authors addressed the problem of obtaining bases of a given anisotropic Besov space, and proposed different solutions, depending on the precise definition of anisotropic spaces they started with, cf. e.g. [9, 10, 21, 30, 31, 36, 50] and also the recent papers [26] by G. Garrigós and A. Tabacco, and [29] by D. Haroske and E. Tamás, which contain numerous references on the subject. Note also that, in several cases, the study was restricted to a specific anisotropy parameter: Typically, parabolic anisotropy, where a contraction by \( \lambda \) in one direction is associated with a contraction by \( \lambda^2 \) along the orthogonal direction, (see [35, 40, 44], and references therein, in particular for applications to directional regularity). The corresponding dictionaries are in that case curvelets or contourlets (corresponding to the Hart–Smith decomposition in the continuous setting, see [47]). This choice was motivated from application to PDEs where this specific anisotropy has a physical relevance (see e.g., [18, 27] where curvelets and ridgelets are used for the study of Fourier integral operators, with applications to the wave equation); however, it is no longer justified when dealing with images, where no particular form of anisotropy can be postulated a priori. Actually, except perhaps in very specific types of applications, the particular type of anisotropy present in a image is not a priori known, and figuring it out is part of the problem. Therefore a key practical issue for us is that we must deal simultaneously with the whole collection of anisotropic Besov spaces, for all possible anisotropies. This rises the mathematical question of finding specific bases which would serve as a common “dictionary” for all of these spaces simultaneously, and obtaining simple characterizations of them on this “dictionary”. Clearly, the dictionary used should display all possible types of anisotropy; among the variants of
directional wavelets that are not tailored to a specific anisotropy, two play a prominent role because of their numerical simplicity (as tensor products of 1-D wavelets) and their good functional properties:

- One is the collection of anisotropic Triebel bases, see [50]; each basis is constructed from the standard wavelet case through a multiresolution procedure, tailored to a specific anisotropy. Simple characterizations of Besov spaces (with the same anisotropy) have been supplied within this system. Such characterizations can thus be used as a building step to construct a multifractal formalism, see [6] for first results in this direction, and also Section 3. Triebel bases provide a powerful tool to deduce results on a given anisotropic Besov space; in particular, they enable to show that these spaces are isomorphic to the corresponding isotropic Besov spaces. Further, some results such as embeddings or profiles of Besov characteristics can be obtained, via the transference method proposed by H. Triebel. However, the knowledge of the expansion of a function in one basis gives a priori neither information about its expansion in an other basis nor about its belonging to other anisotropic Besov spaces. Therefore, this tool does not seem to give insight in the understanding of the link between different forms of anisotropy - in term of function spaces by example.

In order to deal with situations where no a priori anisotropy is prescribed, a natural idea would consist in using the union of these bases, for all possible anisotropies. One easily checks that this dictionary is too redundant to constitute a frame. However, because many of their elements coincide (and therefore need not be duplicated), one cannot exclude the possibility that the union of these bases could nonetheless prove an efficient tool for analyzing anisotropic fields with no a priori anisotropy.

- Another possible decomposition system is supplied by hyperbolic wavelets, introduced in various settings under various denominations (standard, rectangular or hyperbolic wavelet analysis) notably in image coding (see [52]), numerical analysis, see [11, 12], and in [21, 31] for the purpose of approximation theory. They are simply defined as tensor products of 1D wavelets, allowing different dilations factors along different directions, as opposed to the classical discrete wavelet transform that relies on a single isotropic dilation factor. This key difference enables to use them as a tool in the study of anisotropy. Hyperbolic wavelet bases form a non-redundant system by construction, and contain all possible anisotropies. They have been used in statistics for the purpose of adaptive estimation of multidimensional curves. Notably, it has been proven in two seminal articles [38, 39] that nonlinear thresholding of noisy hyperbolic wavelet coefficients leads to (near)—optimal minimax rates of convergence over a wide range of anisotropic smoothness classes; see also the recent work of F. Autin, G. Claeskens, J.M. Freyermuth [3] where this problem is considered from the maxiset point of view. Other interesting applications of hyperbolic analysis can also be founded in [4, 5] where the decompositions of Fractional Brownian Sheets and Linear Fractional Stable Sheets are derived and are used to prove sample paths properties of these random fields (regularity, Hausdorff dimension of the graph).

A key feature of hyperbolic wavelet bases is that they provide a common dictionary for anisotropic Besov spaces. This result is stated in Theorem 2.2 of Section 2. The critical exponent in anisotropic Besov spaces will be related to some $\ell^p$ norms of the hyperbolic
wavelet coefficients. These mathematical results yield an efficient method for the detection of anisotropy, as detailed in a companion article, where numerical investigations are conducted, see [43].

We now briefly discuss the pros and cons of both dictionaries. For a fixed anisotropy, Triebel bases display slightly better mathematical properties: An exact characterization of anisotropic Besov spaces, as shown in [50], and a characterization of pointwise smoothness as sharp as in the isotropic case, as shown by H. Ben Braiek and M. Ben Slimane in [7]. A first purpose of the present contribution is to show that these two important properties hold almost as well for hyperbolic wavelets: In Section 2 “almost characterizations” (i.e., necessary and sufficient conditions that differ by a logarithmic correction) of anisotropic Besov spaces are obtained. Furthermore, if one is not only interested in analysis, but also in simulation, this slight disadvantage (a logarithmic loss, which in applications is irrelevant) is overcompensated by the advantage of using a basis instead of an overcomplete system. Indeed, generating a random field with prescribed regularity properties requires the use of a basis (using an overcomplete system cannot guarantee a priori that the simulated field with coefficients of specific sizes has the expected properties, since nontrivial linear combinations of the building blocks may vanish). A contrario, with the hyperbolic wavelet basis, one can easily provide toy examples with different multifractal spectra depending on the anisotropy. Our being jointly motivated by analysis and synthesis motivates the choice of a system that permits an interesting trade-off among directional wavelets, in terms of mathematical efficiency and numerical simplicity and robustness, both on the analysis and synthesis sides. The practical relevance of the mathematical tools introduced and studied here are assessed in a companion paper [43].

We now focus the comparison in terms of pointwise directional smoothness. First, note that this notion has been the subject of few investigations so far: To our knowledge, the natural definition which allows for a wavelet characterization was first introduced by M. Ben Slimane in the 90s, see [8], in order to investigate the multifractal properties of anisotropic selfsimilar functions. Partial results when using parabolic basis (i.e., curvelets and Hart–Smith transform) have been obtained by J. Sampo and S. Sumetkijakan see [35, 40, 41] and references therein. A generalization and implications in terms of sizes of coefficients on directional wavelets (the so-called “anisets”, which are a mixture of of the wavelet and Gabor transform, where the wavelets can be arbitrarily shrunk in certain directions) were also worked out in [34]. Finally, an “almost ” characterization of pointwise directional regularity was recently obtained by H. Ben Braiek and M. Ben Slimane in [7] on the Triebel basis coefficients, where the basis is picked so that its anisotropy parameter is fitted to the type of directional regularity considered. In Section 3 we will obtain a similar result, but relying on the coefficients of the hyperbolic wavelet basis, thus paving the way to the construction of a multifractal formalism. An important difference with [7] is that, here again, a single basis fits all anisotropies. Therefore, as in the case of Besov spaces, the advantage is that no a priori needs to be assumed on the particular considered anisotropy. This thus can be used as a way to detect the specific anisotropy which exists in data at hand, rather than assuming a priori its particular form beforehand. Note that other decomposition systems have also been used for the detection of local singularities, see for instance [22, 28] where shearlets and wedgelets are used for the detection of discontinuities along smooth edges.
Let us now come back to the anisotropic self–similar fields considered in [19, 43]. Such exactly selfsimilar models are somewhat toy examples, and, though testing regularity indices on their realizations is an important validation step, their study could prove misleadingly simple (just as, in 1D, fractional Brownian motion is too simple a model to fit the richness of situations met in real-world data). Natural images are indeed likely to consist of patchworks of different kinds of deformed pieces and therefore, can be expected to exhibit more complex scale invariance properties, and only in an approximate way. A natural setting to describe such properties, where different kinds of singularities are mixed up, is supplied by multifractal analysis. The next step is therefore to combine both anisotropy and multifractality. To this end, a new form of multifractal analysis is introduced, based on the hyperbolic wavelet coefficients, and relating the global and local characterizations of regularity. It allows to take into account both scale invariance properties and local anisotropic features of an image. Thus, it provides a new tool for image classification, seen as a refinement of texture classification based on the usual isotropic multifractal analysis, as proposed for instance in [1, 33]. Section 3 is devoted to the introduction of this new framework: A new multifractal formalism, referred to as the hyperbolic multifractal formalism is introduced. It allows to relate local anisotropic regularity of the analyzed image to global quantities called hyperbolic structure functions as commonly done in multifractal analysis. Note that alternative multifractal analysis and multifractal formalism were introduced by H. Ben Braiek and M. Ben Slimane in [6], based on Triebel basis coefficients. In their approach, a particular anisotropy is picked, and the corresponding basis is used. As above, the main difference between both approaches is that the approach proposed here is more flexible and can thus be used when anisotropies of several types are simultaneously present in data.

In the present article, we explore the possibilities supplied by the hyperbolic wavelet transform in order to investigate directional regularity, both in global (anisotropic Besov spaces) and local (directional pointwise regularity) forms. The underlying motivation is to develop a multifractal formalism relating these two notions (just as the standard multifractal formalism relates the usual Besov spaces with the notion of (anisotropic) Hölder pointwise smoothness, see [33] and references therein). It also aims at obtaining a numerically stable procedure that thus permits to extract the anisotropic features existing in natural images as well as information related to the size (fractional dimensions) of the corresponding geometrical sets.

Our results on the characterization of regularity spaces in terms of hyperbolic wavelets are stated and commented in Section 2. Applications to multifractal analysis are detailed in Section 3. We will show how to recover information on the Hausdorff dimensions of the sets of points where a given directional regularity occurs, from the knowledge of global directional quantities such as the Besov regularity of the data. Such ideas were initially introduced by Parisi and Frisch in the context of the study of hydrodynamic turbulence [41], and extended to the setting of a fixed anisotropy by Ben Braiek and Ben Slimane, see [6]. The novelty over previous works in this direction is that hyperbolic wavelets allow to drop the assumption of a unique, a priori given, anisotropy in the data. The next step, which we intend to investigate in the future, is to relax another simplifying assumption and allow for local rotations in the anisotropy axes. Detailed proofs of all the results stated in Sections 2 and 3 are provided in Section 4.
2 Anisotropic global regularity and hyperbolic wavelets

We first focus on the measure of anisotropic global regularity using a common analyzing dictionary: hyperbolic wavelet bases. Here, we start by providing the reader with a brief account of the corresponding functional spaces. Thereafter, we recall some well-known facts about hyperbolic wavelet analysis (cf. Section 2.2). The main result of the present section consists of Theorem 2.2, proven in Section 4.1, which allows to determine the critical directional Besov indices of data by regressions on log-log plot of quantities based on hyperbolic wavelet coefficients (see Section 2.2 for a precise statement).

2.1 Anisotropic Besov spaces

Anisotropic Besov spaces were introduced in a completely different context: For the study of semi-elliptic pseudo-differential operators whose symbols have different degrees of smoothness along different directions, cf. e.g. [49]; see also [2], and references therein, for a recent use of such spaces for optimal regularity results for the heat equation.

Anisotropic Besov spaces generalize classical (isotropic) Besov spaces, and many results concerning isotropic spaces have been extended in this setting, see [16, 15] for a complete account on the results used in this section, and [51, 53, 54] for a detailed overview on anisotropic spaces.

A key property of anisotropic Besov spaces is that they verify (asymptotically in the limit of small scales) norm invariances with respect to anisotropic scaling; we start by recalling this notion. Let \( \alpha = (\alpha_1, \alpha_2) \) denote a fixed couple of parameters, with \( \alpha_1, \alpha_2 > 0 \) and \( \alpha_1 + \alpha_2 = 2 \). In the remainder, such couples will be referred to as admissible anisotropies. Such couples quantify the degree of anisotropy of the space \( \alpha_1 = \alpha_2 = 1 \) corresponding to the isotropic case). For any \( t \geq 0 \) and \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \), we define anisotropic scaling by \( t^\alpha \xi = (t^{\alpha_1} \xi_1, t^{\alpha_2} \xi_2) \). Note that, in this definition and in the following, the coordinate axes are chosen as anisotropy directions. This particular choice can of course be modified by the introduction of an additional rotation (as envisaged e.g., in [43]).

Anisotropic Besov spaces may be introduced using an anisotropic Littlewood Paley analysis: let \( \varphi_0^\alpha \geq 0 \) belong to the Schwartz class \( \mathcal{S}(\mathbb{R}^2) \) and be such that

\[
\varphi_0^\alpha(x) = 1 \quad \text{if} \quad \sup_{i=1,2} |\xi_i| \leq 1 ,
\]

and \( \varphi_0^\alpha(x) = 0 \quad \text{if} \quad \sup_{i=1,2} |2^{-\alpha_i} \xi_i| \geq 1 .\)

For \( j \in \mathbb{N} \), we define

\[
\varphi_j^\alpha(x) = \varphi_0^\alpha(2^{-j\alpha} \xi) - \varphi_0^\alpha(2^{-(j-1)\alpha} \xi) .
\]

Then

\[\sum_{j=0}^{+\infty} \varphi_j^\alpha \equiv 1 ,\]

and \( (\varphi_j^\alpha)_{j \geq 0} \) is called an anisotropic resolution of the unity. It satisfies

\[\text{supp} (\varphi_0^\alpha) \subset R_1^\alpha, \quad \text{supp} (\varphi_j^\alpha) \subset R_j^\alpha \setminus R_{j-1}^\alpha ,\]

where

\[
R_j^\alpha = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2; \sup_{i=1,2} |\xi_i| \leq 2^{\alpha_i j} \} .
\]
For $f \in \mathcal{S}'(\mathbb{R}^2)$ let
\[
\Delta_j^s f = \mathcal{F}^{-1} \left( \varphi_j^s \hat{f} \right).
\]
The sequence $(\Delta_j^s f)_{j \geq 0}$ is called an anisotropic Littlewood–Paley analysis of $f$. The anisotropic Besov spaces are then defined as follows (see [16, 15]).

**Definition 2.1** The Besov space $B_{p,q,\log}^{s,\alpha}(\mathbb{R}^2)$, for $0 < p \leq +\infty$, $0 < q \leq +\infty$, $s, \beta \in \mathbb{R}$, is defined by
\[
B_{p,q,\log}^{s,\alpha}(\mathbb{R}^2) = \{ f \in \mathcal{S}'(\mathbb{R}^2); \left( \sum_{j \geq 0} j^{-\beta q} 2^{jsq} \| \Delta_j^s f \|_p^q \right)^{1/q} < +\infty \}.
\]

This definition does not depend on the resolution of the chosen unity $\varphi_0^\alpha$ and the quantity
\[
\| f \|_{B_{p,q,\log}^{s,\alpha}} = \left( \sum_{j \geq 0} j^{-\beta q} 2^{jsq} \| \Delta_j^s f \|_p^q \right)^{1/q},
\]
is a norm (resp., quasi-norm) on $B_{p,q}^{s,\alpha}(\mathbb{R}^2)$ for $1 \leq p, q \leq +\infty$ (resp., $0 < \min(p,q) < 1$) and with usual modification if $q = +\infty$.

As in the isotropic case, anisotropic Besov spaces encompass a large class of classical anisotropic functional spaces (see [51] for details). For example, when $p = q = 2$ and $(\alpha_1, \alpha_2) \in \mathbb{Q}^2$ is an admissible anisotropy, let us consider $s > 0$ such that $s/\alpha_1$ and $s/\alpha_2$ are both integers, then the anisotropic Sobolev space
\[
H^{s,\alpha}(\mathbb{R}^2) = \{ f \in L^2(\mathbb{R}^2) \text{ such that } \partial^{s/\alpha_1} f / \partial x_1, \partial^{s/\alpha_2} f / \partial x_2 \in L^2(\mathbb{R}^2) \},
\]
coincides with the Besov space $B_{2,2}^{s,\alpha}(\mathbb{R}^2)$.

In the special case where $p = q = \infty$, the spaces $B_{\infty,\infty}^{s,\alpha}(\mathbb{R}^2)$ are called anisotropic Hölder spaces and are denoted $C_{\log}^{s,\alpha}(\mathbb{R}^2)$. These spaces also admit a finite difference characterization that we now recall (see also [51] for details).

Let $(e_1, e_2)$ denote the canonical basis of $\mathbb{R}^2$. For a function $f : \mathbb{R}^2 \to \mathbb{R}$, $\ell \in \{1,2\}$ and $t \in \mathbb{R}$ one defines
\[
\Delta_{t,\ell}^s f(x) = f(x + te_\ell) - f(x).
\]
The difference of order $M \geq 2$ of a function $f$, along direction $e_\ell$, is iteratively defined as
\[
\Delta_{t,\ell}^M f(x) = \Delta_{t,\ell} \Delta_{t,\ell}^{M-1} f(x).
\]
One then has:

**Proposition 2.1** Let $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{R}_+^*)^2$ such that $\alpha_1 + \alpha_2 = 2$, $s > 0$, $u \in \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$. The function $f$ belongs to the anisotropic Hölder space $C_{\log}^{s,\alpha}(\mathbb{R}^2)$ if
\[
\| f \|_{L^\infty(\mathbb{R}^2)} + \sum_{\ell = 1}^2 \sup_{t > 0} \| \Delta_{t,\ell}^{M_\ell} f(x) \|_{L^\infty(\mathbb{R}^2)} < +\infty,
\]
where for any $\ell \in \{1,2\}$, $M_\ell = \lceil s/\alpha_\ell \rceil + 1$. 
2.2 Hyperbolic wavelet characterization of anisotropic Besov spaces

We state our first main result which consists of an hyperbolic wavelet characterization of anisotropic Besov spaces.

We first recall the definition of the hyperbolic wavelet bases as tensorial products of two unidimensional wavelet bases (see [21]) and second state Theorem 2.2, further proven in Section 4.1.

Definition 2.2 Let \( \psi \) denote the unidimensional Meyer wavelet and \( \varphi \) the associated scaling function. The hyperbolic wavelet basis is defined as the system \( \{ \psi_{j_1,j_2,k_1,k_2}, (j_1,j_2) \in (\mathbb{Z}^+ \cup \{-1\})^2, (k_1,k_2) \in \mathbb{Z}^2 \} \) where

- if \( j_1, j_2 \geq 0 \),
  \[ \psi_{j_1,j_2,k_1,k_2}(x_1,x_2) = \psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2). \]
- if \( j_1 = -1 \) and \( j_2 \geq 0 \)
  \[ \psi_{-1,j_2,k_1,k_2}(x_1,x_2) = \varphi(x_1 - k_1)\psi(2^{j_2}x_2 - k_2). \]
- if \( j_1 \geq 0 \) and \( j_2 = -1 \)
  \[ \psi_{j_1,-1,k_1,k_2}(x_1,x_2) = \psi(2^{j_1}x_1 - k_1)\varphi(x_2 - k_2). \]
- if \( j_1 = j_2 = -1 \)
  \[ \psi_{-1,-1,k_1,k_2}(x_1,x_2) = \varphi(x_1 - k_1)\varphi(x_2 - k_2). \]

For any \( f \in S'(\mathbb{R}^2) \), one then defines its hyperbolic wavelet coefficients as follows:

\[
\begin{align*}
c_{j_1,j_2,k_1,k_2} &= 2^{j_1+j_2} \langle f, \psi_{j_1,j_2,k_1,k_2} \rangle \quad \text{if } j_1, j_2 \geq 0, \\
c_{j_1,-1,k_1,k_2} &= 2^{j_1} \langle f, \psi_{j_1,j_2,k_1,k_2} \rangle \quad \text{if } j_1 \geq 0 \text{ and } j_2 = -1, \\
c_{-1,j_2,k_1,k_2} &= 2^{j_2} \langle f, \psi_{j_1,j_2,k_1,k_2} \rangle \quad \text{if } j_1 = -1 \text{ and } j_2 \geq 0, \\
c_{-1,-1,k_1,k_2} &= \langle f, \psi_{j_1,j_2,k_1,k_2} \rangle \quad \text{if } j_1 = j_2 = -1.
\end{align*}
\]

Remark 2.1 We chose a \( L^1 \)-normalization for the wavelet coefficients, which is best suited to scale invariance.

The main result of this section is an hyperbolic wavelet characterization of the spaces \( B_{p,q}^{s,\alpha} \log |\beta| (\mathbb{R}^2) \), up to a logarithmic correction. In the sequel, some notations are needed. For any \( (j_1,j_2) \in (\mathbb{N} \cup \{-1\})^2 \), let us define:

\[
\| c_{j_1,j_2,\ldots} \|_{\ell^p} = \left( \sum_{(k_1,k_2) \in \mathbb{Z}^2} |c_{j_1,j_2,k_1,k_2}|^p \right)^{1/p}.
\]

Let \( \alpha = (\alpha_1, \alpha_2) \) be an admissible anisotropy, we define the following subsets of \( (\mathbb{N} \cup \{-1\})^2 \):

\[
\begin{align*}
\Gamma_j^{(HL)}(\alpha) &= \{(j_1,j_2) \in \mathbb{N}^2, \lfloor (j-1)\alpha_1 \rfloor - 1 \leq j_1 \leq \lfloor j\alpha_1 \rfloor + 1 \text{ and } 0 \leq j_2 \leq \lfloor (j-1)\alpha_2 \rfloor - 1 \}, \\
\Gamma_j^{(LH)}(\alpha) &= \{(j_1,j_2) \in \mathbb{N}^2, 0 \leq j_1 \leq \lfloor (j-1)\alpha_1 \rfloor - 1 \text{ and } \lfloor (j-1)\alpha_2 \rfloor - 1 \leq j_2 \leq \lfloor j\alpha_2 \rfloor + 1 \}, \\
\Gamma_j^{(HH)}(\alpha) &= \{(j_1,j_2) \in \mathbb{N}^2, \lfloor (j-1)\alpha_1 \rfloor - 1 \leq j_1 \leq \lfloor j\alpha_1 \rfloor + 1 \text{ and } \lfloor (j-1)\alpha_2 \rfloor - 1 \leq j_2 \leq \lfloor j\alpha_2 \rfloor + 1 \}.
\end{align*}
\]
\[ \Gamma_j(\alpha) = \Gamma_j^{(HL)}(\alpha) \cup \Gamma_j^{(LH)}(\alpha) \cup \Gamma_j^{(HH)}(\alpha). \]

Let us now state our hyperbolic wavelet characterization of anisotropic Besov spaces:

**Theorem 2.2** Let \( \alpha = (\alpha_1, \alpha_2) \) be an admissible anisotropy, \( (s, \beta) \in \mathbb{R}^2 \) and \( (p, q) \in (0, +\infty)^2 \). Let \( f \in \mathcal{S}'(\mathbb{R}^2) \). If \( p \geq 1 \) let be \( p^* = \max(p, p') \) and \( p_* = \min(p, p') \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \). Set
\[
 r_1 = \begin{cases} 
 q\left(\frac{1}{p} - 1\right) + \max(q - 1, 0) & \text{if } p \leq 1 \\
 \max\left(\frac{q}{p_*} - 1, 0\right) & \text{if } p > 1 
\end{cases}
\]
and
\[
 r_2 = \begin{cases} 
 1 & \text{if } p < 1 \\
 \max(1 - q/p^*, 0) & \text{if } p \geq 1.
\end{cases}
\]

1. Set \( \beta(p, q) = \max(1/p - 1, 0) + \max(1 - 1/q, 0) \). If
\[
\left( \sum_{j \in \mathbb{N}_0} j^{-r_1-\beta q_2 j^2} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} 2^{-\frac{(j_1 + j_2)q}{p}} \|c_{j_1, j_2, \cdot, \cdot}\|^q_{\ell^p} \right)^{1/q} < +\infty,
\]
then \( f \in B^{s, \alpha}_{p, q, \log |\beta|}(\mathbb{R}^2) \) (with usual modifications when \( q = \infty \)).

2. Conversely,
   (a) If \( q < \infty \) and \( f \in B^{s, \alpha}_{p, q, \log |\beta|}(\mathbb{R}^2) \) then
   \[
   \left( \sum_{j \in \mathbb{N}_0} j^{-r_2-\beta q_2 j^2} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} 2^{-\frac{(j_1 + j_2)q}{p}} \|c_{j_1, j_2, \cdot, \cdot}\|^q_{\ell^p} \right)^{1/q} < +\infty.
   \]
   (b) If \( f \in B^{s, \alpha}_{p, \infty, \log |\beta|}(\mathbb{R}^2) \) then
   \[
   \max_{j \in \mathbb{N}_0} j^{-\beta q j^2} \max_{(j_1, j_2) \in \Gamma_j(\alpha)} 2^{-\frac{(j_1 + j_2)q}{p}} \|c_{j_1, j_2, \cdot, \cdot}\|_{\ell^p} < +\infty.
   \]

Remark that in the special case where \( p = q = 2 \), that is for anisotropic Sobolev spaces, there logarithmic correction desappears.

**Theorem 2.3** Let \( \alpha = (\alpha_1, \alpha_2) \) an admissible anisotropy, \( s \in \mathbb{R} \). Let \( f \in \mathcal{S}'(\mathbb{R}^2) \). The two following assertions are equivalent:

(i) \( f \in H^{s, \alpha}(\mathbb{R}^2) = B_{2,2}^{s, \alpha}(\mathbb{R}^2) \).

(ii)
   \[
   \left( \sum_{j \in \mathbb{N}_0} 2^{2js} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} 2^{-(j_1 + j_2)} \|c_{j_1, j_2, \cdot, \cdot}\|^2_{\ell^2} \right)^{1/2} < +\infty.
   \]
Proof. Theorem 2.2 is proven in Section 4.1.

In particular, for \( p = q = \infty \), the following “almost” characterization of anisotropic Hölder spaces by means of hyperbolic wavelets holds:

**Proposition 2.4** Let \( \alpha = (\alpha_1, \alpha_2) \) an admissible anisotropy, \( (s, \beta) \in \mathbb{R}^2 \) and \( f \in S'(\mathbb{R}^2) \).

(i) If \( f \in C^{s,\alpha}(\mathbb{R}^2) \) then there exists some \( C > 0 \) such that for all \( j \in \mathbb{N} \cup \{-1\} \) and any \( (j_1, j_2) \in \Gamma_j(\alpha) \),

\[
\|c_{j_1,j_2,\cdot,\cdot}\|_{\ell^\infty} \leq C 2^{-js}.
\]

(ii) Conversely, assume that there exists some \( C > 0 \) such that for all \( j \in \mathbb{N} \cup \{-1\} \) and any \( (j_1, j_2) \in \Gamma_j(\alpha) \)

\[
\|c_{j_1,j_2,\cdot,\cdot}\|_{\ell^\infty} \leq C \frac{2^{-js}}{j},
\]

then \( f \in C^{s,\alpha}(\mathbb{R}^2) \).

3 Hyperbolic multifractal analysis

We are now interested in the simultaneous analysis of local regularity properties and of anisotropic features of a function. To that end, we construct a new multifractal analysis, referred to as the hyperbolic multifractal analysis. Recall that in the isotropic case, the purpose of multifractal analysis is to provide information on pointwise singularities of functions. Multifractal functions are usually such that their local regularity erratically jumps from point to point, so that it is not possible to estimate the regularity index (defined below) of a function at a given point. Instead, the relevant information consists of the “sizes” of the sets of points where the regularity index takes the same value. This “size” is mathematically formalized as the Hausdorff dimension. The function that associates the dimension of the set of points sharing the same regularity index with this index is referred to as the spectrum of singularities (or multifractal spectrum). The goal of a multifractal formalism is to provide a method that allows to measure the spectrum of singularities from quantities that can actually be computed on real-world data. We extend this approach to the anisotropic setting. Let us first recall that, in the case where the anisotropy of the analyzing space is fixed, this has already been achieved: See [7] for anisotropic pointwise regularity analysis using Triebel bases and [6] for the corresponding anisotropic multifractal formalism. Here, we generalize these two previous works, providing a multifractal analysis which does not rely on the a priori knowledge of the regularity and takes into account all possible anisotropies. Note that for a fixed anisotropy, both formalisms coincide: Indeed they are derived from wavelet characterizations of the same functional spaces. Nevertheless, the formalism based on hyperbolic wavelet allows to deal simultaneously with all possible anisotropies, thus providing more useful algorithms for analyzing real-world data. In addition, the use of hyperbolic wavelet bases offers the possibility to define and synthesize deterministic and stochastic mathematical objects with prescribed anisotropic behavior.

In Section 3.1 the different concepts related to pointwise regularity are first recalled. An hyperbolic wavelet criterion is then devised in Section 3.1.2. Our main result, Theorem 3.2, is stated in Section 3.1.2 and proven in Section 4. Hyperbolic wavelet analysis is defined in Section 3.2.2 and the validity of the proposed multifractal formalism is investigated in Theorem 3.3.
3.1 Anisotropic pointwise regularity and hyperbolic wavelet analysis

3.1.1 Definitions

Let us start by recalling the usual notion of pointwise regularity (cf. [33] for a complete review).

Definition 3.1 Let $f$ be in $L^\infty_{\text{loc}}(\mathbb{R}^2)$ and $s > 0$. The function $f$ belongs to the space $C^s_{|\log|^\beta}(x_0)$ if there exist some $C > 0$, $\delta > 0$ and $P_{x_0}$ a polynomial with degree less than $s$ such that

$$
|f(x) - P_{x_0}(x)| \leq C|x - x_0|^s \cdot |\log(|x - x_0|)|^\beta,
$$

where $|\cdot|$ is the usual Euclidean norm on $\mathbb{R}^2$. If $\beta = 0$, the space $C^s_{|\log}|(x_0)$ is simply denoted $C^s(x_0)$.

Anisotropic pointwise regularity is further defined as follows. Let $P$ denote a polynomial of the form:

$$P(t_1, t_2) = \sum_{(\beta_1, \beta_2) \in \mathbb{N}^2} a_{\beta_1, \beta_2} t_1^{\beta_1} t_2^{\beta_2},$$

and let $\alpha = (\alpha_1, \alpha_2)$ be an admissible anisotropy. The $\alpha$–homogeneous degree of the polynomial $P$ is defined as:

$$d_\alpha(P) = \sup\{\alpha_1 \beta_1 + \alpha_2 \beta_2, a_{\beta_1, \beta_2} \neq 0\};$$

Finally, for any $(t_1, t_2) \in \mathbb{R}^2$, the $\alpha$–homogeneous norm reads:

$$|t|_\alpha = |t_1|^{1/\alpha_1} + |t_2|^{1/\alpha_2}.$$

We can now define the spaces $C^{s,\alpha}_{|\log|^\beta}(x_0)$.

Definition 3.2 Let $f \in L^\infty_{\text{loc}}(\mathbb{R}^2)$, $\alpha = (\alpha_1, \alpha_2)$ be an admissible anisotropy, $|\cdot|_\alpha$, $s > 0$ and $\beta \in \mathbb{R}$. The function $f$ belongs to $C^{s,\alpha}_{|\log|^\beta}(x_0)$ if there exists some $C > 0$, $\delta > 0$ and $P_{x_0}$ a polynomial with $\alpha$–homogeneous degree less than $s$ such that

$$|f(x) - P_{x_0}(x)| \leq C|x - x_0|^s \cdot |\log(|x - x_0|)|^\beta.$$  

If $\beta = 0$, the space $C^{s,\alpha}_{|\log}|(x_0)$ is simply denoted $C^{s,\alpha}(x_0)$.

The anisotropic pointwise exponent of a locally bounded function $f$ at $x_0$ can be thus be defined as:

$$h_{f,\alpha}(x_0) = \sup\{s, f \in C^{s,\alpha}(x_0)\}.$$  

The link between global and pointwise anisotropic regularity is given by:

Proposition 3.1  

$\bullet$ If $f \in B^{s,\alpha}_{\infty,\infty}(\mathbb{R}^2) = C^{s,\alpha}(\mathbb{R}^2)$ then $f$ belongs to $C^{s,\alpha}(x_0)$ for all $x_0 \in \mathbb{R}^2$.

$\bullet$ Conversely, if $f$ belongs to $C^{s,\alpha}(x_0)$ for all $x_0 \in \mathbb{R}^2$ with a constant $C$ in (5) independant of $x_0$ then $f \in B^{s,\alpha}_{\infty,\infty}(\mathbb{R}^2)$.

This proposition is a direct consequence of the proof of Proposition 2 of [7] which gives the wavelet characterization of $C^{s,\alpha}(x_0)$ using the Triebel wavelet basis, where it is easy to check that constants appearing in the wavelet characterizations are independant of $x_0$ as soon it is the case for $C$ in (5).

The reader is refered to [7],[34] for more details about the material of this section.
3.1.2 An hyperbolic wavelet criterion

As in the usual isotropic setting (see [33]), the anisotropic pointwise Hölder regularity of a function is closely related to the rate of decay of its wavelet leaders. The usual definition of wavelet leaders needs to be tuned to the hyperbolic setting:

For any \((j_1, j_2, k_1, k_2)\), let \(\lambda(j_1, j_2, k_1, k_2) = \left[\frac{k_1}{2^{j_1}}, \frac{k_1 + 1}{2^{j_1}}\right] \times \left[\frac{k_2}{2^{j_2}}, \frac{k_2 + 1}{2^{j_2}}\right]\), and let \(c_{\lambda}\) stand for \(c_{j_1, j_2, k_1, k_2}\). The hyperbolic wavelet leaders \(d_{\lambda}\), associated with \(\lambda\), can now be defined as:

\[
3_{j_1, j_2}(x_0) = \left[\frac{[2^{j_1}a] - 1}{2^{j_1}}, \frac{[2^{j_1}a] + 2}{2^{j_1}}\right] \times \left[\frac{[2^{j_2}b] - 1}{2^{j_2}}, \frac{[2^{j_2}b] + 2}{2^{j_2}}\right],
\]

(where \([\cdot]\) denotes the integer part) and

\[
d_{j_1, j_2}(x_0) = \sup_{\lambda' \subset 3_{j_1, j_2}(x_0)} |c_{\lambda'}|.
\]

The hyperbolic wavelet leaders criterion for pointwise regularity can now be stated as:

**Theorem 3.2** Let \(s > 0\) and \(\alpha = (\alpha_1, \alpha_2) \in (\mathbb{R}_+^*)^2\) such that \(\alpha_1 + \alpha_2 = 2\).

1. Assume that \(f \in C^{s, \alpha}(x_0)\). There exists some \(C > 0\) such that for any \(j_1, j_2 \in \mathbb{N} \cup \{-1\}\) one has

\[
|d_{j_1, j_2}(x_0)| \leq C2^{-\max(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2})s}.
\]

2. Conversely, assume that \(f\) is uniformly Hölder, that is there exists some \(\varepsilon_0^s > 0\) such that \(f \in C^{s, \alpha}_{\log^2}(\mathbb{R}^2)\). If (8) holds, then \(f \in C^{s, \alpha}(x_0)\).

Proofs are postponed to Section 4.

3.2 Anisotropic multifractal analysis

3.2.1 Hausdorff dimension and spectrum of singularities

The Hausdorff dimension is defined through the Hausdorff measure (see [23] for details). The best covering of a set \(E \subset \mathbb{R}^d\) with sets subordinated to a diameter \(\varepsilon\) can be estimated as follows,

\[
\mathcal{H}^\delta_\varepsilon(E) = \inf \left\{ \sum_{i=1}^{\infty} |E_i|^\delta : E \subset \bigcup_{i=1}^{\infty} E_i, \sum_{i=1}^{\infty} |E_i| \leq \varepsilon \right\}.
\]

Clearly, \(\mathcal{H}^\delta_\varepsilon\) is an outer measure. The Hausdorff measure is defined as the (possibly infinite or vanishing) limit \(\mathcal{H}^\delta_\varepsilon\) as \(\varepsilon\) goes to 0.

The Hausdorff measure is decreasing as \(\delta\) goes to infinity. Moreover, \(\mathcal{H}^\delta(E) > 0\) implies \(\mathcal{H}^{\delta'}(E) = \infty\) if \(\delta' < \delta\). The following definition is thus meaningful.
Definition 3.3  The Hausdorff dimension $\dim_H(E)$ of a set $E \subset \mathbb{R}^d$ is defined as follows,

$$\dim_H(E) = \sup \{ \delta : H^\delta(E) = \infty \} .$$

With this definition, $\dim_H(\emptyset) = -\infty$.

We now define the hyperbolic spectrum of singularities of a locally bounded function using the Hausdorff dimension.

Definition 3.4  Let $f$ be a locally bounded function and $\alpha$ be an admissible anisotropy. The iso–anisotropic–Hölder set are defined as

$$E_f(H, \alpha) = \{ x \in \mathbb{R}^2, h_{f,\alpha}(x) = H \}$$

where the anisotropic pointwise Hölder $h_{f,\alpha}(x)$ has been defined in (6).

The hyperbolic spectrum of singularities of $f$ is then defined as:

$$d : (\mathbb{R}^+ \cup \{ \infty \}) \times (0, 2) \rightarrow \mathbb{R}^+ \cup \{-\infty\} \quad (H, a) \mapsto \dim_H(E_f(H, (a, 2-a))) .$$

3.2.2 Hyperbolic wavelet leader multifractal formalism

It is not always possible to compute theoretically the spectrum of singularities of a given function. A multifractal formalism thus consists of a practical procedure that yields the convex hull of the function $d$, through the construction of structure functions and the use of the Legendre transform. In the classical case, these formalisms are variants of a seminal derivation, proposed by Parisi and Frisch [41]. The hyperbolic wavelet leader multifractal formalism described below aimed at extending the procedure to where both anisotropy and singularities are studied jointly.

Hyperbolic partition functions of a locally bounded function are defined as follows.

$$S(j, p, \alpha) = 2^{-2j} \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \sum_{(k_1,k_2) \in \mathbb{Z}^2} d_{j_1,j_2,k_1,k_2}^p,$$

where $\Gamma_j(\alpha)$ has already been defined in Section 2.2 with Eq. (2).

From the definition of an anisotropic scaling function (or scaling exponents)

$$\omega_f(p, \alpha) = \lim inf_{j \rightarrow \infty} \frac{\log S(j, p, \alpha)}{\log 2^{-j}},$$

let us further define the Legendre hyperbolic spectrum:

$$L_f(H, \alpha) = \inf_{p \in \mathbb{R}^*} \{ Hp - \omega_f(p, (\alpha, 2-\alpha)) + 2 \} .$$

Qualitatively, the Legendre hyperbolic spectrum and the hyperbolic spectrum of singularities $d_f(H, a)$ are expected to coincide, while the theorem below actually provides an upper bound relationship.

Theorem 3.3  Let $f$ be a uniform Hölder function. Then the following inequality holds

$$\forall (H, a) \in (\mathbb{R}^+_+) \times (0, 2), \quad d_f(H, a) \leq L_f(H, a).$$

Note that, if equality holds in (12), i.e., if

$$\forall (H, a) \in (\mathbb{R}^+_+) \times (0, 2), \quad d(H, a) = L_f(H, a),$$

then $f$ is said to satisfy the hyperbolic multifractal formalism.

From an applied perspective, (9) , (10) and (11) constitute the core of the practical procedure enabling to compute the Legendre hyperbolic spectrum from the hyperbolic wavelet leaders computed on the data to be analyzed.
4 Proofs

4.1 Proof of Theorem 2.2

4.1.1 Hyperbolic Littlewood-Paley characterization of $B_{p,q}^{s,\alpha}(\mathbb{R}^2)$

Let $\theta_0 \in \mathcal{S}(\mathbb{R}, \mathbb{R}^+)$ be supported on $[-2, 2]$ such that $\theta_0 = 1$ on $[-1, 1]$. For any $j \in \mathbb{N}$, let us define

$$\theta_j = \theta_0(2^{-j} \cdot) - \theta_0(2^{-(j-1)} \cdot).$$

such that $\sum_{j \geq 0} \theta_j(\cdot) = 1$ is a 1-D resolution of the unity.

Observe that, for any $j \geq 1$, $\text{supp} (\theta_j) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$.

Remark 4.1 In the following, the function $\theta_0$ can be chosen with an arbitrary compact support. It does not change the main results even if technical details of proofs and lemmas have to be adapted. It allows to choose the Fourier transform of a Meyer scaling function for $\theta_0$.

Definition 4.1 1. For any $j, \ell \geq 0$, and any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ set

$$\phi_{j_1,j_2}(\xi_1, \xi_2) = \theta_{j_1}(\xi_1) \theta_{j_2}(\xi_2).$$

For any $j_1, j_2 \geq 0$, the function $\phi_{j_1,j_2}$ belongs to $\mathcal{S}(\mathbb{R}^2)$ and is compactly supported on $\{2^{j_1} \leq |\xi_1| \leq 2^{j_1+1}\} \times \{2^{j_2} \leq |\xi_2| \leq 2^{j_2+1}\}$. Further $\sum_{j_1 \geq 0} \sum_{j_2 \geq 0} \phi_{j_1,j_2} = 1$ and $(\phi_{j_1,j_2})_{(j_1,j_2) \in \mathbb{N}^2}$ is called an hyperbolic resolution of the unity.

2. For $f \in \mathcal{S}'(\mathbb{R}^2)$ and $j_1, j_2 \geq 0$ set

$$\Delta_{j_1,j_2} f = \mathcal{F}^{-1} \left( \hat{\phi}_{j_1,j_2} \right).$$

The sequence $(\Delta_{j_1,j_2} f)_{j_1,j_2 \geq 0}$ is called an hyperbolic Littlewood-Paley analysis of $f$.

In the remainder of the section, we are given $\alpha = (\alpha_1, \alpha_2)$ a fixed admissible anisotropy and $(\varphi_{j_1,j_2}^\alpha)_{j_1,j_2 \geq 0}$ an anisotropic resolution of the unity. One then defines the following functions for any $j \geq 0$,

$$g_j^\alpha = \sum_{j_1,j_2 \in \Gamma_j(\alpha)} \phi_{j_1,j_2},$$

where the sets $\Gamma_j(\alpha)$ have been defined in (2).

Remark 4.2 Hyperbolic Littlewood-Paley analysis is used in the definition of spaces of mixed smoothness. We refer to [43] for a study of these spaces and to [44] for their link with tensor products of Besov spaces and their hyperbolic wavelet characterizations.

We now provide the reader with the following hyperbolic Littlewood-Payley characterization of anisotropic Besov spaces:

Theorem 4.1 Let $s \in \mathbb{R}$ and $(p, q) \in (0, +\infty]^2$. If $p \geq 1$ let be $p^* = \max(p, p')$ and $p_* = \min(p, p')$ with $\frac{1}{p} + \frac{1}{p'} = 1$. 

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1. (a) If $q < \infty$ and
\[
\left(\sum_{j \geq 0} j^{r_1} \cdot j^{-\beta q} 2^{jsq} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \|\Delta_{j_1,j_2}(f)\|_p^q\right)^{1/q} < +\infty,
\]
with $r_1 = \begin{cases} q \left(\frac{1}{p} - 1\right) + \max(q - 1, 0) & \text{if } p \leq 1 \\ \max\left(q \frac{p}{p^*} - 1, 0\right) & \text{if } p > 1 \end{cases}$
then $f \in B^{s,\alpha}_{p,q,|\log|}(\mathbb{R}^2)$.

(b) If
\[
\max_{j \geq 0} \left(j^{\max(1/p-1,0)+1} \cdot j^{-\beta 2^js} \max_{(j_1,j_2) \in \Gamma_j(\alpha)} \|\Delta_{j_1,j_2}(f)\|_p\right) < +\infty,
\]
then $f \in B^{s,\alpha}_{p,\infty,|\log|}(\mathbb{R}^2)$.

2. (a) If $q < \infty$ and $f \in B^{s,\alpha}_{p,q,|\log|}(\mathbb{R}^2)$ then
\[
\left(\sum_{j \geq 0} j^{-r_2} \cdot j^{-\beta q} 2^{jsq} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \|\Delta_{j_1,j_2}(f)\|_p^q\right)^{1/q} < +\infty,
\]
with $r_2 = \begin{cases} 1, & \text{if } p < 1 \\ \max(1 - q/p^*, 0) & \text{if } p \geq 1. \end{cases}$

(b) If $f \in B^{s,\alpha}_{p,\infty,|\log|}(\mathbb{R}^2)$ then
\[
\max_{j \geq 0} \left(j^{-\beta 2^js} \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \|\Delta_{j_1,j_2}(f)\|_p\right) < +\infty. \]

Note in particular that for $p = q = 2$ the logarithmic correction disappears and Theorem 4.1 gives an exact characterization of anisotropic Sobolev spaces in terms of hyperbolic Littlewood–Paley analysis.

The proof of Theorem 4.1 consists of several steps, beginning with

Lemma 4.1

1. For any $j \geq 0$ and any $(j_1, j_2) \not\in \Gamma_j(\alpha)$, one has
\[
\text{supp}(\varphi^0_j) \cap \text{supp}(\phi_{j_1, j_2}) = \emptyset.
\]

2. For any $j \geq 0$ and any $\ell \not\in \{j-1, j, j+1\}$, one has
\[
\text{supp}(g^0_\ell) \cap \text{supp}(\varphi^0_j) = \emptyset.
\]

Proof. It can easily be checked since the support of both functions are known (cf. Eq.1 and Definition 4.1).

\[\square\]

From Lemma 4.1 an intermediate hyperbolic Littlewood–Paley characterization of anisotropic Besov spaces is obtained.
Proposition 4.2 Let \((p, q) \in (0, +\infty]^2\), \(s, \beta \in \mathbb{R}\). the two following assertions are equivalent:

1. \(f \in B_{p,q,|\log|^\beta}(\mathbb{R}^2)\).

2. 

\[
(18) \quad \left( \sum_{j \geq 0} j^{-\beta q} 2^{jsq} \left\| \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} [\Delta_{j_1,j_2}(f)] \right\|_p^q \right)^{1/q} < +\infty .
\]

Proof of Proposition 4.2.

Let us first show that Inequality (18) of Proposition 4.2 implies \(f \in B_{p,q,|\log|^\beta}(\mathbb{R}^2)\). To this end, Point 1 of Lemma 4.1 is used to deduce that for any \(j\)

\[
\phi^\alpha_j \hat{f} = \varphi^\alpha_j \left( \sum_{(j_1,j_2) \in \mathbb{N}^2} \phi_{j_1,j_2} \right) \hat{f} = \varphi^\alpha_j \left( \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \phi_{j_1,j_2} \right) \hat{f} = \varphi^\alpha_j \left( g_j^\alpha \hat{f} \right),
\]

where \(g_j^\alpha\) is defined by Equation (13). Observe now that replacing the usual dilation with an anisotropic one gives an anisotropic version of Equation (13) in Section 1.5.2 in [49]. More precisely assume that we are given \(p \in (0, +\infty], b > 0\) and \(M \in \mathcal{S}(\mathbb{R}^2)\). There exists some \(C > 0\) not depending on \(b\) nor \(M\) such that for any \(h \in L^p(\mathbb{R}^2)\) such that \(\text{supp}(\hat{h}) \subset \{ \xi \in \mathbb{R}^2, \sup_i |\xi_i| \leq b^\alpha \}\), one has

\[
(19) \quad \| \mathcal{F}^{-1}(M \mathcal{F} h) \|_{L^p(\mathbb{R}^2)} \leq C \| M(b^\alpha \cdot) \|_{H_2^s(\mathbb{R}^2)} \| h \|_{L^p(\mathbb{R}^2)}
\]

where \(H_2^s\) is the usual Bessel potential space and \(s > 2/(1/(p, 1) - 1/2)\).

Set now \(b = 2^j\), \(M = \varphi_j^\alpha\) and \(\hat{h} = g_j^\alpha \hat{f}\). Since \(\varphi_j^\alpha(2^j \cdot) = \varphi_1^\alpha\), there exists some \(C > 0\) not depending on \(j\) such that for any \(p \in (0, +\infty]\) and any \(f \in L^p(\mathbb{R}^2)\)

\[
\| \Delta_j^\alpha f \|_{L^p} \leq C \left\| \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \Delta_{j_1,j_2} f \right\|_{L^p} = C \| (\mathcal{F}^{-1} g_j^\alpha * f) \|_{L^p}.
\]

Then

\[
(20) \quad \| f \|_{B_{p,q,|\log|^\beta}} = \left( \sum_{j \geq 0} \| j^{-\beta q} 2^{jsq} \| \Delta_j^\alpha f \|_{L^p}^q \right)^{1/q} \leq C \left( \sum_{j \geq 0} \| j^{-\beta q} 2^{jsq} \| (\mathcal{F}^{-1} g_j^\alpha * f) \|_{L^p}^q \right)^{1/q},
\]

which gives Point 2 of Proposition 4.2.

Let us now prove the converse assertion. Assume that \(f\) belongs to \(B_{p,q,|\log|^\beta}(\mathbb{R}^2)\). Point 2 of Lemma 4.1 gives for any \(j \geq 0\)

\[
g_j^\alpha \hat{f} = g_j^\alpha (\varphi_j^{\alpha-1} + \varphi_j^\alpha + \varphi_j^{\alpha+1}) \hat{f}.
\]

Hence, Inequality (19) applied with \(b = 2^j\), \(M = g_j^\alpha\) and \(\hat{h} = (\varphi_j^{\alpha-1} + \varphi_j^\alpha + \varphi_j^{\alpha+1}) \hat{f}\) gives the existence of some \(C > 0\) not depending on \(j\) nor \(f\) such that for any \(p \in (0, +\infty]\)

\[
\| (\mathcal{F}^{-1} g_j^\alpha * f) \|_{L^p} \leq c \| g_j^\alpha(2^j \cdot) \|_{H_2^s} \| (\mathcal{F}^{-1} \varphi_j^{\alpha-1} + \mathcal{F}^{-1} \varphi_j^\alpha + \mathcal{F}^{-1} \varphi_j^{\alpha+1}) * f \|_{L^p}
\]

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Since $\| \cdot \|_{L^p}$ is either a norm or a quasi–norm (according to the value of $p$), there exists some $C > 0$ such that
\[
\| (F^{-1} g_j^0) * f \|_{L^p} \leq C \| g_j^0 (2^{j\alpha} \cdot) \|_{H^s}
\]
\[
= \| (F^{-1} \varphi_j^{a} \cdot) * f \|_{L^p} + \| (F^{-1} \varphi_j^{a} \cdot) * f \|_{L^p} + \| (F^{-1} \varphi_j^{a} \cdot) * f \|_{L^p} .
\]
Let us first bound $\| g_j^0 (2^{j\alpha} \cdot) \|_{H^s}$. To this end, observe that
\[
F[g_j^0 (2^{j\alpha} \cdot)](\xi) = 2^{-j(\alpha_1 + \alpha_2)} \hat{g}_j (2^{-j\alpha} \xi) = 2^{-2j} \hat{g}_j (2^{-j\alpha} \xi)
\]
\[
= \sum_{(j_1, j_2) \in \Gamma_j (\alpha)} 2^{j_1 + j_2 - 2j} \hat{\theta}_1 (2^{j_1 - j\alpha} \xi_1) \hat{\theta}_1 (2^{j_2 - j\alpha} \xi_2) .
\]
Hence
\[
\| g_j^0 (2^{j\alpha} \cdot) \|_{H^s}^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^s \left[ \sum_{(j_1, j_2) \in \Gamma_j (\alpha)} 2^{j_1 + j_2 - 2j} \hat{\theta}_1 (2^{j_1 - j\alpha} \xi_1) \hat{\theta}_1 (2^{j_2 - j\alpha} \xi_2) \right]^2 d\xi
\]
\[
\leq \int_{\mathbb{R}^2} (1 + |\xi|^2)^s \left[ \sum_{(j_1, j_2) \in \Gamma_j (\alpha)} 2^{j_1 + j_2 - 2j} \hat{\theta}_1 (2^{j_1 - j\alpha} \xi_1) \hat{\theta}_1 (2^{j_2 - j\alpha} \xi_2) \right]^2 d\xi .
\]
Since $\hat{\theta}_1 \in \mathcal{S}(\mathbb{R})$, for any $M > 1$ there exists some $C > 0$ such that
\[
|\hat{\theta}_1 (\xi)| \leq \frac{C_M}{(1 + |\xi|)^{2M}} .
\]
Finally
\[
\| g_j^0 (2^{j\alpha} \cdot) \|_{H^s}^2 \leq C_M \int_{\mathbb{R}^2} (1 + |\xi|^2)^s \left[ \sum_{(j_1, j_2) \in \Gamma_j (\alpha)} \frac{2^{j_1 + j_2 - 2j}}{(1 + |2^{j_1 - j\alpha} \xi_1|)^{2M} \cdot (1 + |2^{j_2 - j\alpha} \xi_2|)^{2M}} \right]^2 d\xi
\]
\[
\leq C_M \int_{\mathbb{R}^2} (1 + |\xi|^2)^s \left[ \sum_{(j_1, j_2) \in \Gamma_j (\alpha)} \frac{1}{(2^{j\alpha_1 - j_1} + |\xi_1|)^{2M} \cdot (2^{j\alpha_2 - j_2} + |\xi_2|)^{2M}} \right]^2 d\xi .
\]
By the inequality
\[
(a + b)^2 \geq a \max (b, 1) ,
\]
valid for any $a > 1$, $b > 0$ and applied successively with $a = 2^{j\alpha_1 - j_1}$ and $b = |\xi_1|$, $a = 2^{j\alpha_2 - j_2}$ and $b = |\xi_2|$, it comes
\[
\| g_j^0 (2^{j\alpha} \cdot) \|_{H^s}^2 \leq C_M \int_{\mathbb{R}^2} (1 + |\xi|^2)^s \left[ \sum_{(j_1, j_2) \in \Gamma_j (\alpha)} 2^{j_1 + j_2 - j\alpha M} \cdot 2^{j_2 - j\alpha M} \max (1, |\xi_1|)^{2M} \max (1, |\xi_2|)^{2M} \right]^2 d\xi .
\]
With a $M$ sufficiently large it follows that
\[
\sup_j \left( \| g_j^\alpha (2^{j\alpha} \cdot) \|_{H^2} \right) < +\infty .
\]

Going back to an upper bound of $\| (F^{-1} g_j^\alpha) \ast f \|_{L^p}$, there exists some $C > 0$ such that
\[
\| (F^{-1} g_j^\alpha) \ast f \|_{L^p} \leq C_j \left( \| (F^{-1} \varphi_{j-1}) \ast f \|_{L^p} + \| (F^{-1} \varphi_{j}^\alpha) \ast f \|_{L^p} + \| (F^{-1} \varphi_{j+1}^\alpha) \ast f \|_{L^p} \right)
\]
and
\[
\left( \sum_{j \geq 0} j^{-\beta q} 2^{jsq} \| (F^{-1} g_j^\alpha) \ast f \|_{L^p}^q \right)^{1/q} \leq C \| f \|_{B^{s,\alpha}_{p,q,|\log|^\beta}} = \left( \sum_{j \geq 0} j^{-\beta q} 2^{jsq} \| \Delta_j^s f \|_{L^p}^q \right)^{1/q},
\]
the last shows that if (18) holds then $f \in B^{s,\alpha}_{p,q,|\log|^\beta}(\mathbb{R}^2)$.

Note that the proof can be shortened in the case $p \geq 1$ using Young’s inequality
\[
\| F^{-1} (MFh) \|_{L^p(\mathbb{R}^2)} \leq C \| F^{-1} M \|_{L^1(\mathbb{R}^2)} \| h \|_{L^p(\mathbb{R}^2)}
\]
instead of Inequality (19).

**Proof of Theorem 4.1.** Let us first recall that:

- For any $q \in (0, +\infty)$, $n \in \mathbb{N}$, and $(a_1, \cdots, a_n) \in (\mathbb{R}_+)^n$
  \[
  (a_1 + \cdots + a_n)^q \leq n^{\max(q-1,0)} (a_1^q + \cdots + a_n^q) .
  \]

- For any $p \in (0, 1]$, $n \in \mathbb{N}$, and $(f_1, \cdots, f_n) \in L^p(\mathbb{R}^2)^n$
  \[
  \| f_1 + \cdots + f_n \|_{L^p} \leq n^{1/p-1} (\| f_1 \| + \cdots + \| f_n \|)
  \]
Moreover Lemma 7.1 of [48] applied to the hyperbolic Littlewood–Paley analysis provides the following inequality as a consequence of interpolation of Plancherel and Minkowski ones:
\[
\left( \sum_{j_1, j_2 \in J} \| \Delta_{j_1,j_2} f \|_{L^p}^{p^*} \right)^{1/p^*} \leq \| \sum_{j_1,j_2 \in J} \Delta_{j_1,j_2} f \|_{L^p} \leq \left( \sum_{j_1,j_2 \in J} \| \Delta_{j_1,j_2} f \|_{L^p}^{p^*} \right)^{1/p^*}.
\]
for $p \geq 1$ and $J$ any subset of $\mathbb{N}^2$. (Recall that $p_* = \min(p, p')$ and $p^* = \max(p, p')$).

Let us now prove the first point of the theorem in the case where $q \neq \infty$. For this, assume that (14) holds and let us prove that $f \in B^{s,\alpha}_{p,q,|\log|^\beta}(\mathbb{R}^2)$. For $p < 1$, by Inequalities (22), (23) and the fact that Card($\Gamma_j(\alpha)$) $\leq C_j$ there exists $C > 0$ such that
\[
\left\| \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \Delta_{j_1,j_2} f \right\|_{L^p}^q \leq C_j q^{(1/p-1)+\max(q-1,0)} \left( \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1,j_2} f \|_{L^p}^q \right) .
\]
Hence,

\[
\left( \sum_{j \geq 0} j^{-\beta q 2^j s q} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \Delta_{j_1, j_2} f \right)^{1/q} \leq C \left( \sum_{j \geq 0} j^{(1/p-1)+\max(1,0)} j^{-\beta q 2^j s q} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1, j_2} f \|^q_{L^p} \right)^{1/q}.
\]

It proves that if \( q \) holds, one has

\[
\left( \sum_{j \geq 0} j^{-\beta q 2^j s q} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \Delta_{j_1, j_2} f \|_{L^p}^{q} \right)^{1/q} < \infty.
\]

Finally, by Point (1) of Proposition 4.2, it comes that \( f \in B_{p, \log |\cdot|^\beta}^{s, \alpha}(\mathbb{R}^2) \).

If \( p \geq 1 \), Inequality \( 24 \) with \( 22 \) gives

\[
\left\| \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \Delta_{j_1, j_2} f \right\|^q_{L^p} \leq \left( \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1, j_2} f \|_{L^p}^{p^*} \right)^{q/p^*} \leq C_j \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1, j_2} f \|_{L^p}^{q},
\]

which also leads to the desired result.

We now deal with the case \( q = \infty \). In this case, we have

\[
\max_{j \geq 0} j^{-\beta q 2^j s} \left\| \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \Delta_{j_1, j_2} f \right\|_{L^p} \leq C \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1, j_2} f \|_{L^p}.
\]

Hence if \( 18 \) holds, \( f \in B_{p, \log |\cdot|^\beta}^{s, \alpha}(\mathbb{R}^2) \).

To prove the converse assertion, let us assume \( f \in B_{p, \log |\cdot|^\beta}^{s, \alpha}(\mathbb{R}^2) \). Let us first deal with the case \( p < 1 \). Observe that for any \( j \geq 0 \) and any \( (j_1, j_2) \in \Gamma_j(\alpha) \), one has

\[
\phi_{j_1, j_2} \widehat{f} = \phi_{j_1, j_2} \left( g_{j-1}^a + g_j^a + g_{j+1}^a \right) \widehat{f}.
\]

Remark that \( \phi_{j_1, j_2}(2^{j_2}) \) is bounded in \( H_2^\alpha(\mathbb{R}^2) \) independently of \( (j_1, j_2) \in \Gamma_j(\alpha) \). Hence, by \( 19 \), there exists \( C > 0 \) not depending on \( j \) nor \( f \) such that for any \( (j_1, j_2) \in \Gamma_j(\alpha) \)

\[
\| (F^{-1} \phi_{j_1, j_2}) * f \|_{L^p} \leq C \| (F^{-1} \phi_{j_1, j_2}) * f \|_{L^p} + \| (F^{-1} g_{j_1}^a) * f \|_{L^p} + \| (F^{-1} g_{j_1}^a) * f \|_{L^p}.
\]

Again, two cases have to be distinguished according whether \( q \neq \infty \) or \( q = \infty \).

Let us consider the case \( q < \infty \). Observing that \( \text{Card}(\Gamma_j(\alpha)) \leq C_j \), we deduce that

\[
\sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \| (F^{-1} \phi_{j_1, j_2}) * f \|_{L^p}^{q} \leq C_j \sum_{l = j_1 - 1}^{j_1 + 1} \| (F^{-1} g_l^a) * f \|_{L^p}^{q}.
\]

So

\[
\sum_{j} j^{-1} j^{-\beta q 2^j s q} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \| (F^{-1} \phi_{j_1, j_2}) * f \|_{L^p}^{q} \leq \sum_{j} j^{-1} j^{-\beta q 2^j s q} \| (F^{-1} g_l^a) * f \|_{L^p}^{q}.
\]

Finally,

\[
(25) \sum_{j \geq 0} j^{-1} j^{-\beta q 2^j s q} \sum_{(j_1, j_2) \in \Gamma_j(\alpha)} \| (F^{-1} \phi_{j_1, j_2}) * f \|_{L^p}^{q} \leq \sum_{j \geq 0} j^{-1} j^{-\beta q 2^j s q} \| (F^{-1} g_l^a) * f \|_{L^p}^{q}.
\]

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Since in addition the function $f$ is assumed to belong to $B_{p,q,|\log| \beta}^{s,\alpha}(\mathbb{R}^2)$, one has
\[
\sum_{j} j^{-\beta q 2^{jsq}} \| (F^{-1} g_j^\alpha) * f \|_{L^p}^q = \sum_{j} j^{-1} j^{-\beta q 2^{jsq}} \| (F^{-1} g_j^\alpha) * f \|_{L^p}^q < \infty,
\]
which directly yields the required inequality using (25).

The case $p \geq 1$ is simpler. Again, Inequalities (24) and (22) gives
\[
\left\| \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \Delta_{j_1,j_2} f \right\|_{L^p}^q \geq \left( \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1,j_2} f \|_{L^p}^{p^*} \right)^{q/p^*} \geq C_j^{-\max(1-q/p^*,0)} \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1,j_2} f \|_{L^p}^q
\]
which allows to conclude.

In the case $q = \infty$, we have
\[
\max_{(j_1,j_2) \in \Gamma_j(\alpha)} \| (F^{-1} \phi_{j_1,j_2}) * f \|_{L^p}^q \leq C \max_{\ell = j-1,j,j+1} \| (F^{-1} g_\ell^\alpha) * f \|_{L^p},
\]
which leads for some $C > 0$
\[
\max_{j \geq 0} \left( j^{-\beta} 2^{js} \max_{(j_1,j_2) \in \Gamma_j(\alpha)} \| (F^{-1} \phi_{j_1,j_2}) * f \|_{L^p} \right) \leq C \max_{j \geq 0} \left( j^{-\beta} 2^{js} \| (F^{-1} g_j^\alpha) * f \|_{L^p} \right),
\]
that is
\[
\max_{j_1,j_2 \geq 0} \left( \max_{\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}} \right)^{-\beta} 2^{\max(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2})} \| (F^{-1} \phi_{j_1,j_2}) * f \|_{L^p} \leq C \max_{j \geq 0} j^{-\beta} 2^{js} \| (F^{-1} g_j^\alpha) * f \|_{L^p}.
\]
Since in addition $f$ is assumed to belong to $B_{p,|\log| \beta}^{s,\alpha}(\mathbb{R}^2)$, it comes
\[
\max_{j \geq 0} j^{-\beta} 2^{js} \| (F^{-1} g_j^\alpha) * f \|_{L^p} < \infty.
\]
Finally, the required conclusion is obtained by an approach similar to the one used for the previous case.

4.1.2 Proof of the hyperbolic wavelet characterization of anisotropic Besov spaces

Let us first consider the general case where $(p,q) \in (0,\infty)^2, \beta, s \in \mathbb{R}$ and $\alpha = (\alpha_1, \alpha_2)$ a fixed anisotropy. Intermediate spaces $\mathcal{E}^{s,\alpha}_{p,q,|\log| \beta}(\mathbb{R}^2)$ are defined as the collection of functions $f$ of $\mathcal{S}'(\mathbb{R}^2)$ such as
\[
\sum_{j \geq 0} j^{-\beta q 2^{jsq}} \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1,j_2} f \|_{L^p}^q < +\infty.
\]
A norm on $\mathcal{E}^{s,\alpha}_{p,q,|\log| \beta}(\mathbb{R}^2)$ is defined as follows
\[
\| f \|_{\mathcal{E}^{s,\alpha}_{p,q,|\log| \beta}} = \left( \sum_{j \geq 0} j^{-\beta q 2^{jsq}} \sum_{(j_1,j_2) \in \Gamma_j(\alpha)} \| \Delta_{j_1,j_2} f \|_{L^p}^q \right)^{1/q}
\]
such that the embeddings
• if $q < \infty$

$E^{s,\alpha}_{\log^j \beta - \frac{1}{q}}(\mathbb{R}^2) \hookrightarrow B^{s,\alpha}_{p,q,\log^j \beta}(\mathbb{R}^2) \hookrightarrow E^{s,\alpha}_{\log^j \beta + \frac{1}{q}}(\mathbb{R}^2)$.

with $r_1 = \begin{cases} q(\frac{1}{p} - 1) + \max(q - 1, 0) & \text{if } p \leq 1 \\ \max(\frac{1}{p} - 1, 0) & \text{if } p > 1 \end{cases}$

and $r_2 = \begin{cases} 1 & \text{if } p < 1 \\ \max(1 - q/p^*, 0) & \text{if } p \geq 1 \end{cases}$

• if $q = \infty$

$E^{s,\alpha}_{p,\log^j \beta}(\mathbb{R}^2) \hookrightarrow B^{s,\alpha}_{p,q,\log^j \beta}(\mathbb{R}^2) \hookrightarrow E^{s,\alpha}_{p,q,\log^j \beta}(\mathbb{R}^2)$.

are an exact rewriting of Theorem 4.1.

In the special case where $p = 2$, we proved in Proposition ?? that $H^{s,\alpha}_{\log^j \beta}(\mathbb{R}^2) = B^{s,\alpha}_{2,2,\log^j \beta}(\mathbb{R}^2)$

and $E^{s,\alpha}_{2,2,\log^j \beta}(\mathbb{R}^2)$ coincide.

In the following proposition, an hyperbolic wavelet characterization of spaces $E^{s,\alpha}_{p,q}(\mathbb{R}^2)$ is given. Combining Proposition 4.3, Theorems 4.1 and ?? directly implies Theorems 2.2.

Proposition 4.3 Let $(p, q) \in (0, +\infty]^2$, $s, \beta \in \mathbb{R}^2$. The following assertions are equivalent

1. $f \in E^{s,\alpha}_{p,q,\log^j \beta}(\mathbb{R}^2)$

2. $\left( \sum_{j \geq 0} j^{-\beta q} 2^{jsq} \sum_{(j_1, j_2) \in \Gamma_j} 2^{-(j_1+j_2)q/p} \left( \sum_{(k_1, k_2) \in \mathbb{Z}^2} |c_{j_1, j_2, k_1, k_2}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < +\infty$.

3. $\left( \sum_{(j_1, j_2) \in \mathbb{N}_0^2} \left( \max(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}) \right)^{-\beta q} 2^{s - \frac{j_1}{\alpha_1} - \frac{j_2}{\alpha_2}} q \left( \sum_{(k_1, k_2) \in \mathbb{Z}^2} |c_{j_1, j_2, k_1, k_2}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < +\infty$.

Let us prove Proposition 4.3. The equivalence between assertions (2) and (3) holds since for any $(j_1, j_2) \in \Gamma_j(\alpha)$, one has $\max(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}) + 2 - 2 \leq j \leq \max(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}) + 2$ and $\Gamma_j = (\mathbb{N} \cup \{-1\})^2$. The crucial point is the equivalence between assertions (1) and (2).

Proof of implication (1) $\Rightarrow$ (2) of Proposition 4.3

The proof of this implication relies on the following sampling lemma which is a adaptation of Lemma 2.4 of [25] in the case of rectangular support.

Lemma 4.2 Let $p \in (0, +\infty]$ and $j = (j_1, j_2) \in \mathbb{N}_0^2$. Suppose $g \in S'(\mathbb{R}^2)$ and $\text{supp}(\hat{g}) \subseteq \{ \xi, |\xi_1| \leq 2^{j_1+1} \text{ and } |\xi_2| \leq 2^{j_2+1} \}$. Then there exists $C > 0$ such that

$\left( \sum_{(k_1, k_2) \in \mathbb{Z}^2} 2^{-(j_1+j_2)} \left| g \left( \frac{k_1}{2^{j_1}}, \frac{k_2}{2^{j_2}} \right) \right|^p \right)^{1/p} \leq C \|g\|_{L^p}$.

Proof. Let $\psi \in S(\mathbb{R}^2)$ be such that $\text{supp}(\hat{\psi}) \subseteq \{ \xi, \max(|\xi_1|, |\xi_2|) \leq \pi \}$ and $\hat{\psi} \equiv 1$ on $[-2, 2]^2$. Set $\psi_j(x) = 2^{j_1+j_2} \psi(2^{j_1}x_1, 2^{j_2}x_2)$. One has $\hat{\psi}_j \equiv 1$ on $[-2^{j_1+1}, 2^{j_1+1}] \times [-2^{j_2+1}, 2^{j_2+1}]$. 

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By assumption $\text{supp}(\hat{g}) \subset [-2^{j_1+1}, 2^{j_1+1}] \times [-2^{j_2+1}, 2^{j_2+1}]$, so that for any $x = (x_1, x_2) \in \mathbb{R}^2$ and any fixed $y = (y_1, y_2) \in \mathbb{R}^2$

$$g(x + y) = (\psi_j * g)(x + y) = (2\pi)^{-2} \int_{\xi_1 = -2^{j_1+1}}^{2^{j_1+1}} \int_{\xi_2 = -2^{j_2+1}}^{2^{j_2+1}} \hat{\psi}_j(\xi) \hat{g}(\xi) e^{ix\xi} e^{iy\xi} d\xi.$$

Denote $\hat{h}_j$ the periodic extension of $\hat{\psi}_j$ with period $2^{j_i+1} \pi$ for each variable $\xi_i$ ($i = 1, 2$). One has

$$g(x + y) = (2\pi)^{-2} \int_{\xi_1 = -2^{j_1+1}}^{2^{j_1+1}} \int_{\xi_2 = -2^{j_2+1}}^{2^{j_2+1}} \left( \hat{h}_j(\xi) e^{ix\xi} \right) \left( \hat{g}(\xi) e^{iy\xi} \right) d\xi.$$

Using an expansion of $\hat{h}_j e^{ix\xi}$ in two dimensional Fourier series, it comes

$$\hat{h}_j(\xi) e^{ix\xi} = \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \int_{\xi_1 = -2^{j_1+1}}^{2^{j_1+1}} \int_{\xi_2 = -2^{j_2+1}}^{2^{j_2+1}} \hat{h}_j(\xi) e^{i\ell_1 \xi_1} e^{-i2^{j_1} \ell_1 \xi_1} e^{i\ell_2 \xi_2} e^{-i2^{j_2} \ell_2 \xi_2} e^{i\ell_1 \xi_1} e^{i\ell_2 \xi_2} \left( \hat{h}_j(\xi) e^{ix\xi} \right) \left( \hat{g}(\xi) e^{iy\xi} \right) d\xi.$$

where for $j = (j_1, j_2)$ and $\ell = (\ell_1, \ell_2)$, the notation $2^{-j' \ell} = (2^{-j_1} \ell_1, 2^{-j_2} \ell_2)$ is used. Replacing $\hat{h}_j(\xi) e^{ix\xi}$ with the last sum in Equation (27) yields that for any $x = (x_1, x_2) \in \mathbb{R}^2$ and any fixed $y = (y_1, y_2) \in \mathbb{R}^2$

$$g(x + y) = \frac{2^{-(j_1 + j_2)}}{4\pi^2} \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \left( \int_{\xi_1 = -2^{j_1+1}}^{2^{j_1+1}} \int_{\xi_2 = -2^{j_2+1}}^{2^{j_2+1}} \psi_j(x - 2^{-j' \ell}) e^{i\ell_1 \xi_1} e^{i\ell_2 \xi_2} \left( \hat{g}(\xi) e^{iy\xi} \right) d\xi \right)$$

Hence for all $y \in \lambda_{j_1, j_2, k_1, k_2} = [2^{-j_1} k_1, 2^{-j_1} (k_1 + 1)] \times [2^{-j_2} k_2, 2^{-j_2} (k_2 + 1)]$

$$\sup_{|z_1 - 2^{-j_1} k_1| \leq 2^{-j_1}, |z_2 - 2^{-j_2} k_2| \leq 2^{-j_2}} |g(z)| \leq \sup_{|x_1| \leq 2^{-j_1|\ell_1|}, |x_2| \leq 2^{-j_2|\ell_2|}} |g(x + y)| \leq 2^{-(j_1 + j_2)} \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} |g(2^{-j' \ell} + y)| \cdot \sup_{\max(2^{j_1}|\ell_1|, 2^{j_2}|\ell_2|) \leq \sqrt{2}} \left| \psi_j(x - 2^{-j' \ell}) \right|$$

$$\leq 2^{-(j_1 + j_2)} \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} |g(2^{-j' \ell} + y)| \cdot \frac{1}{(1 + |\ell|)^M},$$

where the last inequality follows from the fast decay of $\psi$. Take $M$ sufficiently large and use either the $p$ triangular inequality either the Hölder inequality according whether $p \in (0, 1)$ or $p \in [1, +\infty)$. Hence, one has

$$|g(2^{-j_1} k_1, 2^{-j_2} k_2)|^p \leq \sup_{|z_1 - 2^{-j_1} k_1| \leq 2^{-j_1}, |z_2 - 2^{-j_2} k_2| \leq 2^{-j_2}} |g(z)|^p \leq C 2^{-(j_1 + j_2)} \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} |g(2^{-j' \ell} + y)|^p \cdot \frac{1}{(1 + |\ell|)^{Mp}},$$

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for some $M' > 1$. An integration over $y \in \lambda_{j_1,j_2,k_1,k_2}$ leads to
\[
2^{-(j_1+j_2)}|g(2^{-j_1}k_1,2^{-j_2}k_2)|^p \leq \sum_{(\ell_1,\ell_2) \in \mathbb{Z}^2} \frac{1}{(1 + |\ell|)^M'} \int_{\lambda_{j_1,j_2,k_1,k_2}} |g(y)|^p \, dy
\]
and a sum over $k \in \mathbb{Z}^2$ gives
\[
\sum_k 2^{-(j_1+j_2)}|g(2^{-j_1}k_1,2^{-j_2}k_2)|^p \leq \sum_k \sum_{(\ell_1,\ell_2) \in \mathbb{Z}^2} \frac{1}{(1 + |\ell|)^3} \int_{\lambda_{j_1,j_2,k_1,k_2}} |g(y)|^p \, dy
\]
which ends the proof of Lemma 4.2.

Now, observe that $c_{j_1,j_2,k_1,k_2} = \Delta_{j_1,j_2}f(2^{-j_1}k_1,2^{-j_2}k_2)$. By Lemma 4.2 applied to $g = \Delta_{j_1,j_2}f \in S(\mathbb{R}^2)$, one has
\[
\sum_{(k_1,k_2) \in \mathbb{Z}^2} |c_{j_1,j_2,k_1,k_2}|^p = \sum_{(k_1,k_2) \in \mathbb{Z}^2} |\Delta_{j_1,j_2}f(2^{-j_1}k_1,2^{-j_2}k_2)|^p \leq C 2^{j_1} 2^{j_2} \|\Delta_{j_1,j_2}f\|_p^p,
\]
which is the desired wavelet characterization.

**Proof of implication (2) \(\Rightarrow\) (1) of Proposition 4.3**
To obtain the converse implication, the same approach as in the proof of Theorem 3.1 of [25] is followed.

Since $\phi_{j_1,j_2}$ and $\psi_{m_1,m_2,k_1,k_1}$ are both defined as a tensorial product, Lemma 3.3 of [25] can be applied: there exists some $C > 0$ such that for any $\alpha > 0$ and for all $x = (x_1,x_2) \in \mathbb{R}^2$ one has
\[
(28) \quad |\phi_{j_1,j_2} \ast \psi_{m_1,m_2,k_1,k_1}(x)| \leq C \frac{2^{-((j_1-m_1)+(j_2-m_2))(M+3)}}{(1 + 2^{\inf(j_1,m_1)}|x_1 - 2^{-m_1}k_1|)^\alpha (1 + 2^{\inf(j_2,m_2)}|x_2 - 2^{-m_2}k_2|)^\alpha},
\]
where $M$ denotes the number of vanishing moments of the wavelets.

A lemma analogous to Lemma 3.4 of [25] is now proved:

**Lemma 4.3** Let $p \in [1, +\infty]$, $\ell_1, \ell_2, m_1, m_2$ integers such that $\ell_1 \leq m_1$ and $\ell_2 \leq m_2$. We are also given some functions $g_{k_1,k_2}$ satisfying the following inequality for some $C > 0$

\[
(29) \quad \forall x = (x_1,x_2) \in \mathbb{R}^2, \ |g_{k_1,k_2}(x)| \leq \frac{C}{(1 + 2^{\ell_1}|x_1 - 2^{-m_1}k_1|)^2 (1 + 2^{\ell_2}|x_2 - 2^{-m_2}k_2|)^2}.
\]

Set
\[
F = \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} d_{k_1,k_2}g_{k_1,k_2}
\]
Then
\[
(30) \quad \|F\|_{L^p} \leq C 2^{-(m_1+m_2)/p} 2^{m_1-\ell_1} 2^{m_2-\ell_2} \left( \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} |d_{k_1,k_2}|^p \right)^{1/p}.
\]
Proof. By definition of the $L^p$-norm, one has:

$$\|F\|_{L^p}^p = \int_{\mathbb{R}^2} \left| \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} d_{k_1,k_2} g_{k_1,k_2}(x) \right|^p \, dx$$

$$\leq \sum_{k'(k'_1,k'_2) \in \mathbb{Z}^2} \int_{\lambda_{m_1,m_2,k'_1,k'_2}} \left| \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} d_{k_1,k_2} g_{k_1,k_2}(x) \right|^p \, dx,$$

where the hyperbolic dyadic cube $\lambda_{m_1,m_2,k'_1,k'_2}$ are defined in (7). Observe now that, by the usual triangular inequality and by inequality (29), there exists some $C > 0$ such that for any $(k_1, k_2) \in \mathbb{Z}^2$, $(k'_1, k'_2) \in \mathbb{Z}^2$

$$\sup_{x \in \lambda_{m_1,m_2,k'_1,k'_2}} \left| \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} d_{k_1,k_2} g_{k_1,k_2}(x) \right| \leq \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} \left| d_{k_1,k_2} \right| \prod_{i=1,2} (1 + 2^{\ell_i} |2^{-m_1}k'_i - 2^{-m_1}k_i|^2)$$

Hence one has

$$\|F\|_{L^p}^p \leq C 2^{-(m_1+m_2)} \sum_{(k'_1,k'_2) \in \mathbb{Z}^2} \left( \sum_{(k_1,k_2) \in \mathbb{Z}^2} \frac{|d_{k_1,k_2}|}{(1 + 2^{\ell_1-m_1}|k'_1 - k_1|)(1 + 2^{\ell_2-m_2}|k'_2 - k_2|)} \right)^p,$$

Let us recall the usual convolution inequality, valid for any sequences $s, s' \in \ell^p(\mathbb{Z}^2)$ for $p \geq 1$,

$$\|s * s'\|_{\ell^p(\mathbb{Z}^2)} \leq \|s\|_{\ell^p(\mathbb{Z}^2)} \|s'\|_{\ell^1(\mathbb{Z}^2)}.$$ 

Applied to $s = |d_{k_1,k_2}|$ and $s' = (1 + 2^{\ell_1-m_1}|k'_1 - k_1|)^{-2}(1 + 2^{\ell_2-m_2}|k'_2 - k_2|)^{-2}$, it gives

$$\|F\|_{L^p}^p \leq C 2^{-(m_1+m_2)} \left( \sum_{(k_1,k_2) \in \mathbb{Z}^2} |d_{k_1,k_2}|^p \right) \left( \sum_{(k'_1,k'_2) \in \mathbb{Z}^2} \frac{1}{(1 + 2^{\ell_1-m_1}|k'_1|)(1 + 2^{\ell_2-m_2}|k'_2|)} \right)^p,$$

Recall now the classical result:

$$\sum_{k'(k'_1,k'_2) \in \mathbb{Z}^2} \frac{1}{(1 + 2^{\ell_1-m_1}|k'_1|)^2(1 + 2^{\ell_2-m_2}|k'_2|)^2} \leq C 2^{m_1-\ell_1} 2^{m_2-\ell_2}$$

Hence

$$\|F\|_{L^p}^p \leq C 2^{-(m_1+m_2)} 2^{(m_1-\ell_1)p_2(m_2-\ell_2)p} \left( \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} |d_{k_1,k_2}|^p \right) \times \left( \frac{1}{\prod_{i=1,2} (1 + 2^{\ell_i-m_1}|k'_i|^2)} \right)^p,$$

which directly yields the required result. It ends the proof of Lemma 4.3. □

Let us now go back to Implication (2) $\Rightarrow$ (1) of Proposition 4.3. Two cases are considered: $p \in (0, 1)$ and $p \in [1, +\infty]$.
Let us first assume that \( p \in (0, 1) \).
We have to bound \( \| \Delta_{j_1, j_2} f \|_{L^p} = \| \phi_{j_1, j_2} \ast f \|_{L^p} \).
Observe that

\[
\phi_{j_1, j_2} \ast f = \sum_{m_1, m_2} \sum_{k_1, k_2} c_{m_1, m_2, k_1, k_2} (\phi_{j_1, j_2} \ast \psi_{m_1, m_2, k_1, k_2})
\]

By the \( p \)-triangular inequality, it comes

\[
\forall x = (x_1, x_2) \in \mathbb{R}^2, \ | \phi_{j_1, j_2} \ast f(x)|^p \leq \sum_{m_1, m_2} \sum_{k_1, k_2} |c_{m_1, m_2, k_1, k_2}|^p \| (\phi_{j_1, j_2} \ast \psi_{m_1, m_2, k_1, k_2})(x) \|^p
\]

By Inequality [28], for all \( x = (x_1, x_2) \in \mathbb{R}^2 \), one has

\[
| \phi_{j_1, j_2} \ast f(x) |^p \leq \sum_{m_1, m_2} \sum_{k_1, k_2} |c_{m_1, m_2, k_1, k_2}|^p \frac{2^{-p(j_1 - m_1) + j_2 - m_2}}{(1 + 2^{|j_1 - m_1|} |x_1 - 2^{m_1} k_1|)p\alpha(1 + 2^{|j_2 - m_2|} |x_2 - 2^{m_2} k_2|)p\alpha}
\]

An integration over \( \mathbb{R}^2 \) implies that :

\[
\| \phi_{j_1, j_2} \ast f(x) \|^p_{L^p} \leq \sum_{m_1, m_2} \sum_{k_1, k_2} |c_{m_1, m_2, k_1, k_2}|^p 2^{-p(j_1 - m_1) + j_2 - m_2)(M+3)} .
\]

Hence

\[
\| f \|^q_{\mathcal{E}^{s, \alpha}_{p, q, \log |\beta|}} = \sum_{j_1, j_2} \left( \max\left(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}\right) \right)^{-\beta q} 2^{q\alpha \max\left(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}\right)} \| \phi_{j_1, j_2} \ast f(x) \|^q_{L^p}
\]

\[
\leq \sum_{j_1, j_2} \left( \sum_{m_1, m_2} \| c_{m_1, m_2, \cdot} \|_p \right)^p 2^{-p(j_1 - m_1) + j_2 - m_2)(M+3)} \left( \max\left(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}\right) \right)^{-\beta p} 2^{p\alpha \max\left(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}\right)}
\]

Set for any \( t \in \mathbb{R} \), \((t)_+ = \max(t, 0)\) and

\[
\text{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}
\]

Observe now that for any integers \( j, m \)

\[
m - (m - j)_+ \leq j \leq (j - m)_+ + m,
\]

and that for any integers \( j_1, j_2, m_1, m_2 \)

\[
1 - \frac{\max\left(\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_1}\right)}{\max\left(\frac{m_1 - j_1}{\alpha_1}, \frac{m_2 - j_2}{\alpha_1}\right)} \leq \max\left(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_1}\right) \leq \max\left(\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_1}\right) \left[ 1 + \max\left(\frac{j_1 - m_1}{\alpha_1}, \frac{j_2 - m_2}{\alpha_1}\right) \right],
\]

(except in the case \( m_1 = m_2 = 0 \) which can be treated separately). Hence

\[
\| f \|^q_{\mathcal{E}^{s, \alpha}_{p, q, \log |\beta|}} \leq \sum_{j_1, j_2} \left( \sum_{m_1, m_2} u_{m_1, m_2} (j_1 - m_1)_+ (j_2 - m_2)_+ \right)^{q/p},
\]
with \( s_{m_1,m_2} = \left( \max\left( \frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2} \right) \right)^{-\beta p} 2^{ps \max\left( \frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2} \right)} \| c_{m_1,m_2} \|_{\ell^p} \),

and \( s'_{j_1,j_2} = 2^{-p(|j_1|+|j_2|)(M+3)} \left[ 1 + \max\left( \frac{(j_1)+}{\alpha_1}, \frac{(j_2)+}{\alpha_2} \right) \right]^{-\beta p} 2^{ps \max\left( \frac{(j_1)+}{\alpha_1}, \frac{(j_2)+}{\alpha_2} \right)} \| \varphi \|_{\ell^p} \).

If \( q/p > 1 \) Young’s inequality can be applied, which states that for any sequences \( s, s' \),
\[
\| s \ast s' \|_{\ell^q(\mathbb{Z}^2)} \leq \| s \|_{\ell^q(\mathbb{Z}^2)} \| s' \|_{\ell^q(\mathbb{Z}^2)},
\]
whereas if \( q/p \leq 1 \) the usual \((q/p)\)–triangle inequality and the usual inequality \( \| s \ast s' \|_{\ell^q(\mathbb{Z}^2)} \leq \| s \|_{\ell^q(\mathbb{Z}^2)} \| s' \|_{\ell^q(\mathbb{Z}^2)} \) valid for any sequence \( s, s' \) are applied. In any case, the following inequality is obtained
\[
\| f \|_{L^{q,s,a}_{p,q;\log|\cdot|^\beta}} \leq \left( \sum_{m_1,m_2} \left( \max\left( \frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2} \right) \right)^{-\beta p} 2^{ps \max\left( \frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2} \right)} \| c_{m_1,m_2} \|_{\ell^p} \right) \times \sum_{j_1,j_2} \left( 2^{-p(j_1+j_2)(M+3)} \left( \max\left( \frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2} \right) \right)^{-\beta p} 2^{ps \max\left( \frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2} \right)} \right)^{\max(q/p,1)}.
\]

If the wavelets have sufficiently vanishing moments, we get that
\[
\| f \|_{L^{q,s,a}_{p,q;\log|\cdot|^\beta}} \leq C \left( \sum_{m_1,m_2} \left( \max\left( \frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2} \right) \right)^{-\beta p} 2^{ps \max\left( \frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2} \right)} \| c_{m_1,m_2} \|_{\ell^p} \right),
\]
which is the required result.

We now consider the case \( p \in [1, +\infty) \). In this case, observe that
\[
\Delta_{j_1,j_2} f = \sum_{k_1,k_2} d_{k_1,k_2} g_{k_1,k_2}
\]
with
\[
g_{k_1,k_2} = 2^{(j_1-m_1)+|j_2-m_2|}(\phi_{j_1,j_2} \ast \psi_{m_1,m_2,k_1,k_2}),
\]
and
\[
d_{k_1,k_2} = 2^{-((j_1-m_1)+|j_2-m_2|)(M+3)}c_{j_1,j_2,k_1,k_2}.
\]

We set \( \ell_1 = \inf(j_1, m_1) \) and \( \ell_2 = \inf(j_2, m_2) \). Lemma 4.3 gives
\[
\| \Delta_{j_1,j_2} f \|_{L^p} \leq C 2^{-p(j_1-m_1)+|j_2-m_2|}(M+3) 2^{-(m_1+m_2)/p} \| c_{m_1,m_2} \|_{\ell^p} 2^{m_1-\ell_1} 2^{m_2-\ell_2}
\]
Again two cases \( q \leq 1 \) and \( q > 1 \) are distinguished and the same approach than in the case \( p \in (0,1) \) is followed. It leads to the required conclusion.

4.2 Proof of Theorem 3.2

First a two–microlocal criterion is proved.

**Proposition 4.4**

1. Assume that \( f \in C^{s,a}(x_0) \). Then there exists some \( C > 0 \) such that for any \((j_1,j_2,k_1,k_2) \in (\mathbb{N} \cup \{ -1 \})^2 \times \mathbb{Z}^2 \),
\[
|c_{j_1,j_2,k_1,k_2}| \leq C \min(2^{-\frac{j_1}{\alpha_1}}, 2^{-\frac{j_2}{\alpha_2}}, a \left| \frac{k_1}{2^j} \right|, 2^{-\frac{j_2}{\alpha_2}}, b \left| \frac{k_2}{2^j} \right|).
\]
2. Conversely, assume that $f$ is uniformly Hölder and that (31) holds, then $f \in C^{s,\alpha}_{\log|\cdot|^2}(x_0)$.

**Proof.** Let us first assume that $f \in C^{s,\alpha}(x_0)$ with $x_0 = (a, b)$. Assume that $j_1 \neq -1$ and $j_2 \neq -1$. By definition of the hyperbolic wavelet coefficients one has

$$c_{j_1,j_2,k_1,k_2} = 2^{j_1+j_2} \int_{\mathbb{R}^2} f(x_1, x_2) \psi(2^{j_1} x_1 - k_1) \psi(2^{j_2} x_2 - k_2) dx_1 dx_2$$

Since $\psi$ admits at least one vanishing moment, the two following equalities hold

(32) \hspace{1cm} c_{j_1,j_2,k_1,k_2} = 2^{j_1+j_2} \int_{\mathbb{R}^2} (f(x_1, x_2) - P_{x_0}(a, x_2)) \psi(2^{j_1} x_1 - k_1) \psi(2^{j_2} x_2 - k_2) dx_1 dx_2

and

(33) \hspace{1cm} c_{j_1,j_2,k_1,k_2} = 2^{j_1+j_2} \int_{\mathbb{R}^2} (f(x_1, x_2) - P_{x_0}(x_1, b)) \psi(2^{j_1} x_1 - k_1) \psi(2^{j_2} x_2 - k_2) dx_1 dx_2

Equality (32) and the assumption $f \in C^{s,\alpha}(x_0)$ imply that

$$|c_{j_1,j_2,k_1,k_2}| \leq 2^{j_1+j_2} \int_{\mathbb{R}^2} |x_1 - a^n|^{s/\alpha_1} |\psi(2^{j_1} x_1 - k_1)\psi(2^{j_2} x_2 - k_2)| dx_1 dx_2$$

$$\leq 2^{j_1+j_2} \int_{\mathbb{R}^2} \left( \left| x_1 - \frac{k_1}{2^{j_1}} \right|^{s/\alpha_1} + \left| \frac{k_1}{2^{j_1}} - a \right|^{s/\alpha_1} \right) \psi(2^{j_1} x_1 - k_1) \psi(2^{j_2} x_2 - k_2) dx_1 dx_2$$

We now set $u_1 = 2^{j_1} x_1 - k_1$, $u_2 = 2^{j_2} x_2 - k_2$ and deduce that

$$|c_{j_1,j_2,k_1,k_2}| \leq \left( 2 \frac{\min\{a_1, a_2\}}{\max\{a_1, a_2\}} \int_{\mathbb{R}^2} |u_1|^{s/\alpha_1} |\psi(u_1)\psi(u_2)| du_1 du_2 + \frac{k_1}{2^{j_1}} - a \right)^{s/\alpha_1} \int \left| \psi(u_1)\psi(u_2) \right| du_1 du_2.$$  

Hence for some $C$ depending only on $\psi$, $s$ and $\alpha$ one has

$$|c_{j_1,j_2,k_1,k_2}| \leq C \left( 2^{\frac{s}{\alpha_1}}, \frac{k_1}{2^{j_1}} - a \right)^{s/\alpha_1}$$

A similar approach yields that

$$|c_{j_1,j_2,k_1,k_2}| \leq C \left( 2^{\frac{s}{\alpha_2}}, \frac{k_2}{2^{j_2}} - b \right)^{s/\alpha_2}$$

This shows that (31) can be read as a necessary condition for pointwise regularity of $f$.

Let us now prove the converse result. Assuming that (31) holds, the aim first consists in defining a polynomial approximation of $f$ at $x_0$. To that end, a Taylor expansion is used to investigate the differentiability of $f$ at $x_0$. Let us define $f_j$ as:

$$f_j = \sum_{(j_1,j_2)\in \Gamma(a)} \sum_{(k_1,k_2)\in \mathbb{Z}^2} c_{j_1,j_2,k_1,k_2} \sum_{j_1,j_2} \psi_{j_1,j_2,k_1,k_2}.$$  

where the notations are the same as in the proof of Proposition 2.4. One has

$$|f_j(x)| \leq \sum_{(j_1,j_2)\in \Gamma(a)} \sum_{(k_1,k_2)\in \mathbb{Z}^2} \min(\frac{2^{-j_1s/\alpha_1} + |\frac{k_1}{2^{j_1}} - a|^{s/\alpha_1}}{(1 + |2^{j_1} x_1 - k_1|)^N (1 + |2^{j_2} x_2 - k_2|)^N}, \frac{2^{-j_2s/\alpha_2} + |\frac{k_2}{2^{j_2}} - a|^{s/\alpha_2}}{(1 + |2^{j_1} x_1 - k_1|)^N (1 + |2^{j_2} x_2 - k_2|)^N})$$

$$\leq \sum_{j_1,j_2} \sum_{k_1,k_2} \frac{2^{-js} + |\frac{k_2}{2^{j_2}} - x_2|^{s/\alpha_2} + |x_2 - b|^{s/\alpha_2}}{(1 + |2^{j_1} x_1 - k_1|)^N (1 + |2^{j_2} x_2 - k_2|)^N} + \sum_{j_2 \leq j} \frac{2^{-js} + |\frac{k_1}{2^{j_1}} - x_1|^{s/\alpha_1} + |x_1 - a|^{s/\alpha_1}}{(1 + |2^{j_1} x_1 - k_1|)^N (1 + |2^{j_2} x_2 - k_2|)^N}$$

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Then
\[ |f_j(x)| \leq C(j2^{-js} + j|x_1 - a|^{s/\alpha_1} + j|x_2 - b|^{s/\alpha_2}). \]

In the same way, if \( \beta = (\beta_1, \beta_2) \), one has

\[ |\partial^\beta f_j| \leq \sum_{(j_1, j_2) \in \Gamma_j} 2^{j_1\beta_1 + j_2\beta_2} \sum_{(k_1, k_2) \in \mathbb{Z}^2} \min(2^{-j_1s/\alpha_1} + |k_1 - 1/2|^{s/\alpha_1}, 2^{-j_2s/\alpha_2} + |k_2 - 1/2|^{s/\alpha_2}) \frac{(1 + |2j_1x_1 - k_1|)^N(1 + |2j_2x_2 - k_2|)^N}{(1 + |2j_1x_1 - k_1|)^N(1 + |2j_2x_2 - k_2|)^N}. \]

Then
\[ |\partial^\beta f_j(x)| \leq C2^{j(\beta_1 \alpha_1 + \beta_2 \alpha_2)}(2^{-js} + |x_1 - a|^{s/\alpha_1} + |x_2 - b|^{s/\alpha_2}). \]

So, the function \( f \) is \( \beta \)-differentiable at \( x_0 \) provided that \( \beta_1 \alpha_1 + \beta_2 \alpha_2 \leq s \). The Taylor polynomial of \( f \) at \( x_0 \) is defined by

\[ P_{j,x_0}(x) = \sum_{\beta_1 \alpha_1 + \beta_2 \alpha_2 \leq s} \frac{(x - x_0)^\beta}{\beta!} \partial^\beta f_j(x_0) \]

and \( P_{x_0}(x) = \sum_j P_{j,x_0}(x) \).

We shall now bound \( |f(x) - P_{x_0}(x)| \) in the neighborhood of \( x_0 \). Recall that \( f \) is assumed to be uniformly Hölder, namely there exists some \( \varepsilon_0 > 0 \) such that \( f \in C^{\varepsilon_0}(\mathbb{R}^2) \). The inclusions between Hölder spaces with different anisotropies (see \([51]\)) leads to the existence of \( \varepsilon_0 \) such that \( f \in C^{s_0, \alpha}(\mathbb{R}^2) \). Set \( J_1 = [\alpha J / \varepsilon_0] \). Observe that

\[ |f(x) - P_{x_0}(x)| \leq \sum_{j \leq J} |f_j(x) - P_{j,x_0}(x)| + \sum_{j = J + 1}^{J_1} |f_j(x)| + \sum_{j > J_1} |f_j(x)| + \sum_{j > J} |P_{j,x_0}(x)|. \]

Let us now bound each term of the right hand side of this inequality.

We first deal with the term corresponding to \( j \leq J \). In this case we shall use an anisotropic version of Taylor inequality which can be found in \([17, 24]\) and recalled in \([6]\). It gives the existence of some \( C > 0 \) such that

\[ |f_j(x) - P_{j,x_0}(x)| \leq C \sum_{\beta_1 + \beta_2 \leq k + 1, \alpha_1 \beta_1 + \alpha_2 \beta_2 > s} |x - x_0|^\alpha_1 \beta_1 + \alpha_2 \beta_2 \sup_{z = (z_1, z_2) \in \mathbb{R}^2} |\partial^\beta f_j|. \]

with \( k = [\max(s/\alpha_1, s/\alpha_2)] \). The bound \((35)\) implies that there exists some \( C > 0 \) such that

\[ |f_j(x) - P_{j,x_0}(x)| \leq C \sum_{\beta_1 + \beta_2 \leq k + 1, \alpha_1 \beta_1 + \alpha_2 \beta_2 > s} |x - x_0|^\alpha_1 \beta_1 + \alpha_2 \beta_2 \left(2^{-js} + |x_1 - a|^{s/\alpha_1} + |x_2 - b|^{s/\alpha_2}\right) \]

Hence,

\[ \sum_{j \leq J} |f_j(x) - P_{j,x_0}(x)| \leq C \sum_{\beta_1 + \beta_2 \leq k + 1, \alpha_1 \beta_1 + \alpha_2 \beta_2 > s} |x - x_0|^\alpha_1 \beta_1 + \alpha_2 \beta_2 \left(2^{-j(\beta_1 \alpha_1 + \beta_2 \alpha_2)} + 2^j(\beta_1 \alpha_1 + \beta_2 \alpha_2) \right) |x - x_0|^s. \]

Since \( |x - x_0|^s \leq 2^{-J} \) it comes

\[ \sum_{j \leq J} |f_j(x) - P_{j,x_0}(x)| \leq C |x - x_0|^s. \]
Let us now bound the sum $\sum_{j=J+1}^{J_1} |f_j(x)|$. By (34) and the definition of $J_1$ which depends on $J$, one has

$$(37) \quad \sum_{j=J+1}^{J_1} |f_j(x)| \leq \sum_{j=J}^{J_1} (j2^{-j^3} + j|x - x_0|^h) \leq J2^{-j^3} + J^2|x - x_0|^h.$$ 

To bound the sum $\sum_{j>J_1} |f_j(x)|$ the uniform regularity of $f$ is used, leading to

$$(38) \quad \sum_{j>J_1} |f_j(x)| \leq C2^{-j\varepsilon_0} \leq C2^{-j^3}$$

the last equality following from the definition of $J_1$.

Finally, by (35), the sum $\sum_{j>J} |P_{j,x_0}(x)|$ can be bounded. Indeed, for some $C > 0$, one has

$$\sum_{j>J} |P_{j,x_0}(x)| \leq \sum_{j>J} \frac{|x - x_0|^\beta}{\beta!} \sum_{j>J} |\partial^\beta f_j(x_0)| \leq C \sum_{j>J} \frac{|x_1 - a|\beta_1 |x_2 - b|\beta_2 \sum_{j>J} 2^{j(\beta_1 \alpha_1 + \beta_2 \alpha_2 - s)}}{\beta!}$$

Since $|x_1 - a| \leq |x - x_0|^{\alpha_1} \leq 2^{-J\alpha_1}$ and $|x_2 - b| \leq |x - x_0|^{\alpha_2} \leq 2^{-J\alpha_2}$ it comes

$$(39) \quad \sum_{j> J} |P_{j,x_0}(x)| \leq C \sum_{j> J} 2^{-J(\beta_1 \alpha_1 + \beta_2 \alpha_2)} \sum_{j> J} 2^{j(\beta_1 \alpha_1 + \beta_2 \alpha_2 - s)} \leq C2^{-j^3}.$$ 

Finally, Inequalities (36), (37), (38) and (39) yield that $f \in C^{n,\alpha}_{|\log|} (x_0)$.

Theorem 3.2 is a straightforward consequence of the two–microlocal criterion and of the following lemma:

**Lemma 4.4** The two following properties are equivalent:

(i) Inequality (31) holds.

(ii) Inequality (8) holds.

**Proof.** Assume that (31) holds. If $X \subset 3\lambda_{j_1,j_2}(x_0)$, then

$$j_1' \geq j_1, j_2' \geq j_2,$$

and

$$\frac{k_1'}{2^{j_1'}} - a \leq 2.2^{-j_1'} \text{ and } \frac{k_2'}{2^{j_2'}} - b \leq 2.2^{-j_2'}.$$ 

Condition (31) implies

$$|c_X| \leq \min(2^{-\frac{j_1}{\alpha_1}}, 2^{\frac{j_2}{\alpha_2}}) = 2^{-\max(\frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2})^n}.$$ 

Conversely, assume that (8) holds. Let $X' = \lambda(j_1', j_2', k_1', k_2')$ an hyperbolic dyadic cube. Set

$$j_1 = \sup\{\ell_1, 2^{-j_1'} + \frac{k_1'}{2^{j_1'}} - a \leq 2^{-\ell_1}\}$$

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and
\[ j_2 = \sup \{ \ell_2, 2^{-j_2} + \left| \frac{k'_2}{2^{j_2}} - b \right| \leq 2^{-\ell_2} \} \]

We have \( X \subset 3\lambda_{j_1,j_2}(x_0) \). Since (8) holds one has
\[ |c_X| \leq \min(2^{-\frac{j_1}{\alpha_1}}, 2^{-\frac{j_2}{\alpha_2}}) \leq C \min(2^{-\frac{j_1}{\alpha_1}} + \left| \frac{k'_1}{2^{j_1}} - a \right|^{s/\alpha_1}, 2^{-\frac{j_2}{\alpha_2}} + \left| \frac{k'_2}{2^{j_2}} - b \right|^{s/\alpha_2}) , \]
that is 31 holds. \( \square \)

### 4.3 Proof of Theorem 3.3

The proof of Theorem 3.3 is based on the two following lemmas, analogous to Propositions 7 and 8 of [33]:

**Lemma 4.5** Set \( \alpha = (a, 2 - a) \) and define
\[ G(H, \alpha) = \{ x \in \mathbb{R}^2, f \not\in C^{H, \alpha}_{|\log|} (x) \} . \]
Let \( p > 0 \) and \( s \in (0, \omega(p, \alpha)/p] \). Then for any \( H \geq s - 2/p \)
\[ \dim_H(G(H, \alpha)) \leq Hp - sp + 2 . \]
If \( H < s - 2/p \), \( \dim_H(G(H, \alpha)) = \emptyset \).

**Lemma 4.6** Set \( \alpha = (a, 2 - a) \) and define
\[ B(H, \alpha) = \{ x \in \mathbb{R}^2, f \in C^{H, \alpha}(x) \} . \]
Let \( p < 0 \) and \( s \in (0, \omega(p, \alpha)/p] \). Then
\[ \dim_H(B(H, \alpha)) \leq \dim_P(B(H, \alpha)) \leq Hp - sp + 2 . \]

The proof of Lemma 4.5 in the case \( H \geq s - 2/p \) is exactly the same as this of Proposition 7 of [33], except that the set \( G_{j,H} \) are replaced with the sets
\[ G(j, H, \alpha) = \{ \lambda = \lambda(j_1, j_2, k_1, k_2), (j_1, j_2) \in \Gamma_j(\alpha), |d_\lambda| \geq 2^{-jHp} \} . \]

Lemma 4.5 in the case \( H < s - 2/p \), comes from the hyperbolic wavelet characterization of anisotropic Besov spaces stated in Theorem 2.2 and the Sobolev embeddings which can be proved in the anisotropic case as in the isotropic one (see [51]).

The proof of Lemma 4.6 is exactly the same as this of Proposition 8 of [33], except that the set \( B_H \) are replaced with the sets \( B(H, \alpha) \).

Lemmas 4.5 and 4.6 then imply Theorem 3.3, since for any \( \alpha = (\alpha_1, \alpha_2) \) such that \( \alpha_1 + \alpha_2 = 2 \) one has
\[ E(H, \alpha) \subset (\cap_{H > H} G(H', \alpha)) \cap (\cup_{H < H} B(H', \alpha)) . \]

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References


