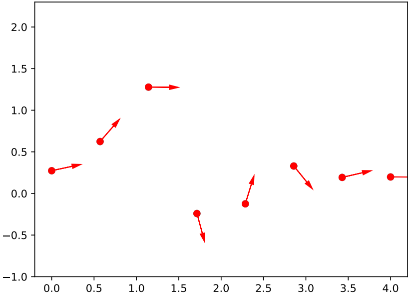
Chapter 4

Hermite interpolation

Objective

- Geometrically, *Hermite interpolation* consists in determining a curve (i.e., a function) that passes through predetermined positions (x_i, y_i) with additional constraints on the derivatives at the interpolating points.



- Data : $(x_i, y_i, y_i'), \qquad i = 0, \ldots, n$
  Problem : find a function f (in a given space F) such that
  $f(x_i) = y_i$ and $f'(x_i) = y_i'$
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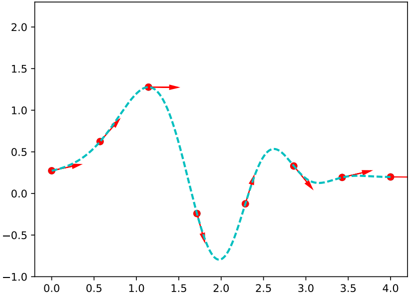

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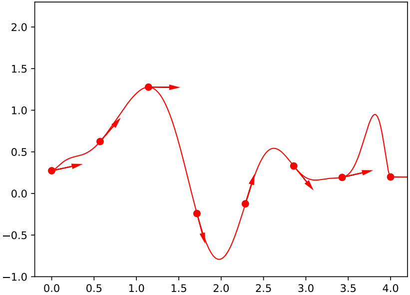

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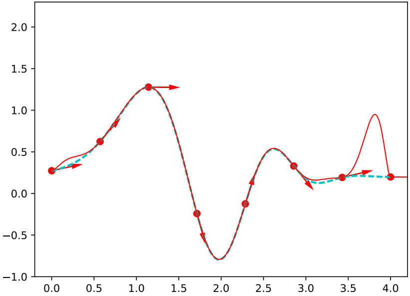
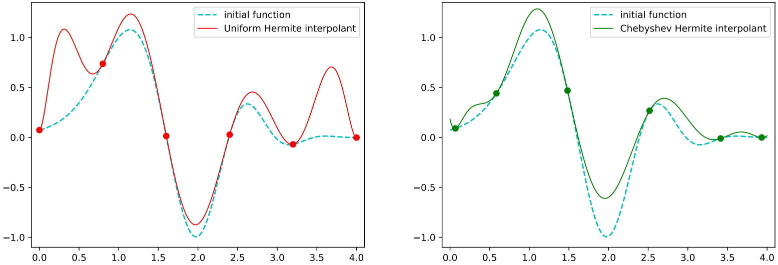


- Data : $(x_i, y_i, y'_i), \quad i = 0, \ldots, n$
  Problem : find a function f (in a given space F) such that
  $f(x_i) = y_i$ and $f'(x_i) = y'_i$
- Choice of an appropriate space F to achieve uniqueness

Choice of space E

- Global solution : *Hermite interpolation over* $n+1$ *data points*
  space F : polynomials of degree $\leq\ 2n+1$



Choice of space E

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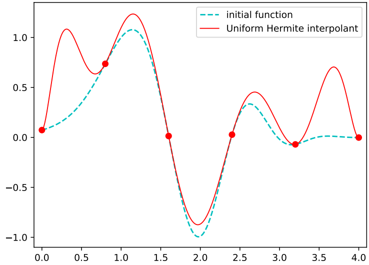
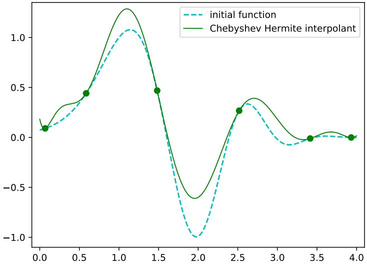
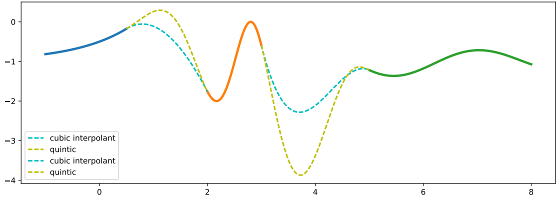




- Local solution : *Hermite interpolation over* 2 *points*
  space F : piecewise cubic, quintic (polynomials)



Statement of the problem

Given a set of $n+1$ distincts points $x_0, x_1, \ldots, x_n$ in an interval $[a, b]$ and two set of $n+1$ real values :

$$y_0,\ y_1, \ldots,\ y_n \qquad \text{and} \qquad y'_0,\ y'_1, \ldots,\ y'_n\,,$$

we look for a polynomial $p(x)$ satisfying

$$\begin{cases} p(x_i) &= y_i\,, \qquad i = 0, 1, \ldots, n, \\ p'(x_i) &= y'_i\,, \qquad i = 0, 1, \ldots, n. \end{cases} \tag{1}$$

This *Hermite interpolating polynomial* will be denoted $P_H(f)$ or $P_H(x,f)$ if the data values y_i and y'_i come from a function $f \in C^1[a, b]$:

$$y_i := f(x_i) \quad \text{and} \quad y'_i := f'(x_i)\,, \qquad 0 \le i \le n\,.$$

Observing the number of constraints (equal to $2n+2$) induces us to search for a polynomial $p(x)$ of degree $2n+1$.

Construction of an Hermite basis

By analogy with the Lagrange approach we construct a polynomial basis

$$\left\{h_i(x),\ \bar{h}_i(x)\,;\ i=0,1,...,n\right\}$$

of $\mathbb{R}_{2n+1}[x]$ satisfying the constraints

$$\left\{\begin{array}{lcl} h_i(x_j) & = & \delta_{ij} \\ h_i'(x_j) & = & 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} \bar{h}_i(x_j) & = & 0 \\ \bar{h}_i'(x_j) & = & \delta_{ij} \end{array}\right. \quad \text{for} \quad \begin{array}{c} 0 \le i \le n \\ 0 \le j \le n \end{array} \tag{2}$$

Such a basis will then make it possible to write the Hermite interpolating polynomial in the form

$$P_H(x) \ = \ \sum_{i=0}^{n} y_i\ h_i(x) \ + \sum_{i=0}^{n} y_i'\ \bar{h}_i(x).$$

Construction of an Hermite basis

We make explicit constraints (2) in the following table where the constraints on each polynomial $h_i(x)$ and $\bar{h}_i(x)$ are specified on the associate column (i.e., labelled by h_i or $\bar{h}_i$).

		h_0	h_1	$\cdots$	h_i	$\cdots$	h_n	$\bar{h}_0$	$\bar{h}_1$	$\cdots$	$\bar{h}_i$	$\cdots$	$\bar{h}_n$
value at	x_0	1	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0
	x_1	0	1	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$	$\ddots$			$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	x_i	0	0		1		0	0	0	$\cdots$	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$			$\ddots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	x_n	0	0	$\cdots$	0	$\cdots$	1	0	0	$\cdots$	0	$\cdots$	0
derivative at	x_0	0	0	$\cdots$	0	$\cdots$	0	1	0	$\cdots$	0	$\cdots$	0
	x_1	0	0	$\cdots$	0	$\cdots$	0	0	1	$\cdots$	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$			$\vdots$
	x_i	0	0	$\cdots$	0	$\cdots$	0	0	0		1		0
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$			$\ddots$	$\vdots$
	x_n	0	0	$\cdots$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$	1

Construction of polynomials $h_i(x)$

By relations (2) we have $h_i(x_j) = h_i'(x_j) = 0$ for $0 \le j \le n,\ j \neq i$, so that polynomial $h_i(x)$ admits a double root at each point $x_j \neq x_i$, and thus is on the form

$$h_i(x) = L_i^2(x)\ r_i(x)$$

where $r_i(x)$ is a polynomial of degree less than or equal to 1.
The two additional constraints that must satisfy $h_i(x)$ leads to

$$\begin{cases} 1 = h_i(x_i) = L_i^2(x_i)\ r_i(x_i) = r_i(x_i) \\ 0 = h_i'(x_i) = L_i^2(x_i)\ r_i'(x_i) + 2\,L_i(x_i)\,L_i'(x_i)\ r_i(x_i) = r_i'(x_i) + 2\,L_i'(x_i) \end{cases}$$

so that

$$r_i(x) = 1 - 2\,(x - x_i)\,L_i'(x_i).$$

Finally, with the formula of the derivative of $L_i(x)$ determined in section *Lagrange form* of chapter *Lagrange interpolation*, we get

$$h_i(x) = L_i^2(x)\left(1 - 2\,(x - x_i)\sum_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x_i - x_j}\right) \tag{3}$$

Construction of polynomials $\bar{h}_i(x)$

By relations (2) we have $\bar{h}_i(x_j) = \bar{h}_i'(x_j) = 0$ for $0 \le j \le n,\ j \ne i$, so that polynomial $\bar{h}_i(x)$ admits a double root at each point $x_j \ne x_i$, and thus is on the form

$$\bar{h}_i(x) = L_i^2(x)\ s_i(x)$$

where $s_i(x)$ is a polynomial of degree less than or equal to 1.
The two additional constraints that must satisfy $\bar{h}_i(x)$ leads to

$$\begin{cases} 0 &= \bar{h}_i(x_i) &= L_i^2(x_i)\ s_i(x_i) &= s_i(x_i) \\ 1 &= \bar{h}_i'(x_i) &= L_i^2(x_i)\ s_i'(x_i) + 2\,L_i(x_i)\,L_i'(x_i)\ s_i(x_i) &= s_i'(x_i) \end{cases}$$

so that

$$s_i(x) = x - x_i$$

and finally

$$\bar{h}_i(x) = (x - x_i)\ L_i^2(x) \tag{4}$$

Proposition 4.1 (Hermite basis)

The set of polynomials $\{h_i(x),\, \bar{h}_i(x);\, i = 0, 1, \ldots, n\}$ *form a basis of the vector space* $\mathbb{R}_{2n+1}[x]$.
This basis is called the polynomial Hermite interpolation basis relative to data points x_i.
Polynomials $h_i(x)$ *and* $\bar{h}_i(x)$ *are named Hermite interpolation basis polynomials.*

Proof

This family contains $2n+2$ *polynomials, each of them being of degree* $2n+1$ *by construction, so that we just need to verify that this family is linearly independent.*
So, consider real values α_i *and* $\bar{\alpha}_i$, $i = 0, 1, \ldots, n$, *such that*

$$\alpha_0\, h_0(x) + \cdots + \alpha_n\, h_n(x) + \bar{\alpha}_0\, \bar{h}_0(x) + \cdots + \bar{\alpha}_n\, \bar{h}_n(x) = 0$$

for any real number x.
This relation and its derivative applied to each of the data point x_i *lead to the nullity of each of the coefficients* α_i *and* $\bar{\alpha}_i$, *from which we deduce the result.*

Hermite interpolation over n data points — Solution

Proposition 4.2 (Hermite interpolating polynomial)

There exists a unique polynomial in $\mathbb{R}_{2n+1}[x]$ satisfying the Hermite constraints (1) defined as follows

$$P_H(x) = \sum_{i=0}^{n} y_i \, h_i(x) + \sum_{i=0}^{n} y_i' \, \bar{h}_i(x)$$
$$= \sum_{i=0}^{n} y_i \, L_i^2(x) \left(1 - 2\,(x - x_i) \sum_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x_i - x_j}\right) + \sum_{i=0}^{n} y_i' \, (x - x_i)\, L_i^2(x) \quad (5)$$

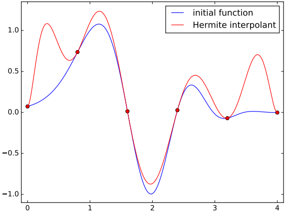
This polynomial is the Hermite interpolating polynomial of the data (x_i, y_i, y_i').

Proof

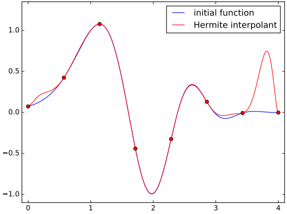
One can easily check that the polynomial defined by (5) satisfy all the constraints (1). Now, assume there exist two polynomials $p_{2n+1}(x)$ and $q_{2n+1}(x)$ in $\mathbb{R}_{2n+1}[x]$ satisfying these constraints. Then, polynomial $p_{2n+1}(x) - q_{2n+1}(x) \in \mathbb{R}_{2n+1}[x]$ admits $n+1$ distinct double roots and is thus zero, which gives the result.

Hermite interpolation over n data points — Examples

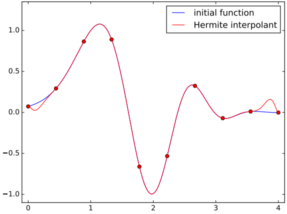
Example : uniform & Chebyshev Hermite interpolation



6 points

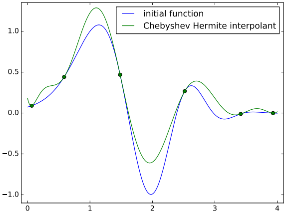


8 points

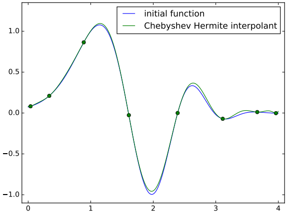


10 points

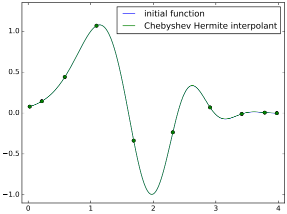
Hermite : Uniform distribution of points



6 points



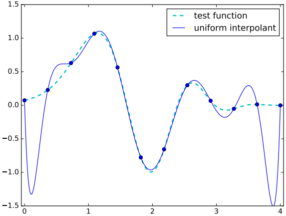
8 points



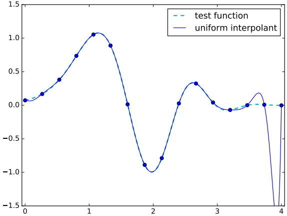
10 points

Hermite : Chebyshev distribution of points

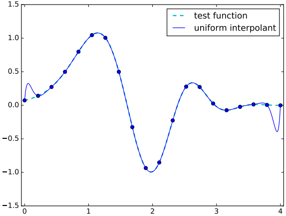
Example : comparison with Lagrange interpolation



12 points

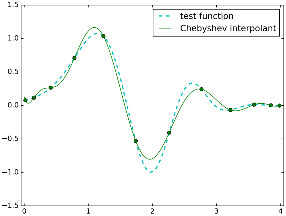


16 points

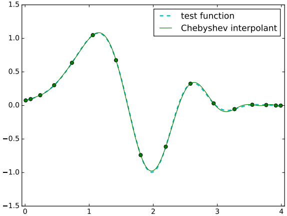


20 points

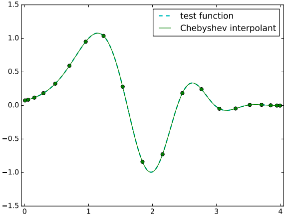
Lagrange : Uniform distribution of points



12 points



16 points



20 points

Lagrange : Chebyshev distribution of points

Objective

We consider the simple case of two interpolation data points α and β $(\alpha < \beta)$.
Precisely, given the two Hermite data $(\alpha, y_\alpha, y'_\alpha)$ and $(\beta, y_\beta, y'_\beta)$, we know by the previous section that there exists a unique cubic polynomial $p(x)$ interpolating these data, that is satisfying

$$p(\alpha) = y_\alpha\,, \quad p'(\alpha) = y'_\alpha\,, \quad p'(\beta) = y'_\beta\,, \quad p(\beta) = y_\beta\,.$$

This *cubic Hermite interpolating polynomial* can be written as follows

$$p(x) = y_\alpha\, h_\alpha(x) + y_\beta\, h_\beta(x) + y'_\alpha\, \bar{h}_\alpha(x) + y'_\beta\, \bar{h}_\beta(x)$$

with the cubic Hermite interpolation basis $h_\alpha, h_\beta, \bar{h}_\alpha, \bar{h}_\beta$ relative to data points α, β.

- We will reduce this Hermite interpolation process to a *standard Hermite process* relative to the two points 0 and 1. This will then make it possible to apply this process simply to n points taken 2 by 2 in the situation of cubic splines.
  Note that Hermite process can also be reduced relative to the two points -1 and 1.
- Hermite interpolation on any interval $[\alpha, \beta]$ is then deduced from the Hermite interpolation on $[0, 1]$ by the affine transformation

$$x \in [\alpha, \beta] \longmapsto t = \frac{x - \alpha}{\beta - \alpha} \in [0, 1]$$

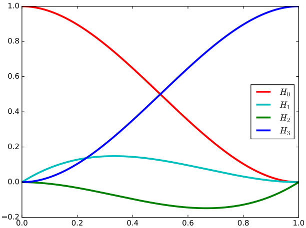
Cubic Hermite basis on $[0, 1]$

The previous cubic Hermite interpolating polynomial $p(x)$ can be rewritten as follows

$$p(x) = y_\alpha \, H_0\left(\frac{x-\alpha}{\beta-\alpha}\right) + y'_\alpha \, (\beta-\alpha) \, H_1\left(\frac{x-\alpha}{\beta-\alpha}\right) + y'_\beta \, (\beta-\alpha) \, H_2\left(\frac{x-\alpha}{\beta-\alpha}\right) + y_\beta \, H_3\left(\frac{x-\alpha}{\beta-\alpha}\right) \tag{6}$$

where H_0, H_1, H_2, H_3 are four cubic polynomials forming the *standard cubic Hermite basis over* $[0, 1]$ and characterized by the following table.

	H_0	H_1	H_2	H_3
$H_i(0)$	1	0	0	0
$H'_i(0)$	0	1	0	0
$H'_i(1)$	0	0	1	0
$H_i(1)$	0	0	0	1



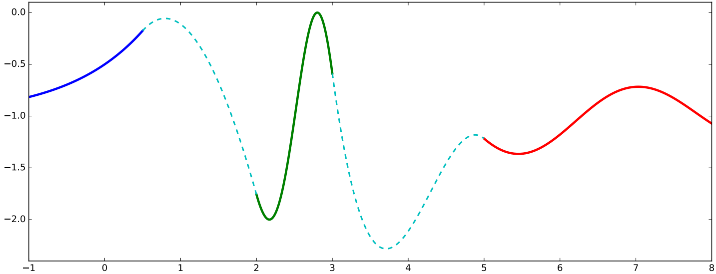
$$H_0(t) = 1 - 3\,t^2 + 2\,t^3$$
$$H_1(t) = t - 2\,t^2 + t^3$$
$$H_2(t) = -t^2 + t^3$$
$$H_3(t) = 3\,t^2 - 2\,t^3$$

Exercise 4.1 — Consider the three following functions which are plotted below respectively in blue, green and red.

$$f_1(x) = \frac{\exp(x)}{2} - 1, \qquad x \in [-1, 0.5],$$

$$f_2(x) = \sin(x^2) - 1, \qquad x \in [2, 3],$$

$$f_3(x) = -1 + 2\,\frac{\sin(2\,x)}{x}, \qquad x \in [5, 8].$$



Determine two functions p_1 and p_2 respectively defined on intervals $[0.5, 2]$ and $[3, 5]$, and plot all these functions, such that the concatenation of the five functions f_1, p_1, f_2, p_2, f_3 provides a C^1 function over the interval $[-1, 8]$.

The following technical result will be essential for the construction of C^2 cubic interpolation splines in the next chapter.

Proposition 4.3 (second derivatives at extremities)

Let $p(x)$ be the cubic Hermite interpolating polynomial relative to data $\left(\alpha, y_\alpha, y'_\alpha\right)$ and $\left(\beta, y_\beta, y'_\beta\right)$ and let $h = \beta - \alpha$.
Then, the second derivatives of $p(x)$ at points α and β can be expressed with respect to the interpolation data as follows.

$$p''(\alpha) = \frac{2}{h^2}\left(3\,y_\beta - 3\,y_\alpha - 2\,h\,y'_\alpha - h\,y'_\beta\right) \quad \textit{and} \quad p''(\beta) = \frac{2}{h^2}\left(3\,y_\alpha - 3\,y_\beta + 2\,h\,y'_\beta + h\,y'_\alpha\right) \tag{7}$$

The following technical result will be essential for the construction of C^2 cubic interpolation splines in the next chapter.

Proof

The two formulas are identical up to a data permutation and by replacing h with $-h$. So we just need to prove the first one. Then, since the Hermite interpolating polynomial $p(x)$ and its derivative $p'(x)$ are respectively of degree 3 and 2, they coincide with their Taylor expansion respectively of order 3 and 2 at point α.

$$p(x) = y_\alpha + (x-\alpha)\, y'_\alpha + \frac{(x-\alpha)^2}{2}\, p''(\alpha) + \frac{(x-\alpha)^3}{6}\, p'''(\alpha),$$
$$p'(x) = y'_\alpha + (x-\alpha)\, p''(\alpha) + \frac{(x-\alpha)^2}{2}\, p'''(\alpha).$$

For $x = \beta$, we get

$$p(\beta) = y_\alpha + h\, y'_\alpha + \frac{h^2}{2}\, p''(\alpha) + \frac{h^3}{6}\, p'''(\alpha) \qquad \text{and} \qquad p'(\beta) = y'_\alpha + h\, p''(\alpha) + \frac{h^2}{2}\, p'''(\alpha),$$

from which we deduce the result after eliminating the term $p'''(\alpha)$

$$p''(\alpha) = \frac{2}{h^2}\left(3\, y_\beta - 3\, y_\alpha - 2\, h\, y'_\alpha - h\, y'_\beta\right).$$

Error bound

We propose to estimate the error associated with *the cubic Hermite interpolation over two points* in the form of a problem. — Let $f \in C^4[\alpha, \beta]$ and let $p(x) = P_H(x,f)$ be the cubic Hermite interpolating polynomial of the function f at points α and β.
Considering a fixed value x in $]\alpha, \beta[$, we introduce the function ϕ defined by

$$u \in [\alpha, \beta] \quad \longmapsto \quad \phi(u) = f(u) - p(u) - \frac{(u-\alpha)^2\,(u-\beta)^2}{(x-\alpha)^2\,(x-\beta)^2}\,\Big(f(x) - p(x)\Big)\,.$$

1. Prove that ϕ cancels at points α, β and x. Deduce that ϕ' cancels at two distinct points in $]\alpha, \beta[$.
2. Prove that $\phi'(\alpha) = \phi'(\beta) = 0$.
3. Deduce that there exists $\zeta_x \in]\alpha, \beta[$ such that $\phi^{(4)}(\zeta_x) = 0$.
4. Prove that

$$f(x) - p(x) = \frac{(x-\alpha)^2\,(x-\beta)^2}{24}\,f^{(4)}(\zeta_x) \quad \text{and that} \quad |(x-\alpha)(x-\beta)| \leq \frac{(\beta-\alpha)^2}{4}.$$

5. Finally, deduce that for all $x \in [\alpha, \beta]$ we have

$$\left|f(x) - p(x)\right| \leq \frac{(\beta-\alpha)^4}{384}\,\max_{\zeta\in[\alpha,\beta]}\left|f^{(4)}(\zeta)\right|,$$

from which we get the upper bound for the error

$$\left|\left|f - P_H(.,f)\right|\right| \leq \frac{(\beta-\alpha)^4}{384}\,\left|\left|f^{(4)}\right|\right|.$$

Objective

We now consider Hermite interpolation of *order two* — which means that constraints involve the value together with the first and second derivatives at the data points — over two points α and β.

Precisely, we look for a polynomial $p(x)$ such that

$$\begin{array}{lll} p(\alpha) = y_\alpha & p'(\alpha) = y'_\alpha & p''(\alpha) = y''_\alpha \\ p(\beta) = y_\beta & p'(\beta) = y'_\beta & p''(\beta) = y''_\beta \end{array}$$

where $y_\alpha,\ y'_\alpha,\ y''_\alpha$ and $y_\beta,\ y'_\beta,\ y''_\beta$ are prescribed real numbers.

The approach is similar to that in the previous section (for Cubic Hermite interpolation over 2 points).

- We construct a *standard quintic Hermite basis* relative to the two points 0 and 1.
- Hermite interpolation on any interval $[\alpha, \beta]$ is then deduced from the quintic Hermite interpolation on $[0, 1]$ by the affine transformation

$$x \in [\alpha, \beta] \longmapsto t = \frac{x - \alpha}{\beta - \alpha} \in [0, 1]$$

This Hermite interpolation process is developed in the form of a problem.

Existence and uniqueness

1. *Hermite interpolation of order* 2 *on* $[0,1]$
   Prove that there exists a unique quintic polynomial $q(x)$ such that

$$
\begin{array}{lll}
q(0) = y_0 & q'(0) = y_0' & q''(0) = y_0'' \\
q(1) = y_1 & q'(1) = y_1' & q''(1) = y_1''
\end{array}
$$

   where $y_0,\ y_0',\ y_0''$ and $y_1,\ y_1',\ y_1''$ are prescribed real numbers.

2. *Hermite interpolation of order* 2 *on* $[\alpha,\beta]$
   Deduce that there exists a unique quintic polynomial $p(x)$ such that

$$
\begin{array}{lll}
p(\alpha) = y_\alpha & p'(\alpha) = y_\alpha' & p''(\alpha) = y_\alpha'' \\
p(\beta) = y_\beta & p'(\beta) = y_\beta' & p''(\beta) = y_\beta''
\end{array}
$$

   where $y_\alpha,\ y_\alpha',\ y_\alpha''$ and $y_\beta,\ y_\beta',\ y_\beta''$ are prescribed real numbers.

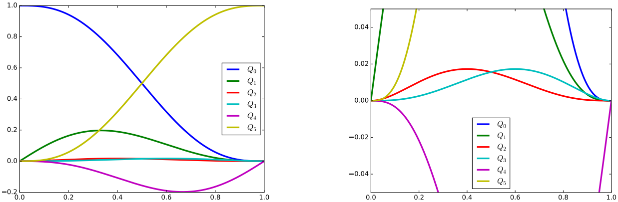
Quintic Hermite basis

3. *Quintic Hermite basis on* $[0,1]$. Consider the following quintic polynomials.

$$Q_0(x) = -6x^5 + 15x^4 - 10x^3 + 1 \qquad Q_1(x) = -3x^5 + 8x^4 - 6x^3 + x$$

$$Q_2(x) = \frac{1}{2}(-x^5 + 3x^4 - 3x^3 + x^2) \qquad Q_3(x) = \frac{1}{2}(x^5 - 2x^4 + x^3)$$

$$Q_4(x) = -3x^5 + 7x^4 - 4x^3 \qquad Q_5(x) = 6x^5 - 15x^4 + 10x^3$$



Quintic Hermite polynomials $Q_i(x)$ *on the interval* $[0,1]$ *& zoom on the right figure.*

Calculate the following matrix which gathers the values of the polynomials $Q_i(x)$, $Q_i'(x)$, $Q_i''(x)$ at points 0 and 1.

$$\begin{bmatrix} Q_0(0) & Q_0'(0) & Q_0''(0) & Q_0''(1) & Q_0'(1) & Q_0(1) \\ Q_1(0) & Q_1'(0) & Q_1''(0) & Q_1''(1) & Q_1'(1) & Q_1(1) \\ Q_2(0) & Q_2'(0) & Q_2''(0) & Q_2''(1) & Q_2'(1) & Q_2(1) \\ Q_3(0) & Q_3'(0) & Q_3''(0) & Q_3''(1) & Q_3'(1) & Q_3(1) \\ Q_4(0) & Q_4'(0) & Q_4''(0) & Q_4''(1) & Q_4'(1) & Q_4(1) \\ Q_5(0) & Q_5'(0) & Q_5''(0) & Q_5''(1) & Q_5'(1) & Q_5(1) \end{bmatrix}$$

The solution

4. Determine the unique solution $q(x)$ to the Hermite interpolation problem of order 2 on the interval $[0, 1]$, as a combination of the polynomials $Q_i(x)$.
5. Determine the unique solution $p(x)$ to the Hermite interpolation problem of order 2 on the interval $[\alpha, \beta]$, as a combination of the polynomials $Q_i(x)$.

Hint : consider the following combination of polynomials $Q_i(x)$:

$$p(x) = \Big(Q_0(t), Q_1(t), Q_2(t), Q_3(t), Q_4(t), Q_5(t)\Big) \begin{pmatrix} y_\alpha \\ (\beta-\alpha)\ y'_\alpha \\ (\beta-\alpha)^2\ y''_\alpha \\ (\beta-\alpha)^2\ y''_\beta \\ (\beta-\alpha)\ y'_\beta \\ y_\beta \end{pmatrix}$$

with $t = \dfrac{x-\alpha}{\beta-\alpha}$

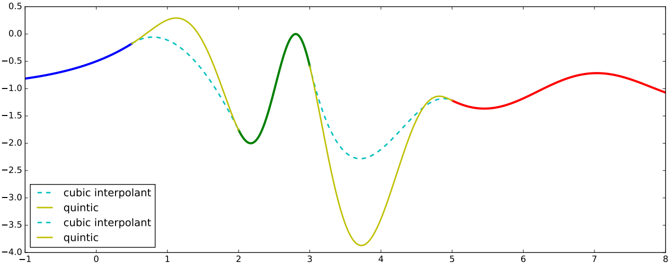
Exercise 4.2 — *Comparison with the cubic Hermite interpolation.*
Consider again the three following functions which are plotted below respectively in blue, green and red.

$$f_1(x) = \frac{\exp(x)}{2} - 1, \qquad x \in [-1, 0.5],$$

$$f_2(x) = \sin(x^2) - 1, \qquad x \in [2, 3],$$

$$f_3(x) = -1 + 2\,\frac{\sin(2\,x)}{x}, \qquad x \in [5, 8].$$

Determine the two quintic polynomials $q_1(x)$ and $q_2(x)$ respectively defined on intervals $[0.5, 2]$ and $[3, 5]$, such that the concatenation of the five functions f_1, q_1, f_2, q_2, f_3 provides a C^2 function over the interval $[-1, 8]$.



Exercise 4.3 — *Hermite interpolation of order n at one point*
Let $n \in \mathbb{N}$, a a fixed real number as well as $n+1$ real numbers

$$y_0,\ y_1,\ \dots,\ y_n.$$

Prove that there exists a unique polynomial $p(x)$ of degree n such that

$$p^{(k)}(a) = y_k, \quad k = 0, 1, \dots, n.$$

Exercise 4.4 — *An instructive example*
Let y_0, y_1', y_2 three real numbers. Determine the set of polynomials $p(x) = a_0 + a_1\, x + a_2\, x^2 \in \mathbb{R}_2[x]$ satisfying the constraints

$$p(0) = y_0, \quad p'(1) = y_1', \quad p(2) = y_2\,,$$

according values of parameters y_0, y_1' and y_2.

Exercise 4.5 — *From Lagrange to Hermite*

Let $h \in]0, 1[$.

1. Write the quadratic Lagrange polynomials relative to data points $x_0 = 0,\ x_1 = h,\ x_2 = 1$.
2. Given real values y_0, α, y_2, determine the unique polynomial $p(x)$ of degree less than or equal 2 satisfying the constraints
$$p(x_0) = y_0,\ p(x_1) = y_0 + \alpha h,\ p(x_2) = y_2.$$
3. Write the previous polynomial $p(x)$ on the form
$$p(x) = y_0\, p_0^h(x) + \alpha\, p_1^h(x) + y_2\, p_2^h(x),$$
and specify the polynomials $p_0^h(x),\ p_1^h(x),\ p_2^h(x)$.
4. Prove that polynomials $p_0^h(x),\ p_1^h(x),\ p_2^h(x)$ converge, when h tends to 0, to three polynomials $p_0(x),\ p_1(x),\ p_2(x)$ which satisfy
$$\begin{array}{rclrclrcl} p_0(0) & = & 1, & {p_0}'(0) & = & 0, & p_0(1) & = & 0,\\ p_1(0) & = & 0, & {p_1}'(0) & = & 1, & p_1(1) & = & 0,\\ p_2(0) & = & 0, & {p_2}'(0) & = & 0, & p_2(1) & = & 1. \end{array}$$
5. Comment the result of the previous question and develop a similar process leading to the standard cubic Hermite basis on $[0, 1]$.

Construction

Hermite interpolation process over *two points* naturally allows to interpolate Hermite data

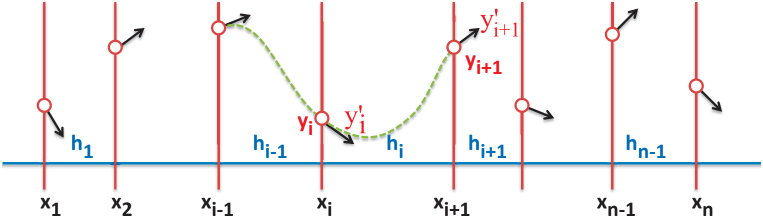
$$(x_i, y_i, y_i'), \quad i = 1, \ldots, n \qquad \text{with} \qquad x_1 < x_2 < \cdots < x_n$$

by a C^1 piecewise cubic function h.

Precisely, the restriction h_i of h on each interval $[x_i, x_{i+1}]$ is the unique *cubic Hermite interpolating polynomial* of data (x_i, y_i, y_i') and $(x_{i+1}, y_{i+1}, y_{i+1}')$.
— The C^1 piecewise cubic function h constructed in that way verifies

$$h(x_i) = y_i \quad \text{and} \quad h'(x_i) = y_i' \qquad \text{for} \qquad i = 1, \ldots, n,$$

and is called the *Hermite interpolating C^1 cubic spline* associated with the Hermite data (x_i, y_i, y_i') and is denoted $s_H := h$

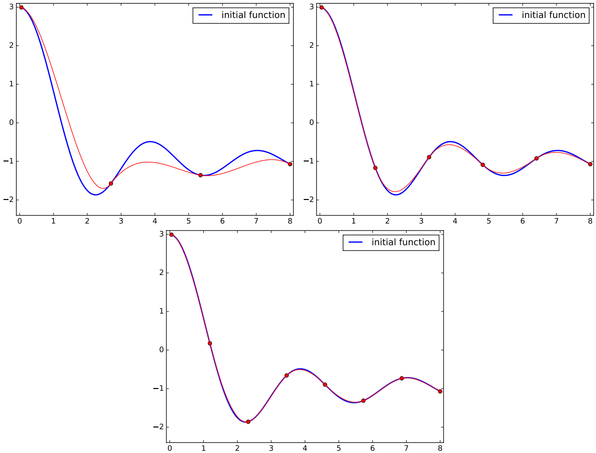


Case of Hermite data from a function

Given a function $f \in C^1[a,b]$, as well as a sequence of n points x_i such that

$$a = x_1 < x_2 < \cdots < x_n = b$$

the *Hermite interpolating* C^1 *cubic spline* $s_H(.,f)$ *of the function* f at points x_i is the Hermite interpolating C^1 cubic spline associated with Hermite data $\big(x_i, f(x_i), f'(x_i)\big)$.



$\sin(2x)$

Error bound

Exercise 4.6 — Consider a function $f \in C^4[a, b]$ as well as its Hermite interpolating C^1 cubic spline $s_{H,n}(.,f)$ over a uniform distribution of n points x_i

$$a = x_1 < x_2 < \cdots < x_n = b \qquad \text{with} \quad x_{i+1} - x_i = \frac{b-a}{n-1}$$

1. Determine the error bound of this C^1 cubic spline Hermite interpolation process, i.e., determine an upper bound of the error

$$\left|\left| s_{H,n}(.,f) - f \right|\right| = \max_{x \in [a,b]} \left| s_{H,n}(x,f) - f(x) \right|$$

2. Prove that this Hermite interpolating process converges to f when the number n of data points x_i tends to $+\infty$
3. Does this result hold for any sequence of n data points at each step ?
4. Consider the case of n data points x_i randomly chosen at each step (with $a = x_1 < x_2 < \cdots < x_n = b$).
5. Consider the case of the Runge function (convergence or not ?).