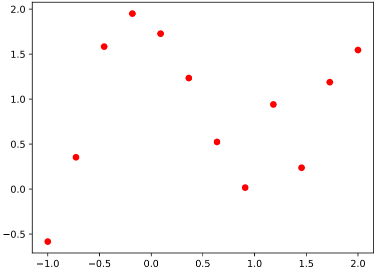
Chapter 3

Lagrange interpolation

Objective

- Geometrically, *Lagrange interpolation* consists in determining a curve (i.e., a function) that passes through predetermined positions (x_i, y_i).



- Data : $(x_i, y_i), \quad i = 0, \ldots, n$
  Problem : find a function p (in a given space E) such that $p(x_i) = y_i$
- No uniqueness (in general)
  $\Rightarrow$ choice of an appropriate space E to achieve uniqueness
- Characterization & construction
- Error analysis ...

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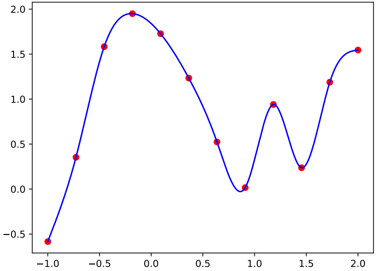

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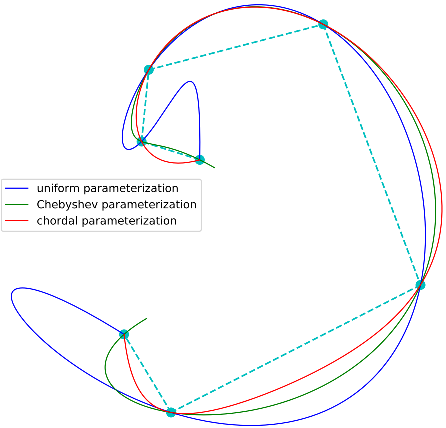
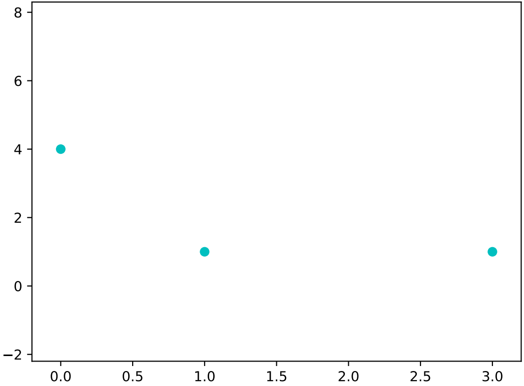
  Parametric case :

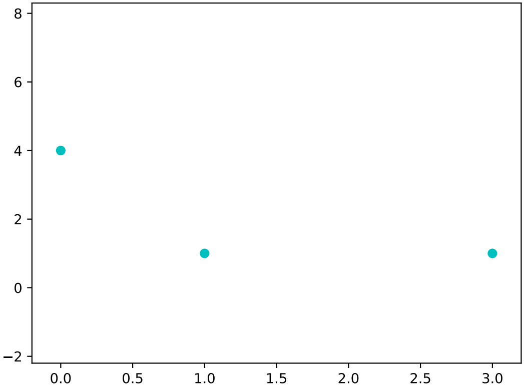


Interpolating space E



- Data : 3 points

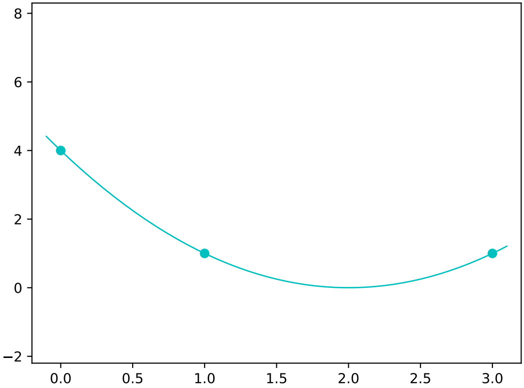
Interpolating space E



- Data : 3 points

- E : polynomials of degree 1
  $\rightarrow$ no solution

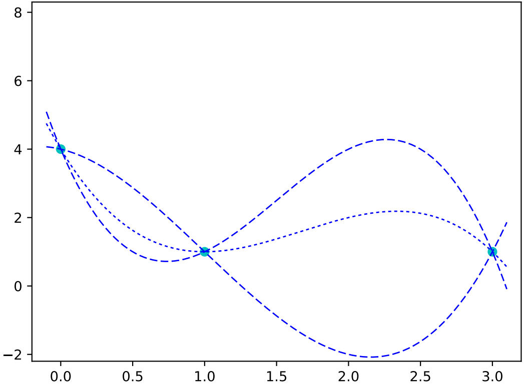
Interpolating space E



- Data : 3 points

- E : polynomials of degree 2
  $\rightarrow$ unique solution

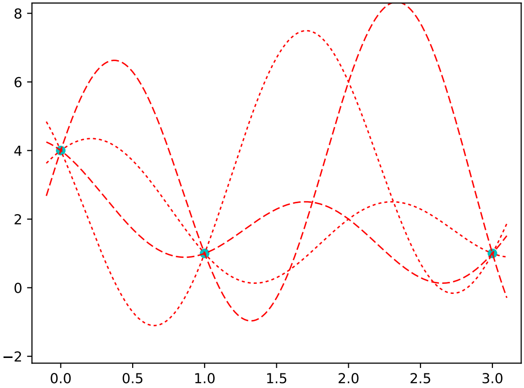
Interpolating space *E*



- Data : 3 points

- E : polynomials of degree 3
  $\rightarrow$ infinity of solutions

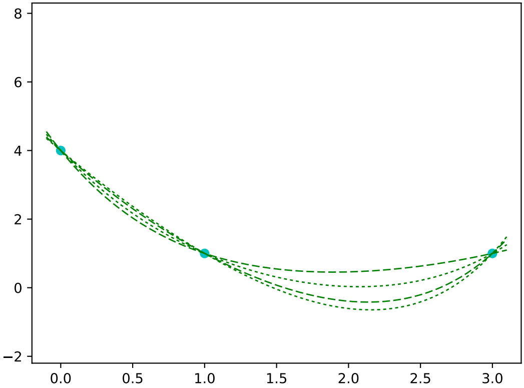
Interpolating space E



- Data : 3 points

- Trigonometric functions
  $E = \text{Vect}\left\{1, \cos(\pi x), \sin(\pi x), \cos(\frac{\pi}{2}x), \sin(\frac{\pi}{2}x)\right\}$
  $\rightarrow$ infinity of solutions

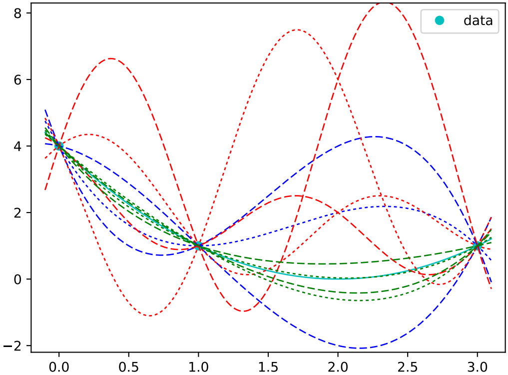
Interpolating space E



- Data : 3 points

- Exponential functions
  $E = \text{Vect}\left\{1, x, \exp(x), \exp(-x)\right\}$
  $\rightarrow$ infinity of solutions

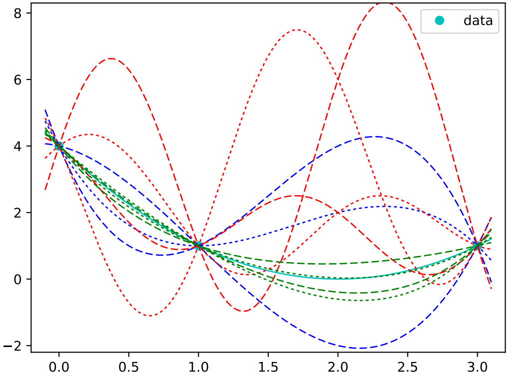
Interpolating space E



- All solutions together

- Data : 3 points

Interpolating space E

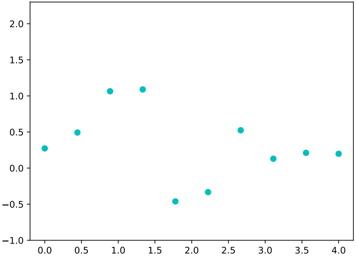


- Data : 3 points

- Choice of E :
  $\rightarrow$ Uniqueness
  $\rightarrow$ Solution easy to evaluate ? Cost ?
  $\rightarrow$ Ability to fit data, etc...

Interpolating data (1)

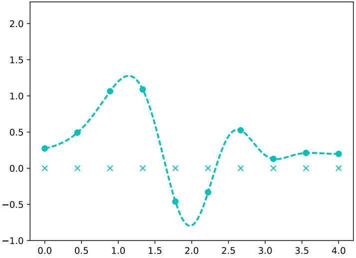
- Interpolation data from constraints, measurements...



- Data (x_i, y_i), $0 \le i \le n$
  $\rightarrow$ sampling of an underlying unknown law
- Interpolant P_n such that

$$P_n(x_i) = y_i$$

- Interpolation data from a function f

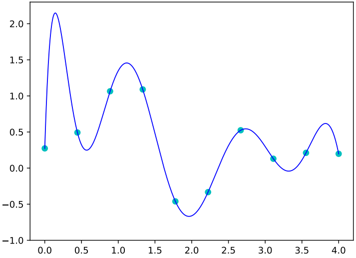


- Data $(x_i, y_i = f(x_i))$, $0 \le i \le n$
  $\rightarrow$ sampling of f
- Interpolant $P_n(.,f)$ such that

$$P_n(x_i, f) = f(x_i)$$

Interpolating data (2)

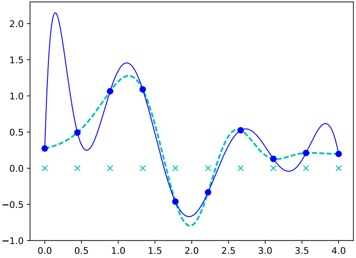
- Interpolation data from constraints, measurements...



- Data (x_i, y_i), $0 \leq i \leq n$
  $\rightarrow$ sampling of an underlying unknown law
- Interpolant P_n such that

$$P_n(x_i) = y_i$$

- Interpolation data from a function f

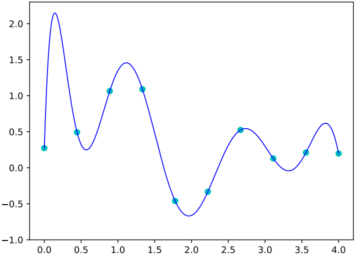


- Data $(x_i, y_i = f(x_i))$, $0 \leq i \leq n$
  $\rightarrow$ sampling of f
- Interpolant $P_n(.,f)$ such that

$$P_n(x_i, f) = f(x_i)$$

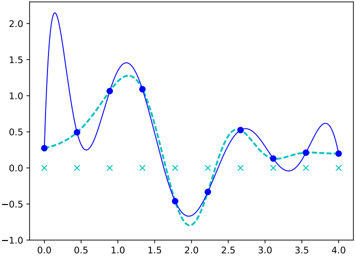
Interpolating data (3)

- Interpolation data from constraints, measurements...



Assumptions on the underlying unknown law are necessary for the qualitative study of the interpolation process

- Interpolation data from a function f



Properties of the function f allow qualitative study of the interpolation process

$\rightarrow$ $P_n(.,f)$ is an interpolant of the function f at points x_i

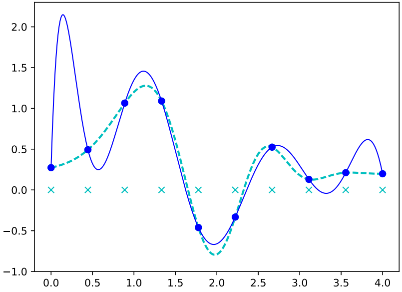
Objective : *Polynomial Lagrange interpolation* consists in determining the polynomial of the lowest possible degree passing through the points of the dataset.

- Data : $(x_i, y_i),\ 0 \leq i \leq n$ **or** $(x_i, f(x_i)),\ 0 \leq i \leq n$
  $\rightarrow n+1$ distinct interpolation points x_i
- Interpolating space E : $\mathbb{R}_n[x] = \{$ polynomials of degree $\leq n$ $\}$
  $\rightarrow \dim \mathbb{R}_n[x] = n+1$
- Problem : find a polynomial

$$P_n(x) \in \mathbb{R}_n[x] \ : \quad P_n(x_i) = y_i, \qquad 0 \leq i \leq n$$

or identically

$$P_n(x,f) \in \mathbb{R}_n[x] \ : \quad P_n(x_i,f) = f(x_i), \quad 0 \leq i \leq n$$



Existence and uniqueness of a solution

Consider the linear map Φ :

$$\begin{array}{ccc} \mathbb{R}_n[x] & \xrightarrow{\Phi} & \mathbb{R}^{n+1} \\ p & \longrightarrow & \Phi(p) = \big(p(x_0), p(x_1), \ldots, p(x_n)\big) \end{array}$$

Proposition 3.1

If points $x_0, x_1, \ldots, x_n$ are all distinct, the linear map Φ is bijective.

Proof

The kernel of Φ consists of polynomials of degree less than or equal to n which cancel at each point x_i, thus admitting $n+1$ zeros. Consequently this kernel only contains the zero polynomial and Φ is injective.
Finally, by the rank-nullity theorem

$$\dim\left(\mathbb{R}_n[x]\right) = \dim\ \mathrm{Ker}\left(\Phi\right) + \dim\ \mathrm{Im}\left(\Phi\right)$$

the linear map Φ is bijective.

Existence and uniqueness of a solution

Proposition 3.2

For any family of $n+1$ real numbers $y_0, y_1, \ldots, y_n$, there exists a unique polynomial $p \in \mathbb{R}_n[x]$ satisfying the constraints

$$p(x_i) = y_i, \quad i = 0, 1, \ldots, n.$$

This polynomial is called the Lagrange interpolating polynomial (or simply, the interpolating polynomial) of the data (x_i, y_i).

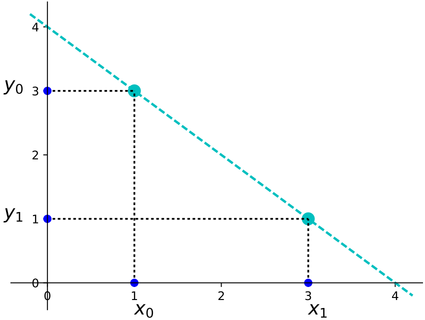
- For example, there exists
  - a unique straight line passing through 2 points with distinct abscissa,
  - a unique parabola passing through 3 points with distinct abscissa, ...
- It is not required for the ordinates to be distinct : *there exists a unique polynomial in $\mathbb{R}_n[x]$ passing through the $n+1$ points $(x_i, y_i = 1)$ where all x_i are distinct.*
- The unique interpolating polynomial of $n+1$ data can be of degree strictly less than n.

Determination of the unique interpolating polynomial

$\rightarrow$ depends on the basis selected to express this polynomial :
$\rightarrow$ monomial basis : general basis
$\rightarrow$ Lagrange basis : specifically dedicated to the Lagrange interpolation
$\rightarrow$ Newton basis : specifically dedicated to the Lagrange interpolation

An example :

- Monomial basis : $\{1, x\}$
  $P_1(x) = \frac{y_0\,x_1 - y_1\,x_0}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0}\,x$
  $P_1(x) = 4 - x$
- Lagrange basis : $\left\{\frac{x - x_1}{x_0 - x_1}, \frac{x - x_0}{x_1 - x_0}\right\}$
  $P_1(x) = y_0\,\frac{x - x_1}{x_0 - x_1} + y_1\,\frac{x - x_0}{x_1 - x_0}$
  $P_1(x) = 3\,\frac{x - 3}{-2} + \frac{x - 1}{2}$
- Newton basis : $\{1, x - x_0\}$
  $P_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$
  $P_1(x) = 3 - (x - 1)$



Let $p(x)$ be a n-degree polynomial expressed in the monomial basis

$$p(x) = a_0 + a_1\,x + a_2\,x^2 + \cdots + a_n\,x^n$$

The $n+1$ interpolation constraints $p(x_i) = y_i$ lead to the $(n+1) \times (n+1)$ linear system

$$\left\{ \begin{array}{ccccccccc} a_0 & + & a_1\,x_0 & + & a_2\,x_0^2 & +\cdots+ & a_n\,x_0^n & = & y_0 \\ a_0 & + & a_1\,x_1 & + & a_2\,x_1^2 & +\cdots+ & a_n\,x_1^n & = & y_1 \\ a_0 & + & a_1\,x_2 & + & a_2\,x_2^2 & +\cdots+ & a_n\,x_2^n & = & y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_0 & + & a_1\,x_n & + & a_2\,x_n^2 & +\cdots+ & a_n\,x_n^n & = & y_n \end{array} \right.$$

which can be written in matrix form

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

The matrix of this system is called the Vandermonde matrix. Its determinant (the Vandermonde determinant) is

$$\prod_{0\leq i<j\leq n}(x_j - x_i)$$

which is obviously non zero if and only if the points x_i are all distinct (*which proves again the existence and the uniqueness of a solution*).

- Unfortunately, the numerical resolution of this Vandermonde system is *intricate, costly and numerically unstable*.
  Indeed, this linear system is *ill-conditioned*, which means that a small error on the coefficients or on the second member leads to a major error on the solution of the system.
- In conclusion, *the monomial basis is not recommended* for the calculation of the interpolating polynomial.
- Note that the evaluation of a polynomial expressed in the canonical basis has to be performed by means of the Horner scheme.

Algorithms involving polynomials, and in particular plotting the graph of a polynomial function over an interval $[a, b]$, require the evaluation of $p(t_i)$ for many values $t_i = a + i * (b - a)/N$, $i = 0, 1, \ldots, N$, which justify the interest of developing efficient algorithms for such evaluations.

- Naive algorithm : requires $2n$ multiplications for the evaluation of $p(x)$. Let

$$p(x) = a_0 + a_1\, x + a_2\, x^2 + \cdots + a_n\, x^n$$

```
#initialization data :
#  x = a real number
#  a = (a[0],a[1],...,a[n]) = vector of polynomial coeffs
xn = 1
p = a[0]
for i = 1 until n do          # n steps
    xn = xn * x               # 1 multiplication
    p = p + a[i] * xn         # 1 multiplication
endfor
return p                      # p = p(x)
```

Monomial form — Horner scheme

Algorithms involving polynomials, and in particular plotting the graph of a polynomial function over an interval $[a, b]$, require the evaluation of $p(t_i)$ for many values $t_i = a + i * (b - a)/N$, $i = 0, 1, \ldots, N$, which justify the interest of developing efficient algorithms for such evaluations.

- Horner scheme : requires only n multiplications for the evaluation of $p(x)$. The Horner scheme (also named the Ruffini-Horner's method) is based on the following factorization, illustrated here with a polynomial of degree 4

$$
\begin{aligned}
p(x) = &\ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\
= &\ a_0 + x\left(a_1 + x\left(a_2 + x\left(a_3 + x\left(a_4\right)\right)\right)\right)
\end{aligned}
$$

```
#initialization data :
#   x = a real number
#   a = (a[0],a[1],...,a[n]) = vector of polynomial coeffs
p = a[n]
for i = n-1 until 0 with step = -1 do    # n steps
    p = a[i] + x * p                     # 1 multiplication
endfor
return p                                 # p = p(x)
```

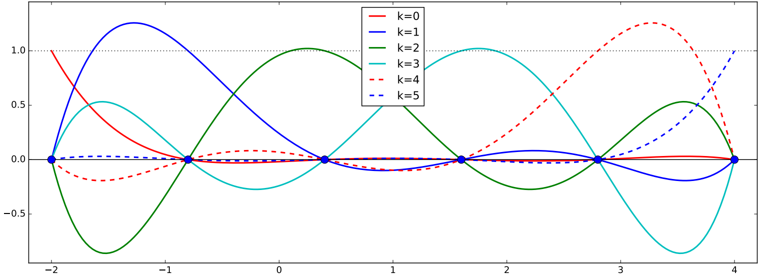
Lagrange polynomials $L_i(x)$ associated with the $n+1$ distinct points

$$x_0,\ x_1,\ x_2,\ldots,\ x_n,$$

are defined by

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}, \qquad 0 \le i \le n.$$

For $n = 0$, the unique Lagrange polynomial is $L_0(x) = 1,\ x \in \mathbb{R}$.



The six Lagrange polynomials $L_k(x)$ of degree 5, associated with 6 points uniformly distributed in the interval $[-2, 4]$.

Proposition 3.3

For $0 \leq i, j \leq n$, *we have*

$$L_i(x_j) = \delta_{ij}.$$

Proposition 3.4 (Lagrange basis)

The $n+1$ *Lagrange polynomials* $L_i(x)$, $0 \leq i \leq n$, *form a basis of* $\mathbb{R}_n[x]$, *referred as the Lagrange basis (relative to points* x_i *or associated with points* x_i*).*

Proof

Clearly $L_i(x) \in \mathbb{R}_n[x]$ *for each* i.
Then, since $\dim \mathbb{R}_n[x] = n+1$, *we just need to prove that the* $n+1$ *polynomials* $L_i(x)$ *are linearly independent. Consider thus a linear combination equal to zero :*

$$\sum_{i=0}^{n} \alpha_i L_i(x) = 0, \quad \forall x \in \mathbb{R}.$$

For $x = x_j$, $0 \leq j \leq n$, *we have* $\sum_{i=0}^{n} \alpha_i L_i(x_j) = \sum_{i=0}^{n} \alpha_i \delta_{ij} = \alpha_j = 0$. *We thus deduce that* $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$, *which proves that polynomials* $L_i(x)$ *are linearly independent and thus form a basis of* $\mathbb{R}_n[x]$.

Proposition 3.5

The interpolating polynomial $p(x)$ *associated with data* (x_i, y_i) *is expressed in the Lagrange basis as follows*

$$p(x) = \sum_{i=0}^{n} y_i \, L_i(x).$$

Proof

A straightforward inspection shows that $p(x) \in \mathbb{R}_n[x]$ *and that* $p(x_i) = y_i$ *for* $i = 0, \ldots, n$

This polynomial is named the *interpolating polynomial in the Lagrange basis*.

- Despite the simple writing of the interpolating polynomial, the basis of Lagrange is not always the best suited to calculations.
- Indeed, each polynomial $L_k(x)$ depends on the set of all interpolation points x_i.
- The use of Lagrange basis is relevant when several interpolations associated with the same interpolation data set x_i have to be performed (which means that only data y_i change).
- If an interpolation point x_i is added or modified, all calculations must be fully resumed.

Lagrange form — Example

Consider the function $f(x) = 4 - x - 4x^2 + x^3$ and the following interpolation data set.

$x_0 = 1$	$x_1 = 3$	$x_2 = -1$	$x_3 = 2$
$y_0 = f(x_0) = 0$	$y_1 = f(x_1) = -8$	$y_2 = f(x_2) = 0$	$y_3 = f(x_3) = -6$

We first interpolate f at point x_0. Then we successively add interpolation points x_1, x_2 and x_3. We thus get the three interpolating polynomials $P_k(x,f)$ ($k = 0, 1, 2, 3$), associated with data $(x_i, f(x_i))$, $0 \leq i \leq k$.

$$P_0(x,f) = y_0 L_0(x) = y_0 = 0 \qquad = 0$$

$$P_1(x,f) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

$$= 0 \frac{x-3}{1-3} + (-8) \frac{x-1}{3-1} \qquad = 4 - 4x$$

$$P_2(x,f) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= 0 \frac{(x-3)(x+1)}{-4} + (-8) \frac{(x-1)(x+1)}{8} + 0 \frac{(x-1)(x-3)}{8} \qquad = 1 - x^2$$

$$P_3(x,f) = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \qquad = 4 - x - 4x^2 + x^3$$

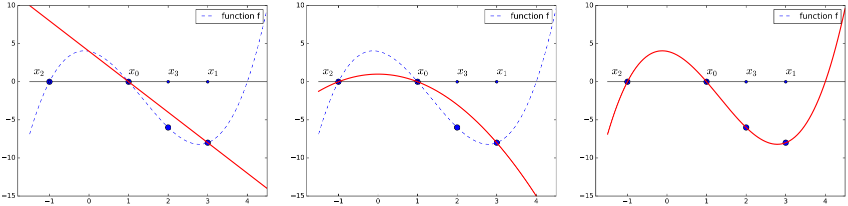
Lagrange form — Example

Consider the function $f(x) = 4 - x - 4x^2 + x^3$ and the following interpolation data set.

$x_0 = 1$	$x_1 = 3$	$x_2 = -1$	$x_3 = 2$
$y_0 = f(x_0) = 0$	$y_1 = f(x_1) = -8$	$y_2 = f(x_2) = 0$	$y_3 = f(x_3) = -6$

We first interpolate f at point x_0. Then we successively add interpolation points x_1, x_2 and x_3. We thus get the three interpolating polynomials $P_k(x,f)$ ($k = 0, 1, 2, 3$), associated with data $(x_i, f(x_i))$, $0 \leq i \leq k$.

- Notice that $P_3(x,f) = f(x)$, which is natural since these two polynomials of degree 3 coincide in 4 distinct values
- There is no apparent link in polynomial expressions of $P_k(x,f)$ and $P_{k+1}(x,f)$ in Lagrange basis



$P_1(x,f) = 4 - 4x$ $\quad$ $P_2(x,f) = 1 - x^2$ $\quad$ $P_3(x,f) = 4 - x - 4x^2 + x^3$

Proposition 3.6 (Additional properties of Lagrange polynomials)

1. *Partition of unity :*

$$\sum_{i=0}^{n} L_i(x) = 1$$

2. *For* $1 \leq j \leq n$ *:*

$$\sum_{i=0}^{n} x_i^j \, L_i(x) = x^j$$

3. *For* $1 \leq j \leq n$ *:*

$$\sum_{i=0}^{n} (x - x_i)^j \, L_i(x) = 0$$

4. *Derivatives for* $0 \leq i \leq n$ *:*

$$L_i'(x) \; = \; L_i(x) \sum_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x - x_j} = \frac{1}{\prod\limits_{j=0,\, j \neq i}^{n} (x_i - x_j)} \sum_{\substack{j=0 \\ j \neq i}}^{n} \left(\prod_{\substack{k=0 \\ k \neq i,j}}^{n} (x - x_k) \right) \tag{1}$$

Specify the values of the derivatives $L_i'(x_i)$ *and* $L_i'(x_j)$, $j \neq i$.

Newton polynomials associated with the $n+1$ distincts points $x_0, x_1, x_2, \ldots, x_n$, are defined as follows :

$$
\begin{aligned}
N_0(x) &= 1, \\
N_1(x) &= x - x_0, \\
N_2(x) &= (x - x_0)(x - x_1), \\
&\vdots \\
N_n(x) &= (x - x_0)(x - x_1)\cdots(x - x_{n-1}).
\end{aligned}
$$

These polynomials are thus determined by the first polynomial $N_0(x)$ and the recurrence relation

$$N_{k+1}(x) = (x - x_k)\,N_k(x), \quad k = 0, 1, \ldots, n-1.$$

- Note that the definition of each polynomial $N_k(x)$ does not involve the *interpolation point* x_k.
- In particular, point x_n does not participate in the definition of the whole base. *But of course*, the point x_n will contribute to the computation of the interpolation polynomial with respect to the Newton base.

Proposition 3.7 (Newton basis)

Newton polynomials

$$N_i(x)\,, \quad 0 \leq i \leq n,$$

form a basis of $\mathbb{R}_n[x]$, *referred as the Newton basis relative to (or associated with) points* x_i.

Proof

Since $\dim \mathbb{R}_n[x] = n+1$, *we just need to prove that the* $n+1$ *polynomials* $N_i(x)$ *are linearly independent. Consider thus a linear combination equal to zero for all* x

$$\sum_{i=0}^{n} \alpha_i\, N_i(x) = 0, \quad \forall\, x \in \mathbb{R}.$$

For $x = x_0$, *we get* $\sum_{i=0}^{n} \alpha_i\, N_i(x_0) = \alpha_0 = 0$.
Then, considering successively values $x_1, x_2, \ldots, x_n$, *we deduce that* $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, *which proves that the family of polynomials* $N_i(x)$ *is linearly independent and thus is a basis of* $\mathbb{R}_n[x]$.

Objective : determination of the coefficients $\delta_0, \delta_1, \ldots, \delta_n$ such that the polynomial

$$P_n(x) = \delta_0 N_0(x) + \delta_1 N_1(x) + \delta_2 N_2(x) + \cdots + \delta_n N_n(x)$$

satisfies $P_n(x_i) = y_i, \quad i = 0, 1, \ldots, n.$

Determination of the first coefficients $\delta_0, \delta_1, \delta_2$:

Interpolating polynomial	constraints	coefficients δ_k
$P_0(x) = \delta_0^0 N_0(x) = \delta_0^0$	$P_0(x_0) = y_0$	$\delta_0^0 = y_0$
$P_1(x) = \delta_0^1 N_0(x) + \delta_1^1 N_1(x) = \delta_0^1 + \delta_1^1 (x - x_0)$	$P_1(x_0) = y_0$ $P_1(x_1) = y_1$	$\delta_0^1 = \delta_0^0$ $\delta_1^1 = \frac{y_1 - y_0}{x_1 - x_0}$
$P_2(x) = \delta_0^2 N_0(x) + \delta_1^2 N_1(x) + \delta_2^2 N_2(x) = \delta_0^2 + \delta_1^2 (x - x_0) + \delta_2^2 (x - x_0)(x - x_1)$	$P_2(x_0) = y_0$ $P_2(x_1) = y_1$ $P_2(x_2) = y_2$	$\delta_0^2 = \delta_0^0$ $\delta_1^2 = \delta_1^1$ $\delta_2^2 = \dfrac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$

• Calculation detail :

$$P_2(x_2) = y_2 \quad \Rightarrow y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0) + \delta_2^2 (x_2 - x_0)(x_2 - x_1) = y_2$$

$$\Rightarrow \delta_2^2 (x_2 - x_0)(x_2 - x_1)(x_1 - x_0) = (y_2 - y_1 + y_1 - y_0)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_1 + x_1 - x_0)$$
$$= (y_2 - y_1)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_1)$$

Newton form — Identification of the interpolant

Objective : determination of the coefficients δ_0, δ_1,. . .,δ_n such that the polynomial

$$P_n(x) = \delta_0\, N_0(x) \;+\; \delta_1\, N_1(x) \;+\; \delta_2\, N_2(x) \;+\cdots+\; \delta_n\, N_n(x)$$

satisfies $P_n(x_i) = y_i, \quad i = 0, 1, \ldots, n.$

Determination of the first coefficients $\delta_0, \delta_1, \delta_2$ (summary) :

	order 0	order 1	order 2
x_0	$y_0 = \delta_0$		
x_1	y_1	$\frac{y_1-y_0}{x_1-x_0} = \delta_1$	
x_2	y_2	$\frac{y_2-y_1}{x_2-x_1}$	$\dfrac{\frac{y_2-y_1}{x_2-x_1} - \frac{y_1-y_0}{x_1-x_0}}{x_2-x_0} = \delta_2$

$$P_2(x) = \underbrace{\underbrace{\underbrace{\delta_0 \,.\, \overbrace{1}^{N_0(x)}}_{P_0(x)} + \delta_1 \overbrace{(x-x_0)}^{N_1(x)}}_{P_1(x)} + \delta_2 \overbrace{(x-x_0)(x-x_1)}^{N_2(x)}}_{P_2(x)}$$

Objective : determination of the coefficients δ_0, δ_1,. . .,δ_n such that the polynomial

$$P_n(x) = \delta_0 N_0(x) + \delta_1 N_1(x) + \delta_2 N_2(x) + \cdots + \delta_n N_n(x)$$

satisfies $P_n(x_i) = y_i, \quad i = 0, 1, \ldots, n.$

Determination of the first coefficients $\delta_0, \delta_1, \delta_2$ (summary & notation) :

	order 0	order 1	order 2
x_0	$y_0 = \delta[x_0]$		
x_1	$y_1 = \delta[x_1]$	$\frac{y_1 - y_0}{x_1 - x_0} = \delta[x_0, x_1]$	
x_2	$y_2 = \delta[x_2]$	$\frac{y_2 - y_1}{x_2 - x_1} = \delta[x_1, x_2]$	$\dfrac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} = \delta[x_0, x_1, x_2]$

$$P_2(x) = \underbrace{\underbrace{\underbrace{\delta[x_0]\,.\,\overbrace{1}^{N_0(x)}}_{P_0(x)} + \delta[x_0, x_1]\,\overbrace{(x - x_0)}^{N_1(x)}}_{P_1(x)} + \delta[x_0, x_1, x_2]\,\overbrace{(x - x_0)(x - x_1)}^{N_2(x)}}_{P_2(x)}$$

Interpolation in Newton basis : coefficients determination

- Interpolating data $(x_i, y_i),\ i = 0, 1, \dots$
- $P_k(x)$ is the interpolating polynomial of data $(x_i, y_i),\ i = 0, \dots, k$

Proposition 3.8 (Newton formula)

For any $k \geq 1$, we have

$$\begin{aligned} P_k(x) &= P_{k-1}(x) + \delta[x_0, \dots, x_k]\, N_k(x) \\ &= \underbrace{\delta[x_0]\, N_0(x) + \delta[x_0, x_1]\, N_1(x) + \cdots + \delta[x_0, \dots, x_{k-1}]\, N_{k-1}(x)}_{P_{k-1}(x)} + \delta[x_0, \dots, x_k]\, N_k(x) \end{aligned} \tag{2}$$

with coefficients $\delta[x_0, \dots, x_j]$ of the main diagonal of the table of divided differences (of order k) defined as follows

	order 0	order 1	order 2		$\cdots$		order k
x_0	$y_0 = \delta[x_0]$						
x_1	$y_1 = \delta[x_1]$	$\delta[x_0, x_1]$					
x_2	$y_2 = \delta[x_2]$	$\delta[x_1, x_2]$	$\delta[x_0, x_1, x_2]$				
x_3	$y_3 = \delta[x_3]$	$\delta[x_2, x_3]$	$\delta[x_1, x_2, x_3]$	$\delta[x_0, x_1, x_2, x_3]$			
$\cdots$	$\cdots$				$\cdots$		
x_{k-1}						$\delta[x_0, \dots, x_{k-1}]$	
x_k	$y_k = \delta[x_k]$	$\delta[x_{k-1}, x_k]$	$\cdots$	$\cdots$		$\delta[x_1, \dots, x_k]$	$\delta[x_0, \dots, x_k]$

with $\delta[x_j, x_{j+1}, \dots, x_{j+p}] := \dfrac{\delta[x_{j+1}, \dots, x_{j+p}] - \delta[x_j, \dots, x_{j+p-1}]}{x_{j+p} - x_j}, \quad 1 \leq p \leq k,\ 0 \leq j \leq k - p$

Proof of Newton formula

Step 1.

Proposition 3.9 (Neville formula)

Let $P_{i,p}(x)$ be the interpolating polynomial at points $x_i, x_{i+1}, \ldots, x_{i+p}$

$$P_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} P_{i+1,p-1}(x) + \frac{x_{i+p} - x}{x_{i+p} - x_i} P_{i,p-1}(x) \qquad \text{for} \quad \begin{array}{l} 1 \leq p \leq n \\ 0 \leq i \leq n-p \end{array}$$

Proof (of Neville formula)

Both sides of Neville formula are of degree p and agree for the $p+1$ points $x_i, x_{i+1}, \ldots, x_{i+p}$, which proves the result by uniqueness of the interpolating polynomial.

Step 2.

Proof of relation $\boxed{P_k(x) = P_{k-1}(x) + \delta_k\, N_k(x)}$:

By definition we have $P_k(x_j) = P_{k-1}(x_j) = y_j$ for $j = 0, \ldots, k-1$,
which proves that $P_k(x) - P_{k-1}(x)$ is a multiple of $(x - x_0)\cdots(x - x_{k-1}) = N_k(x)$.
Finally, by degree analysis we get

$$P_k(x) - P_{k-1}(x) = \delta_k\, N_k(x) \quad \text{with } \delta_k \in \mathbb{R}$$

Step 3. Proof of Newton formula by induction on the degree k

- ◇ Newton formula has already been verified for $k = 0$, $k = 1$, $k = 2$
- ◇ Assume that the formula holds for any interpolating polynomial of degree $k-1$, with $k-1 \geq 0$, which allows to write

$$P_{0,k-1}(x) = P_{0,k-2}(x) + \delta[x_0, \ldots, x_{k-1}]\,(x-x_0)\cdots(x-x_{k-2})$$
$$P_{1,k-1}(x) = P_{1,k-2}(x) + \delta[x_1, \ldots, x_k]\,(x-x_1)\cdots(x-x_{k-1})$$

Then, by Neville's formula, we get

$$\begin{aligned}
P_k(x) = P_{0,k}(x) &= \frac{x-x_0}{x_k-x_0} P_{1,k-1}(x) + \frac{x_k-x}{x_k-x_0} P_{0,k-1}(x) \\
&= \frac{x-x_0}{x_k-x_0}\Big(P_{1,k-2}(x) + \delta[x_1,\ldots,x_k]\,(x-x_1)\cdots(x-x_{k-1})\Big) \\
&+ \frac{x_k-x}{x_k-x_0}\Big(P_{0,k-2}(x) + \delta[x_0,\ldots,x_{k-1}]\,(x-x_0)\cdots(x-x_{k-2})\Big) \\
&= \frac{(x-x_0)P_{1,k-2}(x) + (x_k-x)P_{0,k-2}(x)}{x_k-x_0} \\
&+ \frac{\delta[x_1,\ldots,x_k]\,(x-x_0)(x-x_1)\cdots(x-x_{k-1})}{x_k-x_0} \\
&+ \frac{\delta[x_0,\ldots,x_{k-1}]\,(x-x_0)\cdots(x-x_{k-2})(x_k - x_{k-1} + x_{k-1} - x)}{x_k-x_0} \\
&= \underbrace{\cdots\cdots}_{\text{degree}\,k-1} + \underbrace{\frac{\delta[x_1,\ldots,x_k] - \delta[x_0,\ldots,x_{k-1}]}{x_k-x_0}}_{\delta[x_0,\ldots,x_k]} N_k(x)
\end{aligned}$$

which concludes the proof

Example

Consider the function $f(x) = 4 - x - 4x^2 + x^3$ and the following interpolation data set.

$x_0 = 1$	$x_1 = 3$	$x_2 = -1$	$x_3 = 2$
$y_0 = f(x_0) = 0$	$y_1 = f(x_1) = -8$	$y_2 = f(x_2) = 0$	$y_3 = f(x_3) = -6$

We determine interpolating polynomials $P_k(x,f)$ for $k = 0, 1, 2, 3$, associated with data $(x_i, f(x_i))$, $0 \leq i \leq k$.

We first calculate the associated divided differences that are noted in that case $f[x_j, \ldots, x_{j+p}]$ instead of $\delta[x_j, \ldots, x_{j+p}]$

	order 0	order 1	order 2	order 3
$x_0 = 1$	$y_0 = 0 = f[x_0]$			
$x_1 = 3$	$y_1 = -8 = f[x_1]$	$\frac{-8-0}{3-1} = -4 = f[x_0, x_1]$		
$x_2 = -1$	$y_2 = 0 = f[x_2]$	$\frac{0-(-8)}{-1-3} = -2 = f[x_1, x_2]$	$\frac{-2-(-4)}{-1-1} = -1 = f[x_0, x_1, x_2]$	
$x_3 = 2$	$y_3 = -6 = f[x_3]$	$\frac{-6-0}{2-(-1)} = -2 = f[x_2, x_3]$	$\frac{-2-(-2)}{2-3} = 0 = f[x_1, x_2, x_3]$	$\frac{0-(-1)}{2-1} = 1 = f[x_0, \ldots, x_3]$

$$
\begin{aligned}
P_0(x,f) &= f[x_0]\, N_0(x) && = 0 \\
P_1(x,f) &= P_0(x,f) + f[x_0,x_1]\, N_1(x) && = 0 + (-4)\,(x-1) \\
P_2(x,f) &= P_1(x,f) + f[x_0,x_1,x_2]\, N_2(x) && = 0 + (-4)\,(x-1) + (-1)\,(x-1)(x-3) \\
P_3(x,f) &= P_2(x,f) + f[x_0,x_1,x_2,x_3]\, N_3(x) && = 0 + (-4)\,(x-1) + (-1)\,(x-1)(x-3) + 1\,(x-1)(x-3)(x+1)
\end{aligned}
$$

Proposition 3.10 (Additional properties of DD)

1. *Expanded form :*
$$\delta[x_0, x_1, \ldots, x_k] = \sum_{j=0}^{k} \frac{y_j}{\displaystyle\prod_{i=0, i\neq j}^{k} (x_j - x_i)}, \qquad 0 \leq k \leq n.$$
2. *Symmetry : divided differences do not depend on the order of the data, that is*
$$\delta[x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(k)}] = \delta[x_0, x_1, \ldots, x_k]$$
for all permutation σ of the set $\{0, 1, \ldots, k\}$, $0 \leq k \leq n$.
3. *Linearity : if $f = \lambda\, g + \mu\, h$, where g and h are two functions defined on an interval $[a, b]$ which contains all the points x_i, then*
$$f[x_0, x_1, \ldots, x_k] = \lambda\, g[x_0, x_1, \ldots, x_k] + \mu\, h[x_0, x_1, \ldots, x_k], \quad 0 \leq k \leq n.$$
4. *Derivatives and mean value theorem.*
Assuming that the function f is sufficiently smooth, we have
$$f[x_0, x_1, \ldots, x_k] = \frac{f^{(k)}(\xi)}{k!} \quad \textit{where } \xi \in (\min(x_i), \max(x_i)), \quad 0 \leq k \leq n.$$

Horner scheme for Newton basis

Exercise 3.1 — Write the Horner scheme for a polynomial expressed in Newton basis. For this purpose, consider the following factorization of a polynomial expressed in Newton basis (relative to interpolation points x_i)

$$\begin{aligned} P_3(x) &= \delta_0\, N_0(x) \,+\, \delta_1\, N_1(x) \,+\, \delta_2\, N_2(x) \,+\, \delta_3\, N_3(x) \\ &= \delta_0 \,.\, 1 \,+\, \delta_1\,(x-x_0) \,+\, \delta_2\,(x-x_0)(x-x_1) \,+\, \delta_3\,(x-x_0)(x-x_1)(x-x_2) \\ &= \delta_0 \,+\, (x-x_0)\Big(\delta_1 + (x-x_1)\big(\delta_2 + (x-x_2)(\delta_3)\big)\Big) \end{aligned}$$

```
#initialization data :
#  x = a real number
#  d = (d[0],d[1],...,d[n]) = vector of polynomial coeffs (DD)
#  xi = (xi[0],xi[1],...,xi[n]) = vector of interpolating points
p = ...                          # initialization
for ...
    ...
endfor
return p                         # p = p(x)
```

Calculation of divided differences

Exercise 3.2 —

1) Write a Python function `diffdiv(xi,yi)` enabling the computation of divided differences with a one-dimensional vector `delta`.

`Input data :` `(xi,yi)`, $i = 0, 1, \ldots, n$

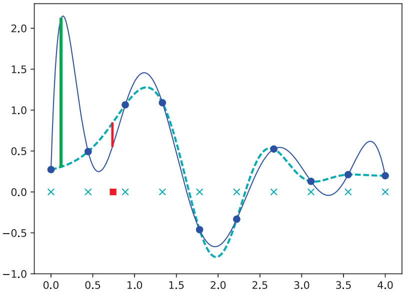
`Output data :` `delta` $= (\underbrace{\delta[x_0], \delta[x_0, x_1], \ldots, \delta[x_0, x_1, \ldots, x_n]}_{\text{divided differences}}, \underbrace{\ldots, \delta[x_{n-1}, x_n], \delta[x_n]}_{\text{for later updates}})$

2) Write a Python function `updateDD(xi,yi,delta,xnew,ynew)` enabling the update of the vector `delta` of divided differences in case of a new data `(xnew,ynew)`.

Hints : (a) Duplicate the vector `yi` into the vector `delta` by inserting a zero between each consecutive element y_i and y_{i+1}, so as to get a vector of size $2n+1$. (b) Take inspiration from the table below, where each calculated divided difference is stored at its line index.

`delta`			
$y_0 = \delta[x_0]$			
0	$\leftarrow \delta[x_0, x_1]$		
$y_1 = \delta[x_1]$		$\leftarrow \delta[x_0, x_1, x_2]$	
0	$\leftarrow \delta[x_1, x_2]$		...
$y_2 = \delta[x_2]$		$\leftarrow \delta[x_1, x_2, x_3]$	
0	$\leftarrow \delta[x_2, x_3]$		...
$y_3 = \delta[x_3]$		$\leftarrow \delta[x_2, x_3, x_4]$	
0	$\leftarrow \delta[x_3, x_4]$		...
$\vdots$	$\vdots$		
0			...
$y_{n-1} = \delta[x_{n-1}]$		$\leftarrow \delta[x_{n-2}, x_{n-1}, x_n]$	
0	$\leftarrow \delta[x_{n-1}, x_n]$		
$y_n = \delta[x_n]$			

- Error :



$$E_n = \max_{x\in[a,b]} \left|P_n(x,f) - f(x)\right|$$

$$a = \min x_i\,,\ b = \max x_i$$

- Choice of interpolation data :

$$n \in \mathbb{N}\,, \quad \min_{x_i \in[a,b]} \left(E_n\right) = \min_{x_i \in[a,b]} \left(\max_{x\in[a,b]} \left|P_n(x,f) - f(x)\right|\right)$$

Optimal distribution : Chebyshev points

- Convergence :

$$\lim_{n\to\infty} E_n \quad ?$$

Proposition 3.11 (error bounds in Lagrange interpolation)

Let $f \in C^{n+1}[a,b]$, $n+1$ *distinct points* $x_0, x_1, \ldots, x_n$ *in* $[a,b]$ *and* $P_n(x,f)$ *the interpolating polynomial of* f *associated with these data points. Then,*

$$\forall x \in [a,b],\ \exists \xi_x \in]a,b[, \quad f(x) - P_n(x,f) = \frac{\prod_{i=0}^{i=n}(x-x_i)}{(n+1)!} f^{(n+1)}(\xi_x)$$

Thus,

$$\forall x \in [a,b], \qquad \left|f(x) - P_n(x,f)\right| \leq \frac{\left|\prod_{i=0}^{i=n}(x-x_i)\right|}{(n+1)!} \max_{a \leq \xi \leq b} \left|f^{(n+1)}(\xi)\right| \qquad (3)$$

Finally, with $\Pi_{n+1}(x) = \prod_{i=0}^{i=n}(x-x_i)$, *we get*

$$\boxed{\left|\left|f - P_n(.,f)\right|\right| := \max_{x \in [a,b]} \left|f(x) - P_n(x,f)\right| \leq \frac{\left|\left|\Pi_{n+1}\right|\right|}{(n+1)!} \left|\left|f^{(n+1)}\right|\right|} \qquad (4)$$

Proof (error bounds in Lagrange interpolation)

Let $x \in [a,b]$ be a real value different from each of the points x_i, and consider the function

$$\phi_x(u) = f(u) - P_n(u,f) - \big(f(x) - P_n(x,f)\big)\,\frac{\prod_{i=0}^{i=n}(u-x_i)}{\prod_{i=0}^{i=n}(x-x_i)}\,, \qquad u \in [a,b],$$

which is clearly of class C^{n+1} on $[a,b]$.
Function ϕ_x vanishes at each point x_i and at point x, and thus admits $n+2$ zeros in the interval $[a,b]$. Therefore, with Rolle's theorem we deduce by induction that the function $\phi^{(n+1)}$ vanishes at one point ξ in $]a,b[$, that is $\exists\,\xi \in]a,b[\,,\ \phi_x^{(n+1)}(\xi) = 0$.
The $(n+1)$-th derivative of the polynomial $P_n(u,f)$ is identically zero and the $(n+1)$-th derivative of the $(n+1)$-degree polynomial $\prod_{i=0}^{i=n}(u-x_i)$ is constant and equal to $(n+1)!$, so that

$$\phi_x^{(n+1)}(\xi) \;=\; f^{(n+1)}(\xi) - \Big(f(x) - P_n(x,f)\Big)\,\frac{(n+1)!}{\prod_{i=0}^{i=n}(x-x_i)} \;=\; 0\,,$$

which leads to

$$f(x) - P_n(x,f) = \frac{\prod_{i=0}^{i=n}(x-x_i)}{(n+1)!}\,f^{(n+1)}(\xi).$$

This relation is also valid for $x = x_i\,,\ i = 0,1,\dots,n$, which concludes the proof.

Error bounds in Lagrange interpolation — Main result

Formula

$$\left|\left|f - P_n(.,f)\right|\right| := \max_{x\in[a,b]} \left|f(x) - P_n(x,f)\right| \leq \frac{\left|\left|\Pi_{n+1}\right|\right|}{(n+1)!} \left|\left|f^{(n+1)}\right|\right|$$

shows that the error depends on the function f and on the norm $\left|\left|\Pi_{n+1}\right|\right|$ which is related to the repartition of the interpolation points x_i in the interval $[a,b]$.

- If we have no information on the distribution of points x_i, the best estimation is

$$\left|\left|\Pi_{n+1}\right|\right| \leq (b-a)^{n+1}.$$

- In case of evenly spaced points x_i in $[a,b]$, one can prove that

$$\left|\left|\Pi_{n+1}\right|\right| \leq \left(\frac{b-a}{e}\right)^{n+1}.$$

- The best possible distribution of the interpolation points consists in the Chebyshev points (that will be introduced later) and leads to the optimal estimation

$$\left|\left|\Pi_{n+1}\right|\right| \leq 2\left(\frac{b-a}{4}\right)^{n+1},$$

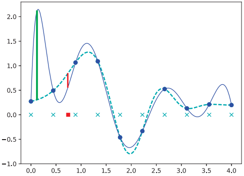
  which represents a substantial gain.

For example, for $n = 20$ the ratio $2\,(\frac{b-a}{4})^{n+1} \,/\, (\frac{b-a}{e})^{n+1} < 6.10^{-4}$, so that interpolation at Chebyshev points significantly improves the accuracy compared with interpolation at evenly spaced points.

- Error :

$$E_n = \max_{x \in [a,b]} \left| P_n(x,f) - f(x) \right|$$

$$a = \min x_i \,, \; b = \max x_i$$



- Choice of interpolation data :

$$n \in \mathbb{N}, \quad \min_{x_i \in [a,b]} \left(E_n \right) = \min_{x_i \in [a,b]} \left(\max_{x \in [a,b]} \left| f(x) - P_n(x,f) \right| \right)$$

$$\leq \min_{x_i \in [a,b]} \frac{|| \Pi_{n+1} ||}{(n+1)!} ||f^{(n+1)}||$$

$$\leq \frac{||f^{(n+1)}||}{(n+1)!} \min_{x_i \in [a,b]} || \Pi_{n+1} ||$$

- Convergence :

$$\lim_{n \to \infty} E_n \quad ?$$

Objective : determine the best distribution of interpolation points x_i in the interval $[a, b]$ so as to minimize the norm $|| \Pi_{n+1} ||$.

- *We first study the problem on the symmetric interval* $[-1, 1]$*, from which we then deduce the solution over any interval* $[a, b]$ *by an affine transformation.*
- *Statement of the problem :*
  determine a strictly increasing sequence of $n + 1$ points x_i :

$$-1 \leq x_0 < x_1 < \cdots < x_{n-1} < x_n \leq 1$$

so as to minimize the norm $|| \Pi_{n+1} ||$, that is

$$\min_{\substack{x_i \\ -1 \leq x_0 < \cdots < x_n \leq 1}} \left(\underbrace{\max_{x \in [-1,1]} \left| (x - x_0)(x - x_1) \cdots (x - x_n) \right|}_{|| \Pi_{n+1} ||} \right)$$

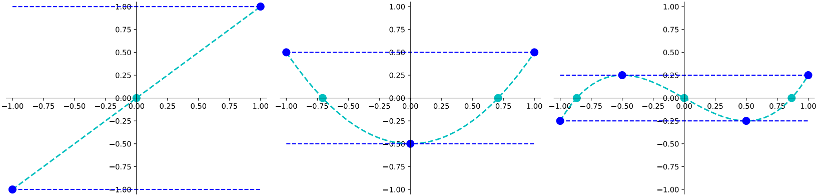
We will first look at some basic examples ($n = 0, n = 1, n = 2$). The general solution is based on Chebyshev [1] polynomials.

1. Pafnuty Lvovich Chebyshev (1821–1894), Russian mathematician.

Chebyshev points — Basic examples

Basic examples :

- Case $n = 0$: $\Pi_{0+1}(x) = (x - x_0)$ with $x_0 \in [-1, 1]$
  Solution : $x_0 = 0$ and $\Pi_{0+1}(x) = x - 0 = x$
- Case $n = 1$: $\Pi_{1+1}(x) = (x - x_0)(x - x_1)$ with $-1 \leq x_0 < x_1 \leq 1$
  Solution : $x_0 = -\frac{1}{\sqrt{2}}, x_1 = \frac{1}{\sqrt{2}}$ and $\Pi_{1+1}(x) = (x + \frac{1}{\sqrt{2}})(x - \frac{1}{\sqrt{2}}) = x^2 - \frac{1}{2}$
- Case $n = 2$: $\Pi_{2+1}(x) = (x - x_0)(x - x_1)(x - x_2)$ with $-1 \leq x_0 < x_1 < x_2 \leq 1$
  Solution : $x_0 = -\frac{\sqrt{3}}{2}, x_1 = 0, x_2 = \frac{\sqrt{3}}{2}$
  and $\Pi_{2+1}(x) = (x + \frac{\sqrt{3}}{2})\, x\, (x - \frac{\sqrt{3}}{2}) = x^3 - \frac{3}{4}x$



$\Pi_{0+1}(x)$
$= x - 0$
$= x - \cos\frac{\pi}{2}$

$\Pi_{1+1}(x)$
$= (x - \frac{1}{\sqrt{2}})(x + \frac{1}{\sqrt{2}})$
$= (x - \cos\frac{\pi}{4})(x - \cos\frac{3\pi}{4})$

$\Pi_{2+1}(x)$
$= (x - \frac{\sqrt{3}}{2})(x - 0)(x + \frac{\sqrt{3}}{2})$
$= (x - \cos\frac{\pi}{6})(x - \cos\frac{3\pi}{6})$
$(x - \cos\frac{5\pi}{6})$

Note that for each optimal solution $\Pi_{n+1}(x)$, $(n = 0, 1, 2)$, the maximum is reached exactly $n + 2$ times over the interval $[-1, 1]$.

Chebyshev polynomials : *The study of Chebyshev polynomials is presented in the form of a problem.* The Chebyshev polynomial of degree n is defined by

$$T_n(x) = \cos\big(n \arccos(x)\big), \quad x \in [-1, 1].$$

1. Determine $T_0(x)$ and $T_1(x)$.
2. Compute $T_{n+1}(x) + T_{n-1}(x)$ and derive the following recursive relation
$$T_{n+1}(x) = 2\,x\,T_n(x) - T_{n-1}(x), \quad n \geq 1.$$
3. Compute the Chebyshev polynomials $T_n(x)$ for $n \leq 5$.
4. Deduce that $T_n(x)$ is a polynomial of degree n whose leading term is $2^{n-1}x^n$ for $n \geq 1$.
5. Show that $T_n(x)$ is even if n is even and is odd if n is odd.
6. Prove that $T_n(x)$ has n simple roots r_k in the interval $[-1, 1]$ and specify these roots as functions of n and k (*the Chebyshev points of order n relative to the interval* $[-1, 1]$).
7. Check that for all x in $[-1, 1]$, $|T_n(x)| \leq 1$, and show that $|T_n(x)| = 1$ for $n+1$ distinct values of x in $[-1, 1]$. Specify these values.
8. Demonstrate that Chebyshev polynomials are orthogonal relative to the following scalar product :
$$\int_{-1}^{1} T_n(x)\ T_p(x)\ \frac{1}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & \text{if } n \neq p \\ \pi & \text{if } n = p = 0 \\ \pi/2 & \text{if } n = p \neq 0 \end{cases}$$
Use the change of variable $x = \cos(u)$ *and we recall that* $\arccos'(x) = \frac{-1}{\sqrt{1-x^2}}$

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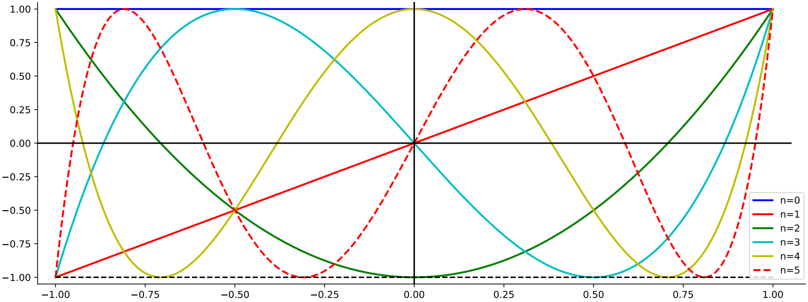
7. Check that for all x in $[-1, 1]$, $|T_n(x)| \leq 1$, and show that $|T_n(x)| = 1$ for $n + 1$ distinct values of x in $[-1, 1]$. Specify these values.



Property of minimum norm :

From, the previous study of Chebyshev polynomials we have

$$T_n(x) = 2^{n-1} \prod_{k=0}^{n-1} (x - r_k) \qquad \text{with} \quad r_k = \cos\left((2k+1)\frac{\pi}{2n}\right), \quad 0 \le k \le n-1 \,,$$

and

$$\left|T_n(x)\right| \;=\; 1 \qquad \Longleftrightarrow \qquad x = m_k = \cos\left(k\frac{\pi}{n}\right), \quad 0 \le k \le n \,,$$

with

$$T_n(m_k) \;=\; \cos(k\pi) \;=\; (-1)^k \,.$$

As already mentioned, the roots r_k are called the *Chebyshev points of order n* relative to the interval $[-1, 1]$.

Proposition 3.12 (property of minimum norm)

Let $h(x)$ be an n-degree polynomial with leading term equal to $2^{n-1}\, x^n$. Assuming that $h(x) \neq T_n(x)$, we have

$$\max_{x\in[-1,1]} \left|h(x)\right| \;>\; \max_{x\in[-1,1]} \left|T_n(x)\right| = 1$$

Proof (of property of minimum norm)

We proceed by contradiction.

- *Assume that* $\max_{x\in[-1,1]} \left|h(x)\right| \leq 1$, *and consider the polynomial* $\gamma(x) = h(x) - T_n(x)$ *which is of degree* $n-1$ *as the leading terms of* $h(x)$ *and* $T_n(x)$ *are identical.*
  We then prove that $\gamma(x)$ *admits* n *roots in* $[-1,1]$, *precisely one root in each interval* $[m_{k+1}, mk]$, *from which we will deduce the result.*
- *If* k *is an even integer, we have by our assumption :*

$$\begin{aligned}\gamma(m_{k+1}) = h(m_{k+1}) - T_n(m_{k+1}) &= h(m_{k+1}) - (-1)^{k+1} &= h(m_{k+1}) + 1 \geq 0 \\ \gamma(m_k) = h(m_k) - T_n(m_k) &= h(m_k) - (-1)^k &= h(m_{k+1}) - 1 \leq 0\end{aligned}$$

  The result is symmetric if k *is an odd integer.*
  Thus, $\gamma(x)$ *admits (at least) one root* ξ_k *in each close interval* $[m_{k+1}, mk]$, $0 \leq k \leq n-1$.
- *Assume now that this root* ξ_k *is at an edge of the interval* $[m_{k+1}, mk]$, *let say* m_k, *(with* $k \neq 0$ *and* $k \neq n$*). Then,* $h(m_k) = T_n(m_k)$ *and* m_k *is an extremum of the polynomial* $h(x)$ *and of course of* $T_n(x)$, *so that* $\gamma'(m_k) = h'(m_k) - T_n'(m_k) = 0$. *Consequently, the polynomial* $\gamma(x)$ *admits a double root in* $\xi_k = m_k$.
  The end of the proof consists in a technical counting (with multiplicity) of the distinct roots.

Proposition 3.13 (Chebyshev points on $[-1,1]$)

The best distribution of $n+1$ points on the interval $[-1,1]$ is the sequence of Chebyshev points of order $n+1$ over the interval $[-1,1]$:

$$r_k = \cos\left((2k+1)\,\frac{\pi}{2n+2}\right), \quad 0 \le k \le n\,,$$

and the minimum norm is $\|\Pi_{n+1}(x)\| = \|\prod_{k=0}^{n}(x-r_k)\| = \|\frac{1}{2^n}\,T_{n+1}(x)\| = \frac{1}{2^n}$.

Proposition 3.14 (Chebyshev points on $[a,b]$)

The best distribution of $n+1$ points on the interval $[a,b]$ is the sequence of Chebyshev points of order $n+1$ over the interval $[a,b]$:

$$\boxed{\hat{r}_k = \frac{a+b}{2} + \frac{b-a}{2}\,\cos\left((2k+1)\,\frac{\pi}{2n+2}\right), \quad 0 \le k \le n\,,} \tag{5}$$

and the minimum norm is

$$\left\|\prod_{k=0}^{n}(x-\hat{r}_k)\right\| = \left(\frac{b-a}{2}\right)^{n+1}\frac{1}{2^n} = 2\left(\frac{b-a}{4}\right)^{n+1} \tag{6}$$

Proof (Chebyshev points on $[-1,1]$ and $[a,b]$)

- *By the property of minimum norm, the optimal solution on* $[-1,1]$ *is*

$$\Pi_{n+1}(x) \;=\; \frac{1}{2^n}\, T_{n+1}(x) \;=\; \prod_{k=0}^{n}(x-r_k) \qquad \textit{and} \qquad ||\,\Pi_{n+1}\,|| = \frac{1}{2^n}$$

- *The solution over a general interval* $[a,b]$ *is derived from the affine transformation*

$$\begin{array}{ccc} [-1,1] & \longrightarrow & [a,b] \\ x & \longmapsto & u=\varphi(x) \end{array} \qquad \textit{with} \qquad \varphi(x) = \frac{a+b}{2} + \frac{b-a}{2}\,x.$$

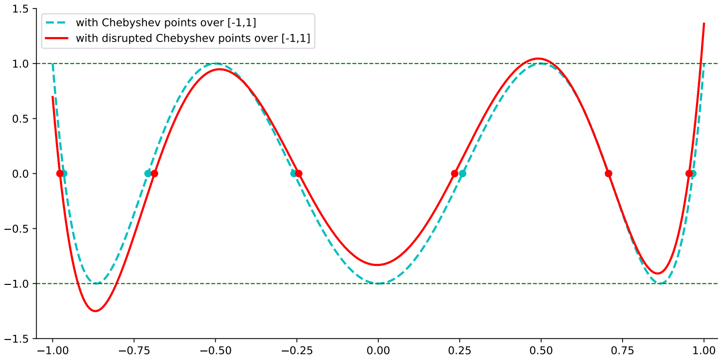
Given a set of $n+1$ *distinct values* u_i *in the interval* $[a,b]$*, associated with* $n+1$ *distinct values* $x_i = \varphi^{-1}(u_i)$ *in* $[-1,1]$*, we have*

$$\prod_{i=0}^{n}(u-u_i) = \prod_{i=0}^{n}\left(\varphi(x)-\varphi(x_i)\right) = \prod_{i=0}^{n}\left(\frac{b-a}{2}\,(x-x_i)\right) = \left(\frac{b-a}{2}\right)^{n+1}\prod_{i=0}^{n}(x-x_i)$$

which concludes the proof.

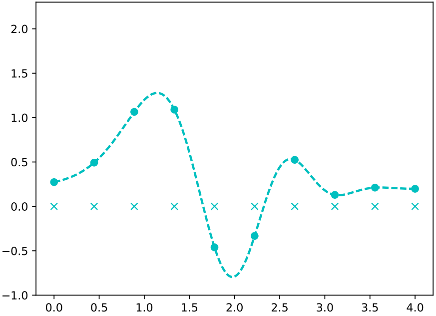
Proposition 3.15 (Alternation/equioscillation property)

The optimal solution $\Pi_{n+1}(x) = \dfrac{1}{2^n}\, T_{n+1}(x) = (x - r_0)(x - r_1)\cdots(x - r_n)$ *is characterized by the fact that the norm* $||\,\Pi_{n+1}\,||$ *is reached at exactly* $n+2$ *distinct points in the interval* $[-1, 1]$ *: once between each point* r_k *and* r_{k+1} *and once at each edge of the interval.*

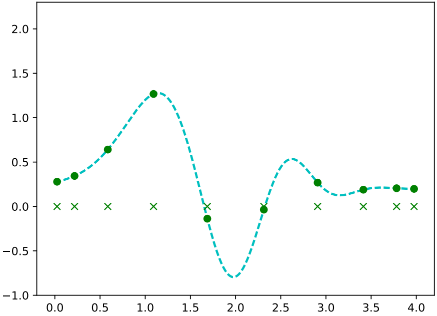


Graphs of $T_{n+1}(x)$ *and of a close polynomial (with disrupted roots)*

Example :

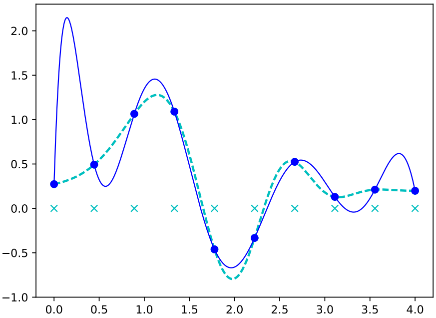


Uniform distribution

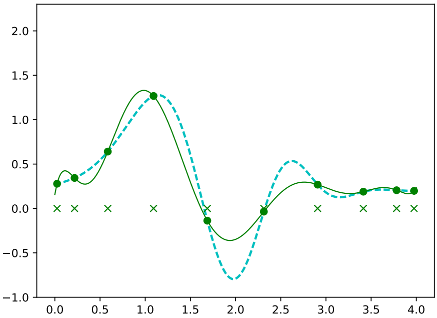


Chebyshev distribution

Example :



Uniform distribution

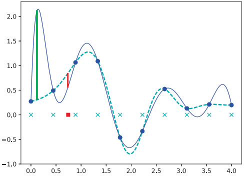


Chebyshev distribution

- Error :

$$E_n = \max_{x \in [a,b]} \left| P_n(x,f) - f(x) \right|$$

$$a = \min x_i \,, \; b = \max x_i$$



- Choice of interpolation data :

$$n \in \mathbb{N} \,, \quad \min_{x_i \in [a,b]} \left(E_n \right) \leq \frac{\left|\left| f^{(n+1)} \right|\right|}{(n+1)!} \min_{x_i \in [a,b]} \left|\left| \Pi_{n+1} \right|\right|$$

Optimal distribution $\rightarrow$ Chebyshev points

- Convergence :

$$\lim_{n \to \infty} E_n \quad \ldots$$

Objective : Given a function $f \in C^p[a, b]$ ($p \geq 0$), and considering for each $n \in \mathbb{N}^*$ a sequence of $n + 1$ distinct points $x_{n,i}$:

$$a \leq x_{n,0} < x_{n,1} < \cdots < x_{n,n} \leq b$$

and the associated interpolating polynomial $P_n(.,f)$ of f at these $n + 1$ points $x_{n,i}$, *we consider the question of the convergence of the sequence of the interpolating polynomials* $P_n(.,f)$ *to* f *when* n *tends to* $+\infty$

- *Convergence may be understood pointwise or uniform*
- *Convergence can only be achieved on a subset of the interval* $[a, b]$
- *We essentially consider here the following two distributions of interpolation data :*

Chebyshev distibution	$\rightarrow$ Chebyshev interpolation
	$\rightarrow$ Chebyshev interpolating polynomial
Uniform distribution	$\rightarrow$ Uniform interpolation
	$\rightarrow$ Uniform interpolating polynomial

References : *Jean-Pierre Demailly, Analyse numérique et équations différentielles, Presses universitaires de Grenoble, 2006*

Runge's phenomenon

Runge's phenomenon [2] is a problem of oscillations that occurs (for certain functions) with *uniform interpolation* when the degree increases.

Carl Runge noticed the behaviour of polynomial interpolation in 1901 when studying the approximation of certain rational functions by polynomials. This discovery was surprising at this time because of the Weierstrass theorem (see chapter on Approximation) it was thought that going to higher degrees would improve accuracy.

Precisely, the classical example of this Runge phenomenon is provided by the rational function

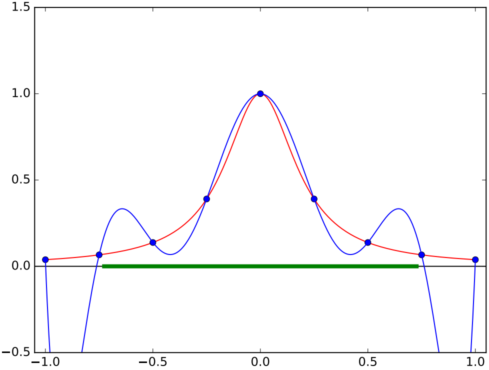
$$f_R(x) = \frac{1}{1 + 25\,x^2}$$

defined on the interval $[-1, 1]$, that we interpolate at uniformly distributed data points in the interval $[-1, 1]$.

2. Carl David Tolmé Runge, 1856-1927, German mathematician

Runge's phenomenon

Runge's phenomenon is a problem of oscillations that occurs (for certain functions) with *uniform interpolation* when the degree increases.



Uniform interpolation : study of convergence when $n \to \infty$

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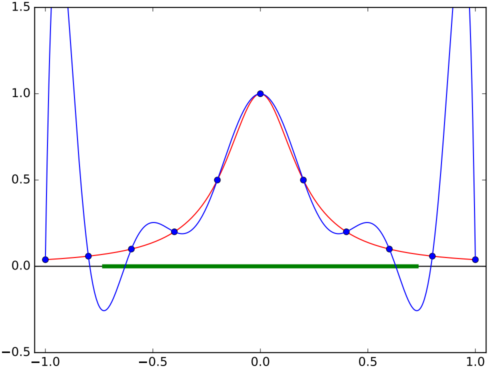

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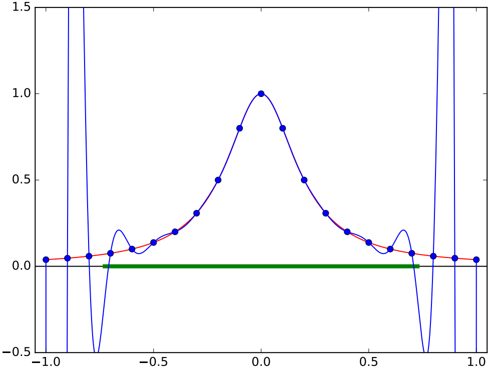

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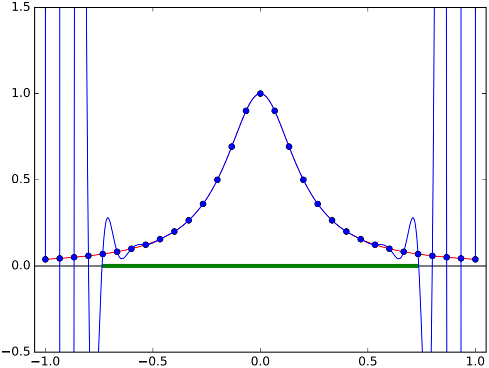

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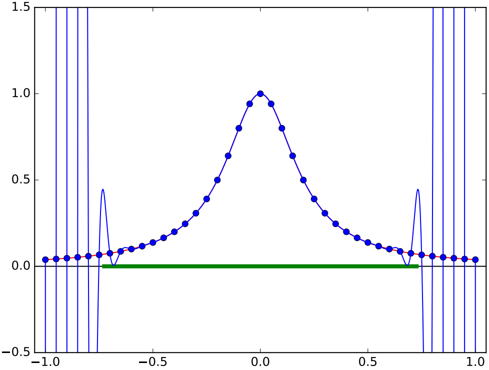

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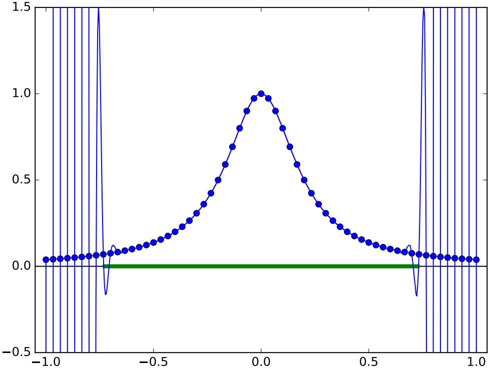


Uniform interpolation : study of convergence when $n \to \infty$

Runge's phenomenon

Runge's phenomenon is a problem of oscillations that occurs (for certain functions) with *uniform interpolation* when the degree increases.

Brief analysis of this phenomenon :

Considering more generally the function $f_\alpha(x) = \frac{1}{\alpha^2+x^2}$, $x \in [-1,1]$ *where* $\alpha > 0$ *is a parameter, and denoting by* $P_n(x,f_\alpha)$ *its uniform interpolating polynomial of degree n, we have the following result.*

$\alpha > \alpha_0 \simeq 0.526$ *: the sequence* $P_n(x,f_\alpha)$ *converges uniformly to* f_α *on* $[-1,1]$.

$\alpha < \alpha_0$ *: the sequence* $P_n(x,f_\alpha)$ *converges uniformly to* f_α *on any close interval contained in the open interval* $]-\bar{x},\bar{x}[$ *and diverges in* $[-1,-\bar{x}[\,\cup\,]\bar{x},1]$, *where* $\bar{x}$ *is solution of the following equation*

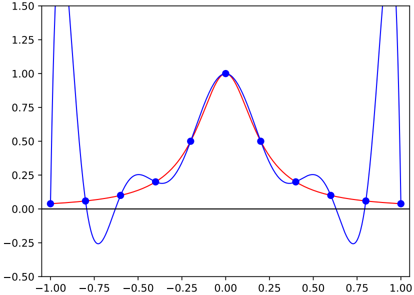
$$(1+x)^{\frac{1+x}{2}}\,(1-x)^{\frac{1-x}{2}} = \sqrt{1+\alpha^2}\,\exp\left(\alpha\,\arctan\left(\frac{1}{\alpha}\right)\right)$$

The classical function f_R *studied by C. Runge corresponds to parameter* $\alpha = 1/5 < \alpha_0$, *which allows to compute the value* $\bar{x} \simeq 0.7266768$. *The interval* $(-\bar{x},\bar{x})$ *has been plotted in* **bold green** *in the previous figures.*

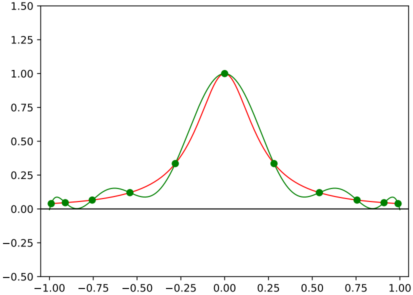
Runge's phenomenon

Runge's phenomenon is a problem of oscillations that occurs (for certain functions) with *uniform interpolation* when the degree increases.

Uniform vs Chebyshev interpolation :



Uniform interpolation
(Runge's phenomenon)



Chebyshev interpolation
(convergence)

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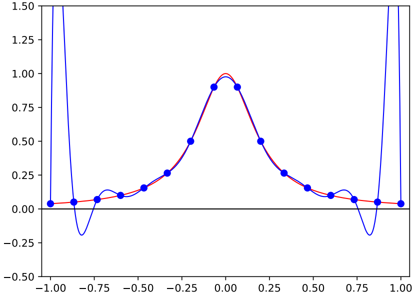
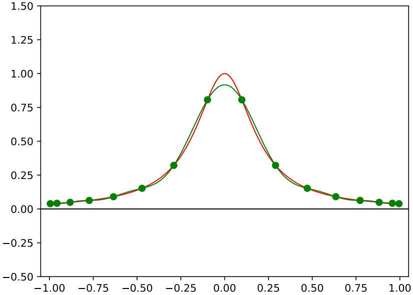
Uniform vs Chebyshev interpolation :



Uniform interpolation
(Runge's phenomenon)



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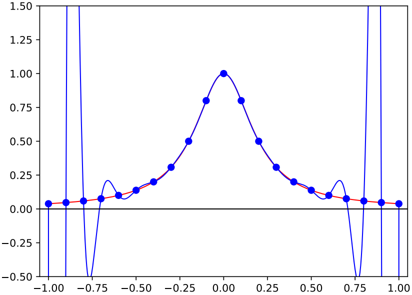
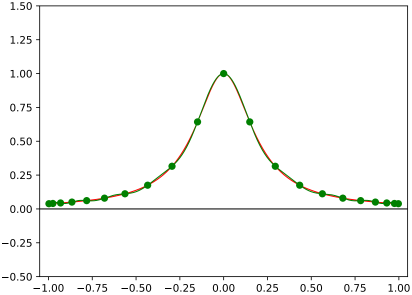


Uniform interpolation
(Runge's phenomenon)



Chebyshev interpolation
(convergence)

Strong convergence

Proposition 3.16 (strong convergence)

Let $f \in C^{\infty}[a,b]$*, such that*

$$\exists M \in \mathbb{R}^{+}, \quad \forall k \in \mathbb{N}, \quad \left|\left|f^{(k)}\right|\right| = \max_{x\in[a,b]} \left|f^{(k)}(x)\right| \leq M .$$

Then, for any sequence $(x_{n,i})$ *of interpolation points in* $[a,b]$*, the sequence of the associated interpolating polynomials* $P_n(.,f)$ *converges uniformly to* f *on the interval* $[a,b]$ *when* n *tends to* $+\infty$*.*

Proof

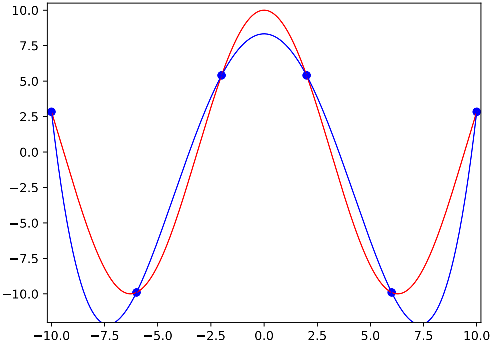
By inequality (4) given in section " Error bounds in Lagrange interpolation "*, we have for any* $n \in \mathbb{N}^{*}$ *:*

$$\left|\left|f - P_n(.,f)\right|\right| = \max_{x\in[a,b]} \left|f(x) - P_n(x,f)\right| \leq \frac{\left|\left|\Pi_{n+1}\right|\right|}{(n+1)!} \left|\left|f^{(n+1)}\right|\right| \leq \frac{(b-a)^{n+1}}{(n+1)!} M$$

which gives the result.

Example 1

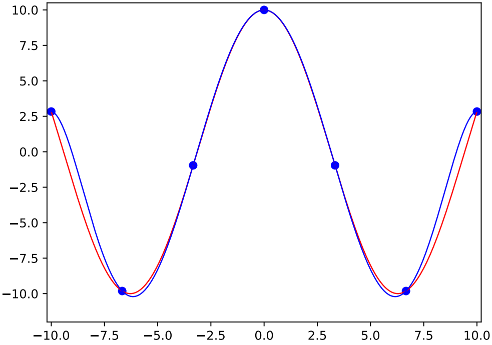
Uniform interpolation on the interval $[a, b]$



$$f(x) = 10\ \cos(x/2), \quad x \in [-10, 10]$$
$$||f^{(k)}|| \leq 10,\ \forall k \in \mathbb{N}$$

Example 1

Uniform interpolation on the interval $[a, b]$



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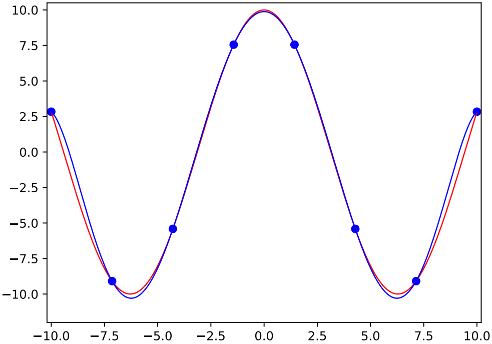

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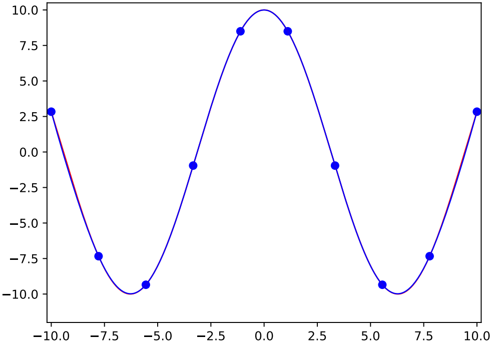
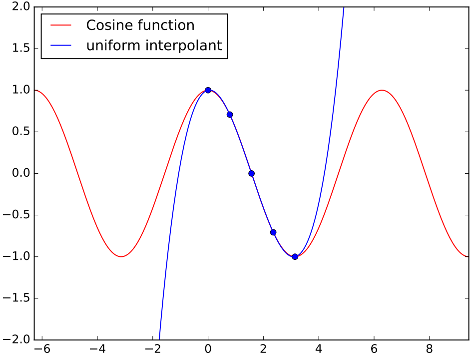


$$f(x) = 10\,\cos(x/2), \quad x \in [-10, 10]$$
$$||f^{(k)}|| \le 10,\ \forall k \in \mathbb{N}$$

Example 2

Uniform interpolation on a sub-interval of $[a, b]$



$$f(x) = \cos(x), \quad x \in [-2\pi, 3\pi]$$
$$||f^{(k)}|| \leq 1, \; \forall\, k \in \mathbb{N}$$

Uniform data points are sampled in the sub-interval $[0, \pi]$
for degrees $4, 8, 12, 16$

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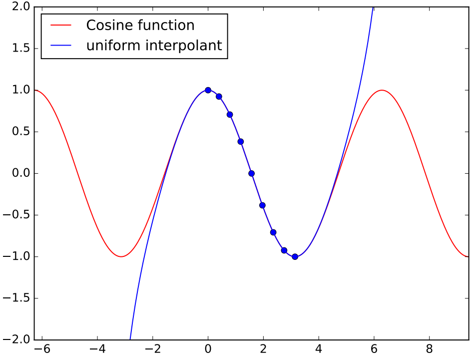

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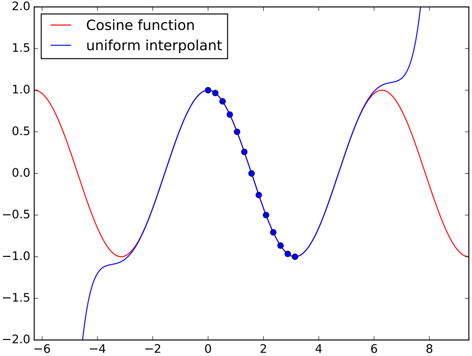

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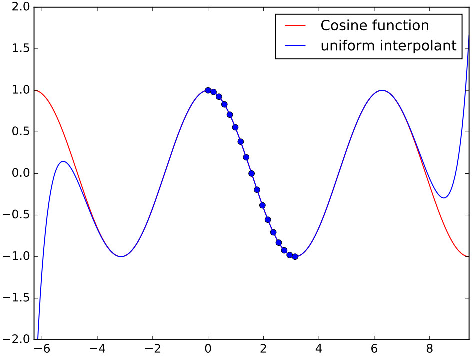

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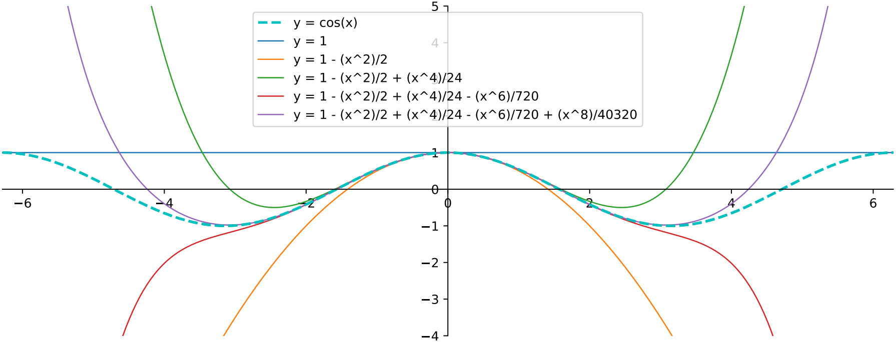
Uniform interpolation on a sub-interval of $[a, b]$

Connection with Taylor expansion

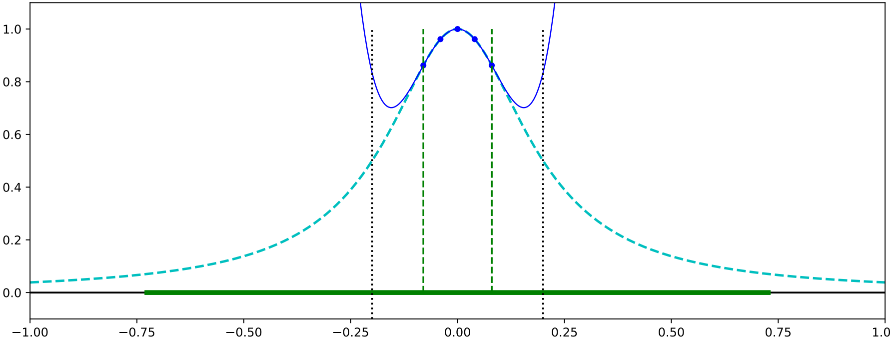


Radius of convergence $R = +\infty$

$$f(x) = \cos(x), \quad x \in [-2\pi, 3\pi]$$

Example 3

Uniform interpolation on a sub-interval of $[a, b]$

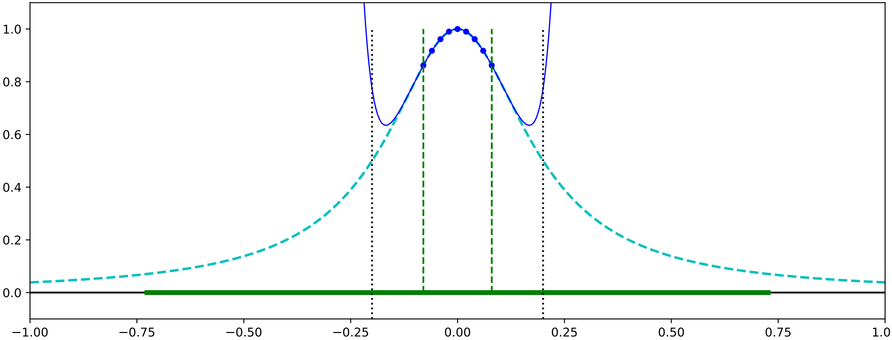


Runge function $f_R(x) = \frac{1}{1+25\,x^2}, \quad x \in [-1, 1]$

Uniform data points are sampled in the sub-interval $[-0.08, 0.08]$
for degrees $4, 8, 12, 16, 20$

Example 3

Uniform interpolation on a sub-interval of $[a, b]$



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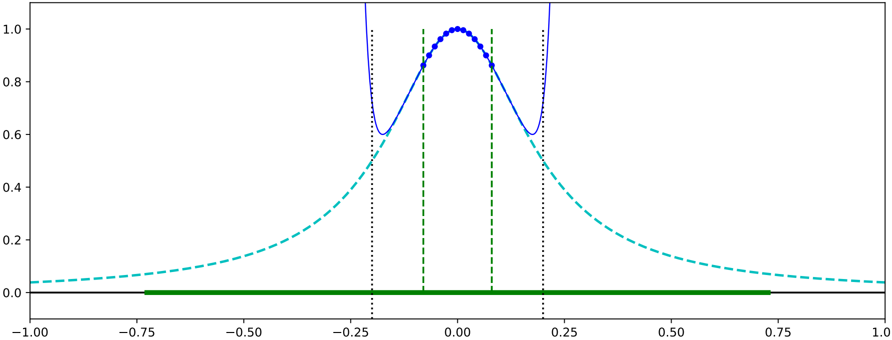

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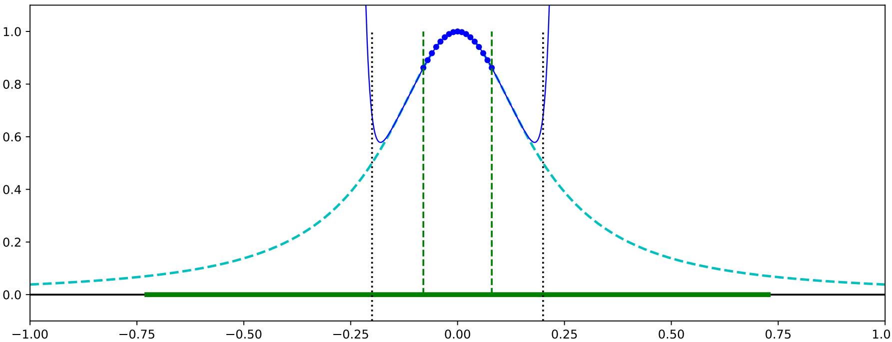

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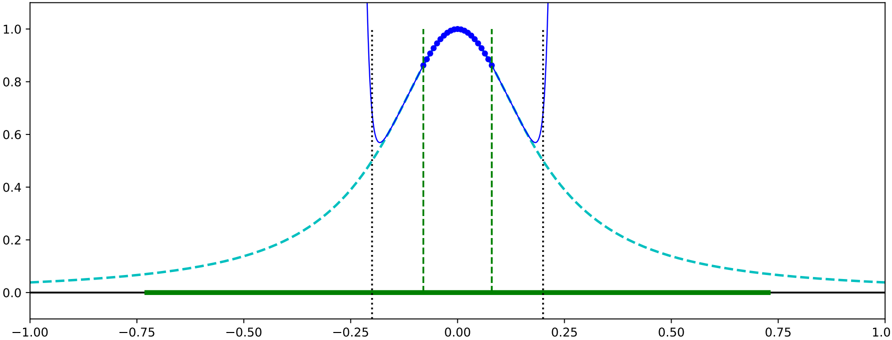

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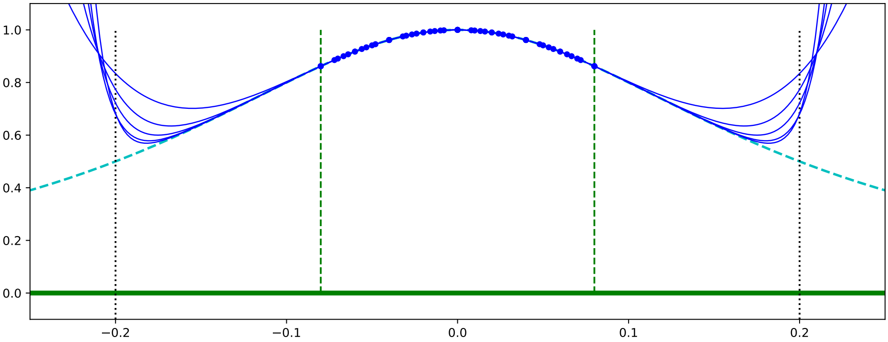

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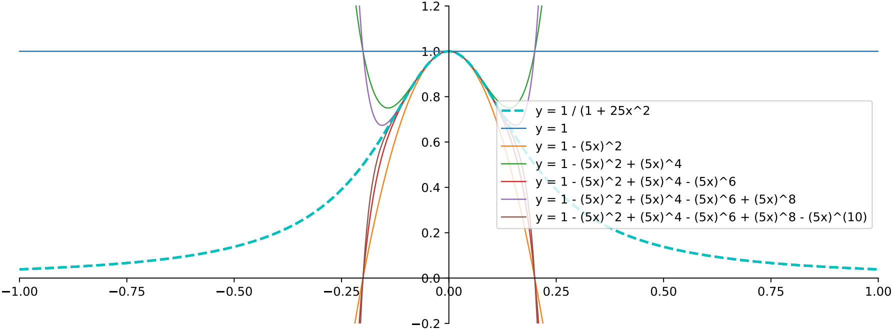
Uniform data points are sampled in the sub-interval $[-0.08, 0.08]$
for degrees 4, 8, 12, 16, 20
ZOOM

Example 3

Uniform interpolation on a sub-interval of $[a, b]$

Connection with Taylor expansion



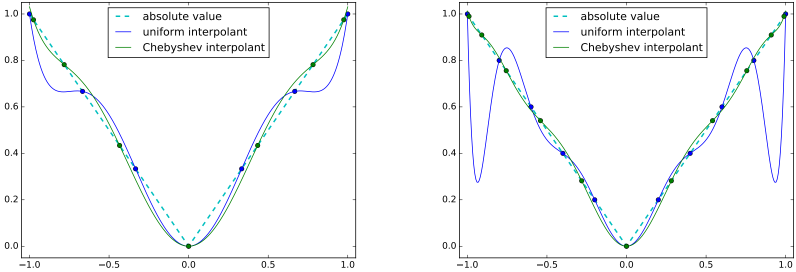
Radius of convergence $R = 0.2$

Runge function $f_R(x) = \frac{1}{1+25\,x^2}, \quad x \in [-1, 1]$

Proposition 3.17 (K-Lipschitz functions)

Assume that the function f is K-Lipschitz, which means that for all $x, y \in [a, b]$, we have $|f(x) - f(y)| \leq K|x - y|$. Then, the sequence of Chebyshev interpolating polynomials converges uniformly to f on $[a, b]$.

Example : we consider the absolute function. Since $\big||x| - |y|\big| \leq |x - y|$ for all $x, y \in \mathbb{R}$, we deduce that the absolute function is 1-Lipschitz, so that the previous proposition applies.



degree 6 *degree* 10

Notice the behaviour of the interpolating polynomials at the non differentiable point

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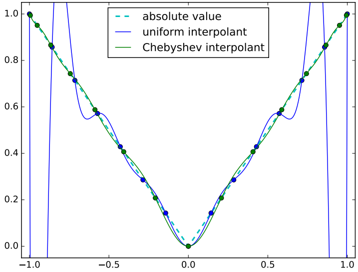
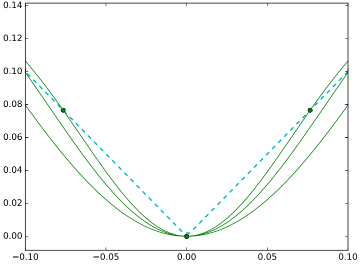


degree 14



zoom on degrees 20 & 30 & 40

Notice the behaviour of the interpolating polynomials at the non differentiable point

The following results provide an interesting insight into the interpolation and convergence process, but their practical relevance is limited.

Proposition 3.18 (Marcinkiewicz)

For any function $f \in C[a,b]$, there exists a triangular sequence of points $x_{n,i} \in [a,b]$, $n = 0, 1, 2, \ldots$ and $0 \leq i \leq n$, so that the associated sequence of interpolating polynomials $P_n(x, f)$ converges uniformly to f on $[a,b]$.

In contrast, we have the following theorem.

Proposition 3.19 (Faber & Bernstein)

For any triangular sequence of points $x_{n,i} \in [a,b]$, $n = 0, 1, 2, \ldots$ and $0 \leq i \leq n$, there exists a continuous function $f \in C[a,b]$ so that the associated sequence of interpolating polynomials $P_n(x, f)$ does not converge uniformly to f on $[a,b]$.

Proof (of Marcinkiewicz proposition)

*The proof is a direct consequence of the Weierstrass approximation theorem which states that there exists a sequence of polynomials $p_n(x)$ of best approximation (*see chapter related to approximation*) which converges uniformly to f.*
Then, due to the equioscillation property, the graph of each polynomial of best approximation $p_n(x)$ intersects the graph of f at $n+1$ distinct values $x_{n,i}$.
Consequently, by uniqueness, the interpolating polynomial $P_n(x,f)$ at these values coincides with $p_n(x)$, which gives the result.

Some quotes... (from Nick Trefethen)

Faber (1914) : “No matter what the grids are, polynomial interpolants diverge for some f”
Stewart (1996), p 153 : “Unfortunately, there are functions for which interpolation at the Chebyshev points fails to converge.”
Polynomial interpolation in Chebyshev points is equivalent to trigonometric interpolation of periodic functions in equispaced points, whose convergence nobody worries about.
Parlett (2010), SIAM Review book review : “You do not want to meet a polynomial of degree 1000 on a dark night.”
Forsythe, Malcolm & Moler (1977), p 68 : “Polynomial interpolation has drawbacks in addition to those of global convergence. The determination and evaluation of interpolating polynomials of high degree can be too time-consuming for certain applications. Polynomials of high degree can also lead to difficult problems associated with roundoff error.”
Froberg (1985), p 234 : “Although Lagrangian interpolation is sometimes useful in theoretical investigations, it is rarely used in practical computations.”

Objective

Given a sequence of points $M_i = (x_i, y_i)$, $0 \leq i \leq n$, we look for a polynomial parametric curve of degree n which interpolates these data points, that is which goes through each point (x_i, y_i) for some parameter t_i .
The main question consists in the choice of the interpolation parameters t_i, which are also called the *interpolation nodes* or simply the nodes.

Essentially, we consider three choices for the interpolation parameters.

- The first one is the *uniform parametrization* : parameters t_i are evenly spaced in the parameter domain.
- The second one is the *Chebyshev parametrization*.
- The last one is the *chordal parametrization* : parameters are chosen in such a way that distances between successive parameters t_i are proportional to the distances between associate successive data points M_i. The chordal parametrization is more reliable and faithful with respect to the geometry of the initial data.

Modelling

Precisely, we look for a n-degree polynomial parametric curve

$$m : \begin{array}{ccc} [a,b] \in \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & m(t) = \begin{pmatrix} m_x(t) \\ m_y(t) \end{pmatrix} \end{array}$$

such that

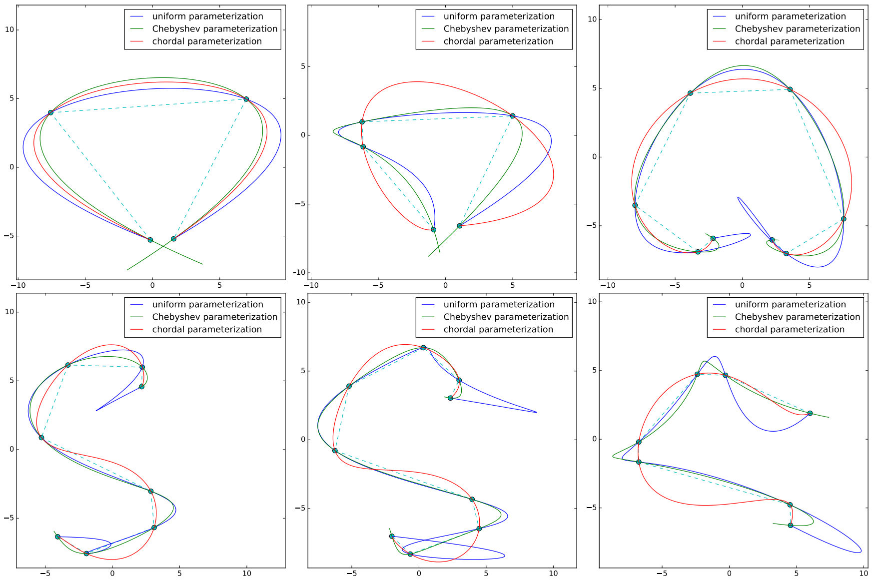
$$m(t_i) = M_i \quad \Leftrightarrow \quad \left\{ \begin{array}{l} m_x(t_i) = x_i \\ m_y(t_i) = y_i \end{array} \right. , \quad 0 \leq i \leq n$$

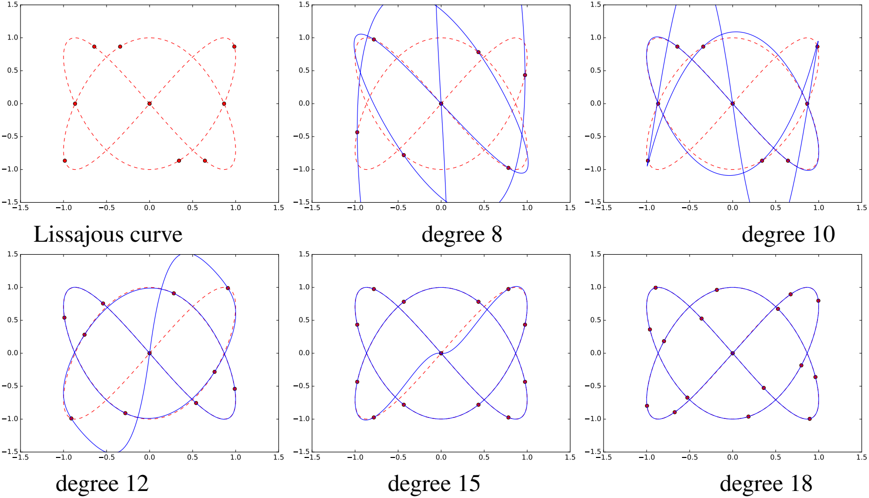
with a sequence of interpolation nodes $a \leq t_0 < t_1 < \cdots < t_n \leq b$, and where $m_x(t)$ and $m_y(t)$ are polynomials of degree n. As a result, we are reduced to solving two separate polynomial interpolation problems.

Considering the parameters domain $[a,b] = [0,1]$ (that does not affect the method), the choices for the interpolation nodes t_i are the following.

- Uniform parameterization : $t_{i+1} - t_i = 1/n$
- Chebyshev parameterization : $\hat{t}_i = \frac{a+b}{2} + \frac{b-a}{2} \cos\left((2i+1)\frac{\pi}{2n+2}\right)$
- Chordal parameterization : $t_{i+1} - t_i = \frac{d_i}{\sum_{k=0}^{n-1} d_k}$ with $d_k = \text{dist}(M_k, M_{k+1})$

Parametric interpolation — Examples





Parametric interpolation of the Lissajous curve : $t \mapsto (x = \sin(2\,t), y = \sin(3\,t)),\ t \in [0, 2\pi]$.
We first choose $n + 1$ *points on the curve through a uniform sampling in the parameter domain. We then interpolate this sampling by polynomial parametric curves of different degrees with the same (uniform) parameters. As the sine function is analytic, the convergence is insured.*