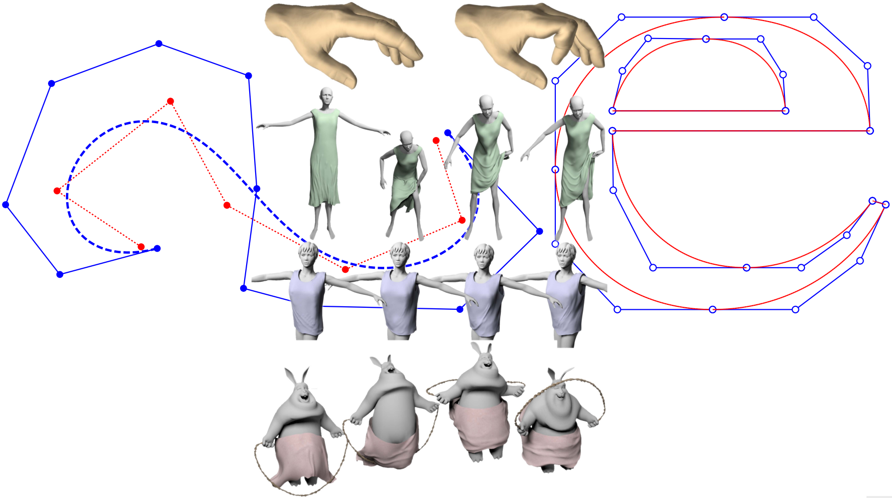
Chapter 9

Bézier curves

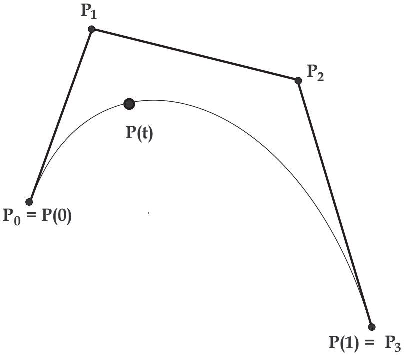
Introduction —

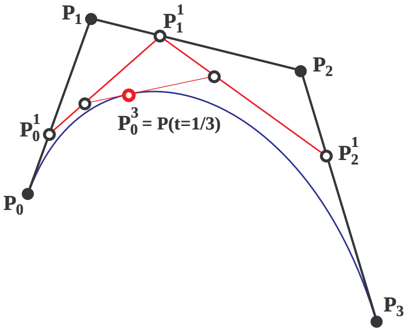
A Bézier curve is a parametric curve whose geometric shape is defined and controlled by means of a *control polygon*.
Bézier curves are totally independent of any frame and are frequently used in computer graphics and related fields.



Completely similar and independent discovery by two French engineers, who gave their names, one to the *curve*, and the other to the *evaluation process*.

- Bézier curves : developed from engineering problems by Pierre Bézier
- De Casteljau algorithm : efficient evaluation of "control points curves" (e.g. Bézier curves) by Paul de Faget De Casteljau





Introduction — A brief history

Pierre Bézier
Engineer from the school "Arts & Métiers", started working at Renault in 1933

Pierre Bézier 1910 - 1999

- 1958 Digital control machine
- 1962-66 Bézier curves and surfaces (UNISURF system)
- this is the beginning of CAGD : Computer Aided Geometric Design

Paul de Faget de Casteljau
Engineer from the school "normal sup", at company Citroen

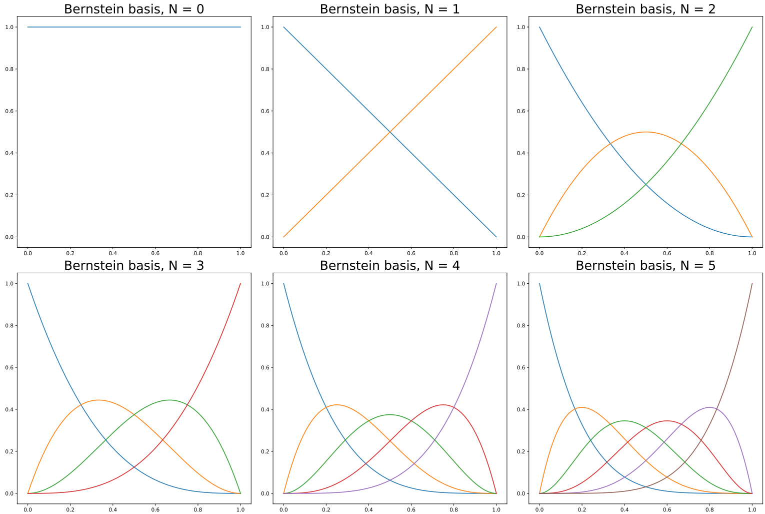
Paul de Faget De Casteljau 1930 -

- Citroen was the first French company working on CAGD (in 1958)
- De Casteljau designed the curves with poles (or Bézier curves) from this date (based on Bernstein polynomials)
- but he had to wait until 1985 to publish his research... (Citroën politic)
- 2012 : Bézier Prize by the Solid Modeling Association comity

Bernstein polynomials — Definition

For $n \in \mathbb{N}$, we define $n+1$ Bernstein polynomials :

$$B_i^n(x) = \binom{n}{i} x^i \, (1-x)^{n-i}, \qquad i = 0, \dots, n,$$



Bernstein polynomials — Properties (1)

– Basis of $\mathbb{R}_n[x]$
For any $n \in \mathbb{N}$, the set of Bernstein polynomials $B_i^n(x)$, $i = 0, \ldots, n$, is a basis of the space $\mathbb{R}_n[x]$ of polynomials of degree less than or equal to n.

– Unit partition
For any $n \in \mathbb{N}$, we have

$$\sum_{i=0}^{n} B_i^n(x) \;=\; 1, \quad \forall\, x \in \mathbb{R}.$$

– Positivity
For any $n \in \mathbb{N}$, and any value of i, $0 \leq i \leq n$, we have

$$\forall\, x \in [0, 1], \quad B_i^n(x) \geq 0.$$

– Values at bounds of the interval $[0, 1]$
For any $n \in \mathbb{N}$, we have

$$\begin{array}{ll} B_0^n(0) = 1 & B_i^n(1) = 0, \;\; i = 0, \ldots, n-1 \\ B_i^n(0) = 0, \;\; i = 1, \ldots, n & B_n^n(1) = 1 \end{array}$$

– Symmetry
For any $n \in \mathbb{N}$, and any value of i, $0 \leq i \leq n$, we have

$$B_{n-i}^n(1-x) \;=\; B_i^n(x), \quad \forall\, x \in \mathbb{R}.$$

- Linear precision
  For any $n \in \mathbb{N}^*$, we have

$$\sum_{i=0}^{n} \frac{i}{n} B_i^n(x) = x, \quad \forall x \in \mathbb{R}.$$

- Recurrence
  For any $n \in \mathbb{N}^*$, and any i, $0 \le i \le n$, we have

$$B_i^n(x) = (1-x) B_i^{n-1}(x) + x B_{i-1}^{n-1}(x), \quad \forall x \in \mathbb{R} \tag{1}$$

  with the rule $B_j^m(x) \equiv 0$ if $j < 0$ or $j > m$.
- Derivatives
  For any $n \in \mathbb{N}^*$, and any i, $0 \le i \le n$, we have

$$DB_i^n(x) = n \left(B_{i-1}^{n-1}(x) - B_i^{n-1}(x) \right), \quad \forall x \in \mathbb{R}$$

  with the same rule as above : $B_j^m(x) \equiv 0$ if $j < 0$ or $j > m$.
- Extremum on $[0, 1]$
  For any $n \in \mathbb{N}^*$, and any i, $0 \le i \le n$, we have

$$\max_{x \in [0,1]} B_i^n(x) = B_i^n\left(\frac{i}{n}\right)$$

  The maximum on the interval $[0, 1]$ of each Bernstein polynomial $B_i^n(x)$ is reached at the value i/n.

Barycentric combinations

Note that the only valid combinations of points in an affine space are *barycentric combinations* (and more specifically convex combinations in our context) and *vector combinations*, as explained thereafter.

Given points $A_1, \ldots, A_p \in E$ and scalars $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$:

- if $\sum_{i=1}^{p} \lambda_i = 1$, then $\sum_{i=1}^{p} \lambda_i A_i \in E$ is the barycentric combination of weighted points (A_i, λ_i),
- if $\sum_{i=1}^{p} \lambda_i = 1$, with $\lambda_i \geq 0$ for all i, then $\sum_{i=1}^{p} \lambda_i A_i \in E$ is a convex combination of points A_i, and is located in the convex hull of points A_i,
- if $\sum_{i=1}^{p} \lambda_i = 0$, then $\sum_{i=1}^{p} \lambda_i A_i \in \overrightarrow{E}$,
- for any point $A \in E$, and any vector $\overrightarrow{u} \in \overrightarrow{E}$, $A + \overrightarrow{u}$ is a point of the affine space E.

Examples

- $2\,A + 3\,B$ is not a valid combination,
- $C = \frac{7}{5}\,A - \frac{2}{5}\,B$ is the barycentric combination of point A with weight $\frac{7}{5}$ and point B with weight $-\frac{2}{5}$, and point C is located on the line (AB),
- $D = \frac{1}{3}\,A + \frac{2}{3}\,B$ is a convex combination of points A and B, and point D is located on the segment $[A, B]$, i.e., in the convex hull of points A and B,
- $-A + B = \overrightarrow{AB}$ is a “vector combination” of points A and B.

A *Bézier curve* is a polynomial parametric curve expressed with respect to the Bernstein basis

$$t \in [0,1] \longmapsto P(t) = \sum_{i=0}^{n} B_i^n(t)\, P_i\,, \qquad P_i \in \mathbb{R}^d$$

→ *Bézier curve* of $\mathbb{R}^d$ of degree n
→ $P(t)$ is a convex combination of points P_i for all $t \in [0,1]$
→ preserves the properties of the curve under affine transformations
→ essential for most applications in Computer Aided Geometric Design (CAGD)

Points P_i are called the *control points* or the *control Bézier points* of the Bézier curve.
Polygon $[P_0 P_1 \ldots P_n]$ is the *control polygon* or the *control Bézier polygon* of the Bézier curve.

Bézier curves are defined on the interval $[0,1]$ because of the property of positivity of Bernstein polynomials.

We sometimes consider the more accurate notation

$$P(t) = B[P_0, \ldots, P_n](t) = \sum_{i=0}^{n} B_i^n(t)\, P_i\,.$$

Consider the polynomial parametric plane curve

$$t \in [0\ 1] \quad \mapsto \quad P(t) = \begin{cases} x(t) = 4\,t - t^2 \\ y(t) = 1 - 2\,t^2 \end{cases}$$

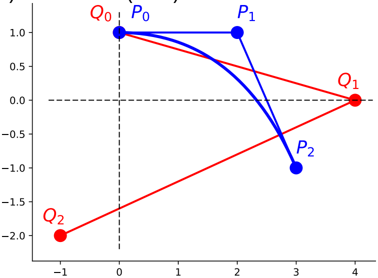
With the following relations between monomial and Bernstein polynomial basis

$$\begin{cases} B_0^2(t) &= (1-t)^2 &= 1 - 2t + t^2 \\ B_1^2(t) &= 2t(1-t) &= 2t - 2t^2 \\ B_2^2(t) &= t^2 &= t^2 \end{cases} \iff \begin{cases} 1 &= B_0^2(t) + B_1^2(t) + B_2^2(t) \\ t &= \frac{1}{2}\,B_1^2(t) + B_2^2(t) \\ t^2 &= B_2^2(t) \end{cases}$$

we can write

$$P(t) = 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \end{pmatrix} + t^2 \begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad = \quad 1\,.\,Q_0 + t\,.\,Q_1 + t^2\,.\,Q_2$$

$$= B_0^2(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + B_1^2(t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + B_2^2(t) \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad = \quad B_0^2(t)\,.\,P_0 + B_1^2(t)\,.\,P_1 + B_2^2(t)\,.\,P_2$$



Consider a Bézier curve $P(t) = \sum_{i=0}^{n} B_i^n(t)\, P_i$ of $\mathbb{R}^d$ as introduced above.

- Interpolation at extremities
  For any $n \in \mathbb{N}$, we have

  $$P(0) = P_0 \quad \text{and} \quad P(1) = P_n$$

  which means that the curve starts at control point P_0 (for $t = 0$) and ends at control point P_n (for $t = 1$).

- Convex hull
  For any $n \in \mathbb{N}^*$, the Bézier curve is included in the convex hull of its control polygon

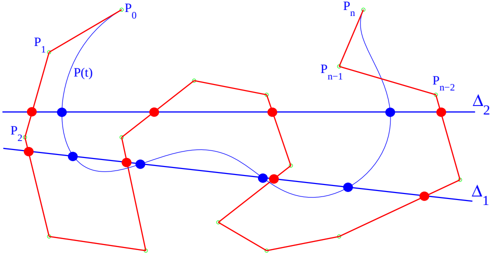
  $$P([0,1]) = \left\{ \sum_{i=0}^{n} B_i^n(t)\, P_i,\;\; t \in [0,1] \right\} \quad \subset \quad \mathrm{Conv}(P_0, \ldots, P_n).$$

- Symmetry
  As a direct consequence of property of Bernstein polynomials we have

  $$\sum_{i=0}^{n} B_i^n(t)\, P_i \;=\; \sum_{i=0}^{n} B_i^n(1-t)\, P_{n-i}$$

  which shows that the curve can be described equivalently in the opposite direction.

– Variation diminishing property
For any hyperplane Δ of $\mathbb{R}^d$, the number of intersections between Δ and the Bézier curve $P(t)$ is less than or equal to the number of intersections between Δ and the control polygon of $P(t)$.



– Invariance by similarity
Let s be a similarity (composition of an homothety, a rotation, a reflection and a translation). We have

$$s\Big(B[P_0, ..., P_n](t)\Big) = B[s(P_0), ..., s(P_n)](t), \quad \forall t \in [0, 1] .$$

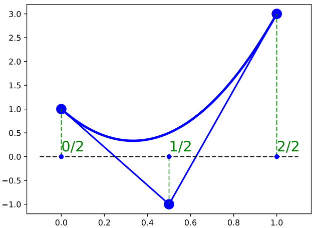
In other words, the image by a similarity of the Bézier curve associated with the control polygon $[P_0, \ldots, P_n]$ is the Bézier curve associated with the image of this control polygon by the similarity.

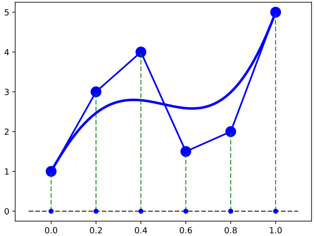
- Influence of control points
  As a direct consequence of property 8 of Bernstein polynomials the influence of point P_i is maximal for $t = i/n$.
- Graph of a Bézier function
  A Bézier function is a polynomial function over the interval $[0, 1]$ expressed with respect to the Bernstein basis

$$t \in [0,1] \longmapsto f(t) = \sum_{i=0}^{n} \lambda_i B_i^n(t)\,, \qquad \lambda_i \in \mathbb{R}.$$

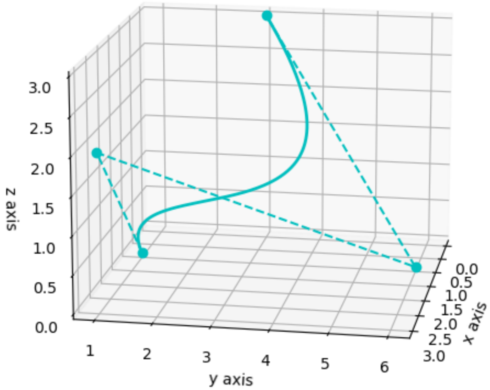
The graph of this real function can be parameterized as a plane curve plane as follows

$$t \in [0,1] \longmapsto F(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{n} i/n\, B_i^n(t) \\ \sum_{i=0}^{n} \lambda_i B_i^n(t) \end{pmatrix} = \sum_{i=0}^{n} \begin{pmatrix} i/n \\ \lambda_i \end{pmatrix} B_i^n(t)$$





As mentioned above, all properties work for 3D curves (and more generally for curves $\mathbb{R}^d$)



For the variation diminishing property, hyperplanes are planes in $\mathbb{R}^3$

Given a Bézier curve $P(t) = \sum_{i=0}^{n} B_i^n(t)\, P_i$ of degree n in $\mathbb{R}^d$, we deduce its derivative from property of Bernstein polynomials :

$$\overrightarrow{P'}(t) = n \sum_{i=0}^{n-1} B_i^{n-1}(t)\, \left(P_{i+1} - P_i\right) = n \sum_{i=0}^{n-1} B_i^{n-1}(t)\, \overrightarrow{\Delta P_i},$$

with the forward differences $\overrightarrow{\Delta P_i} = P_{i+1} - P_i$, which are vectors of the vector space $\overrightarrow{\mathbb{R}^d}$.
Consequently, the derivative $\overrightarrow{P'}(t)$ is a *vector curve* of degree $n-1$ in $\overrightarrow{\mathbb{R}^d}$.

Derivatives of higher order

$$P(t) = \sum_{i=0}^{n} B_i^n(t)\, P_i$$

$$P'(t) = n \sum_{i=0}^{n-1} B_i^{n-1}(t)\, \Delta P_i \qquad \Delta P_i = P_{i+1} - P_i$$

$$P''(t) = n(n-1) \sum_{i=0}^{n-2} B_i^{n-2}(t)\, \Delta^2 P_i \qquad \Delta^2 P_i = \Delta P_{i+1} - \Delta P_i$$

$$\vdots$$

Bézier Curves — Derivatives at extremities

$$\begin{aligned}
P(0) &= P_0 \\
P'(0) &= n\,\Delta P_0 = n\,\left(P_1 - P_0\right) \\
&= n\,\overrightarrow{P_0P_1} \\
P''(0) &= n(n-1)\,\Delta^2 P_0 = n(n-1)\,\left(\Delta P_1 - \Delta P_0\right) = n(n-1)\,\left(P_2 - 2\,P_1 + P_0\right) \\
&= n(n-1)\,\left(\overrightarrow{P_1P_0} + \overrightarrow{P_1P_2}\right) \\
&\vdots \\
P(1) &= P_n \\
P'(1) &= n\,\Delta P_{n-1} = n\,\left(P_n - P_{n-1}\right) \\
&= n\,\overrightarrow{P_{n-1}P_n} \\
P''(1) &= n(n-1)\,\Delta^2 P_{n-2} = n(n-1)\,\left(\Delta P_{n-1} - \Delta P_{n-2}\right) = \cdots \\
&= n(n-1)\,\left(\overrightarrow{P_{n-1}P_{n-2}} + \overrightarrow{P_{n-1}P_n}\right) \\
&\vdots
\end{aligned}$$

We deduce from these calculations that the Bézier curve is tangent to the segment $[P_0P_1]$ at control point P_0 and is tangent to the segment $[P_{n-1}P_n]$ at control point P_n.

A Bézier curve of degree n can be represented as a Bézier curve of degree $n+1$.

$$\begin{aligned}
P(t) &= \sum_{i=0}^{n} \Big((1-t)B_i^n(t) + tB_i^n(t) \Big) P_i \\
&= \sum_{i=0}^{n} \left(\frac{n+1-i}{n+1} B_i^{n+1}(t) + \frac{i+1}{n+1} B_{i+1}^{n+1}(t) \right) P_i \\
&= \sum_{i=0}^{n+1} B_i^{n+1}(t) \underbrace{\left(\frac{i}{n+1} P_{i-1} + \frac{n+1-i}{n+1} P_i \right)}_{\hat{P}_i} \\
&= \sum_{i=0}^{n+1} B_i^{n+1}(t)\, \hat{P}_i
\end{aligned}$$

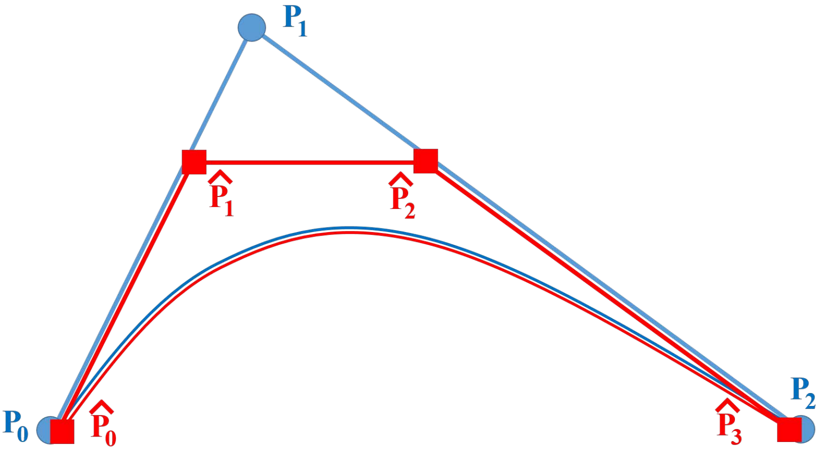
Therefore, the Bézier curve $P(t)$ is now expressed as a Bézier curve of degree $n+1$ with the new control points

$$\hat{P}_i = \frac{i}{n+1} P_{i-1} + \frac{n+1-i}{n+1} P_i, \quad i = 0, 1, \ldots, n+1.$$

Notice that $\hat{P}_0 = P_0$ and $\hat{P}_{n+1} = P_n$.

Repeated degree elevation leads to a sequence of control polygons that slowly converge to the Bézier curve.

Example : degree 2 to degree 3



De-Casteljau's algorithm is a *triangular scheme* that evaluates the curve at a point $P(t)$, $t \in]0,1[$, from convex combinations on the control points P_i.

$$\begin{aligned} P(t) &= \sum_{i=0}^{n} B_i^n(t)\; P_i && \text{step 0 : degree } n \\ &= \sum_{i=0}^{n} \Big((1-t)\, B_i^{n-1}(t) + t\, B_{i-1}^{n-1}(t) \Big)\; P_i && \text{Bernstein recurrence} \\ &= \sum_{i=0}^{n-1} B_i^{n-1}(t) \Big(\underbrace{(1-t)\, P_i + t\, P_{i+1}}_{P_i^1(t)} \Big) \\ &= \sum_{i=0}^{n-1} B_i^{n-1}(t)\; P_i^1(t) && \text{step 1 : degree } n-1 \end{aligned}$$

Then, iterating this process untill step n, we get

$$P(t) = P_0^n(t)$$

which is the De Casteljau algorithm

INITIALIZATION

  FOR $i = 0$ TO n SET

    $P_i^0(t) = P_i$

MAIN BODY

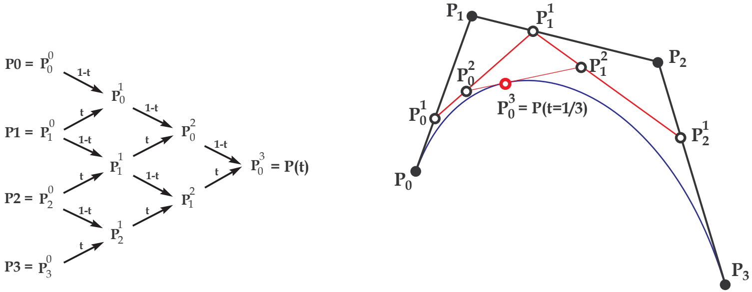
  FOR $j = 1$ TO n DO

    FOR $i = 0$ TO $n - j$ DO

      $P_i^j(t) = (1 - t)\, P_i^{j-1}(t) + t\, P_{i+1}^{j-1}(t)$

RESULT

  $P(t) = P_0^n(t)$



Triangular De Casteljau scheme for a cubic Bézier curve and for parameter $t = 1/3$.

The subdivision algorithm is a direct and remarkable consequence of De Casteljau's algorithm.

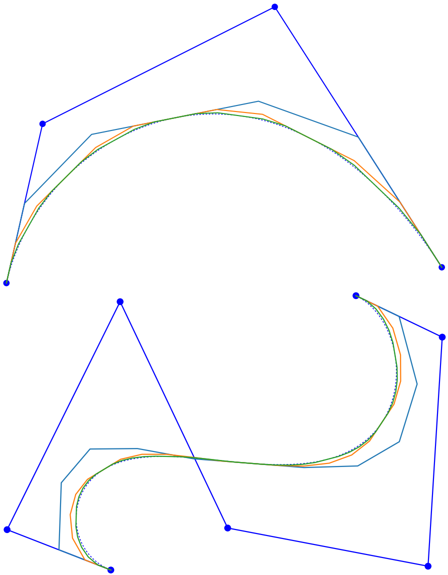
The triangular diagram of De-Casteljau above, carried out for a parameter $\alpha \in]0, 1[$, *subdivides the initial Bézier curve into two Bézier curves of same degree n, associated with each of the descending diagonals of the triangular diagram, namely :*

- $B[P_0^0, P_0^1, P_0^2, \ldots, P_0^n](t) = B[P_0, \ldots, P_n](\alpha\, t)$, $t \in [0, 1]$,
  is the first subdivided Bézier curve, image of $[0, \alpha]$ by the parametrization $P(t)$,

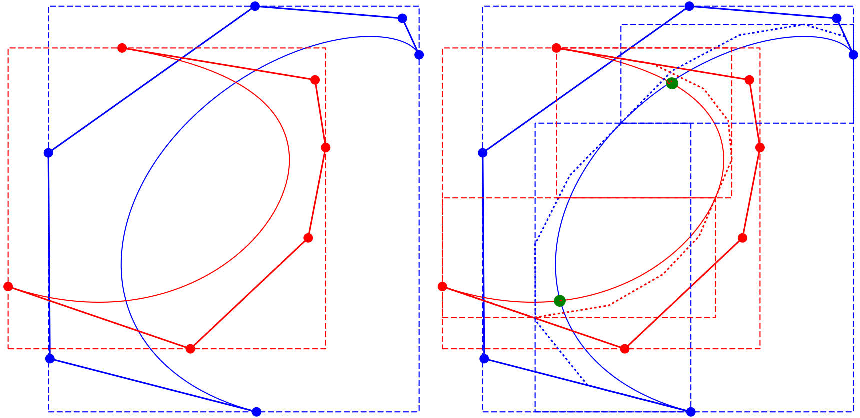
- $B[P_0^n, P_1^{n-1}, P_2^{n-2}, \ldots, P_n^0](t) = B[P_0, \ldots, P_n](\alpha + (1 - \alpha)\, t)$, $t \in [0, 1]$,
  is the second subdivided Bézier curve, image of $[\alpha, 1]$ by the parametrization $P(t)$.

The polygon $[P_0^0, P_0^1, P_0^2, \ldots, P_0^n, P_1^{n-1}, P_2^{n-2}, \ldots, P_n^0]$ obtained by concatenation of the control polygons of these two Bézier curves is the *subdivided polygon of order one*.

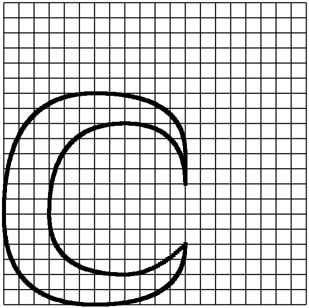
By iteration of this subdivision process, we obtain a sequence of subdivided polygons that converges uniformly towards the initial Bézier curve.

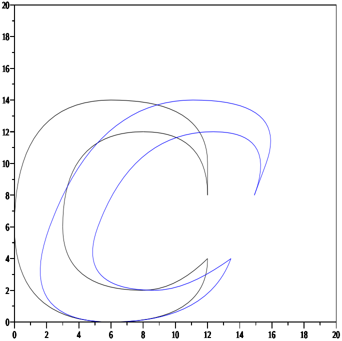


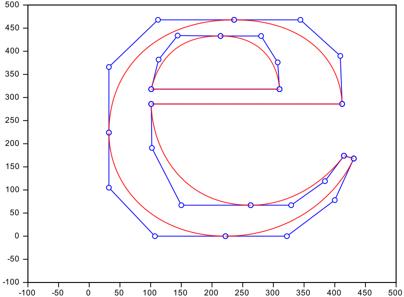
Three steps of subdivision of two Bézier curves with parameter $1/2$.



Some applications — Character modeling







Tensor Bézier surfaces —

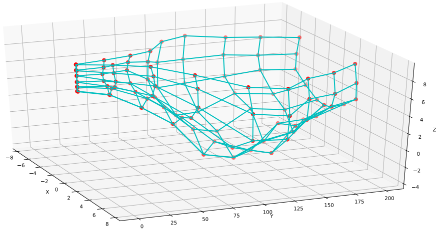
Une surface Bézier polynomiale tensorielle est une surface paramétrique polynomiale définie par *produit tensoriel* dans la base de Bernstein sur $[0,1]^2$

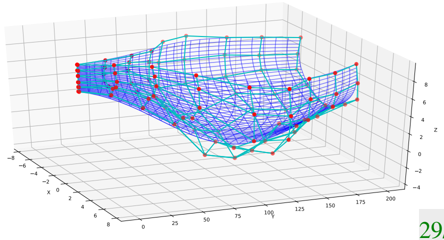
$$(u,v) \in [0,1]^2 \;\mapsto\; S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_i^n(u)\, B_j^m(v)\, P_{i,j}$$

de sorte que chaque point $S(u,v)$ s'exprime comme combinaison convexe d'un réseau de points formant le polyèdre de contrôle de la surface.

$$S(u,v) = \left[B_0^n(u), B_1^n(u), \ldots, B_n^n(u)\right] \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0m} \\ P_{10} & P_{11} & \cdots & P_{1m} \\ \vdots & & & \vdots \\ P_{n0} & P_{n1} & \cdots & P_{nm} \end{bmatrix} \begin{bmatrix} B_0^m(v) \\ B_1^m(v) \\ \vdots \\ B_m^m(v) \end{bmatrix}$$

où les $P_{i,j} \in \mathbb{R}^3$ sont les points de contrôle de la surface $S(u,v)$ dite de bi-degré (n,m).





Tensor Bézier surfaces — Isoparametric curves

L'écriture suivante

$$S(u,v) \;=\; \sum_{i=0}^{n} B_i^n(u) \left[\sum_{j=0}^{m} B_j^m(v)\, P_{i,j} \right] \;=\; \sum_{j=0}^{m} B_j^m(v) \left[\sum_{i=0}^{n} B_i^n(u)\, P_{i,j} \right]$$

montre que :

– chaque courbe iso-paramétrique $v = \hat{v}$ est une courbe de Bézier polynomiale $S(u,\hat{v}) = S_{\hat{v}}(u)$ de degré n dont les points de contrôle sont

$$\hat{P}_i = \sum_{j=0}^{m} B_j^m(\hat{v})\, P_{i,j}, \;\; i = 0, 1, \ldots, n$$

– chaque courbe iso-paramétrique $u = \hat{u}$ est une courbe de Bézier polynomiale $S(\hat{u},v) = S_{\hat{u}}(v)$ de degré m dont les points de contrôle sont

$$\hat{P}_j = \sum_{i=0}^{n} B_i^n(\hat{u})\, P_{i,j}, \;\; j = 0, 1, \ldots, m$$

On en déduit que

– la surface est contenue dans l'enveloppe convexe de son polyèdre de contrôle.

Tensor Bézier surfaces — Propriétés aux bords

En particulier :

- La surface interpole les 4 coins du polyèdre de contrôle :

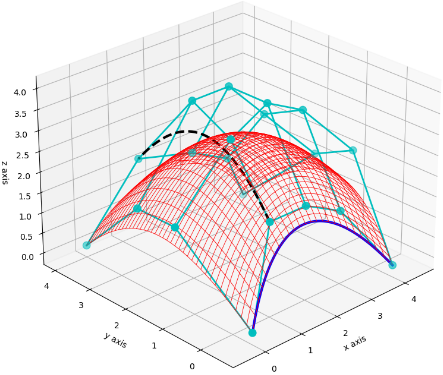
$S(0,0) = P_{0,0}$	$S(0,1) = P_{0,m}$
$S(1,0) = P_{n,0}$	$S(1,1) = P_{n,m}$

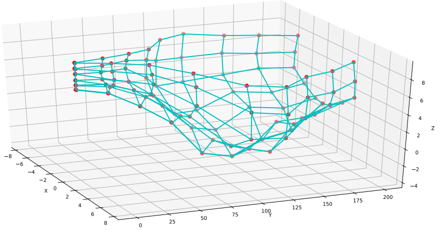
- Les 4 courbes du bord de de la surface $S(u,v)$ sont les courbes de Bézier associées aux bords correspondant du polyèdre, c'est-à-dire aux premières et dernières lignes et colonnes de points de contrôle du polyèdre de contrôle :

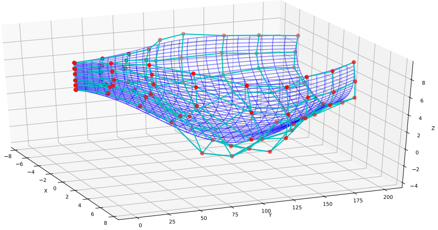
$S(u,0) = BP[P_{0,0}, P_{1,0}, \ldots, P_{n,0}](u)$	$S(u,1) = BP[P_{0,m}, P_{1,m}, \ldots, P_{n,m}](u)$
$S(0,v) = BP[P_{0,0}, P_{0,1}, \ldots, P_{0,m}](v)$	$S(1,v) = BP[P_{n,0}, P_{n,1}, \ldots, P_{n,m}](v)$

- A noter que *les autres lignes et colonnes de points de contrôle ne définissent pas une courbe "sur" la surface $S(u,v)$.*

Tensor Bézier surfaces — Example







Polyèdre de contrôle & Représentation fil de fer de la surface

Remarque :
En pratique, l'évaluation et le tracé d'une courbe ou d'une surface de Bézier sont réalisés à l'aide des algorithmes de *De-Casteljau et de subdivision* directement à partir du polygone/polyèdre de contrôle.

L'utilisation de la base de Bernstein est néanmoins nécessaire pour les algorithmes d'interpolation et d'approximation.

- (+) Simplicité et contrôle : la manipulation des points de contrôle permet de déformer et ajuster aisément la surface.
- (-) Contrôle global : la modification d'un seul point de contrôle modifie l'ensemble de la courbe.
- (-) Pertinent uniquement pour des degrés faibles (6 à 10 maxi)
- (-) Pour l'interpolation ou l'approximation, nécessité de ramener les données sur [0,1], sinon les calculs sont très vite instables numériquement (perte de la propriété de combinaison convexe).