Chapter 1

Hermite interpolation

Introduction

This chapter should be more precisely entitled Polynomial Hermite interpolation. Following Lagrange interpolation which consists in determining a curve which passes through predetermined positions (the interpolation points), we now consider Hermite interpolation which requires to satisfy additional constraints on the derivatives at the interpolation points.

Hermite interpolation is said to be of order one if we consider only first derivatives and of order two if we consider first and second derivatives. In this chapter, we essentially consider first derivatives except in section 3 where we introduce Hermite interpolation of order two over two points.

Figure 1.1: Hermite interpolation of order one of a function $f$ (in blue) over 4, 5 and 10 data points.

1 Hermite interpolation over $n$ data points

Given a set of $n + 1$ distincts points $x_0, x_1, \ldots, x_n$ in an interval $[a, b]$ and two set of $n + 1$ real values :

$y_0, y_1, \ldots, y_n$ and $y'_0, y'_1, \ldots, y'_n$,

we look for a polynomial $p(x)$ satisfying :

\[
\begin{align*}
  p(x_i) &= y_i, & i = 0, 1, \ldots, n, \\
  p'(x_i) &= y'_i, & i = 0, 1, \ldots, n.
\end{align*}
\]

(1.1)

If the values $y_i$ and $y'_i$ come from a function $f \in C^1[a, b] :

\[
y_i := f(x_i) \quad \text{and} \quad y'_i := f'(x_i), \quad 0 \leq i \leq n,
\]

this Hermite interpolating polynomial $p(x)$ will be denoted $P_H(x, f)$. Observing the number of constraints (equal to $2n + 2$) induces us to search for a polynomial $p_{2n+1}(x)$ of degree $2n + 1$.

\footnote{Charles Hermite, 1822-1901, French mathematician}
1.1 Construction of an Hermite basis

By analogy with the Lagrange approach we construct a polynomial basis \( \{ h_i(x), \tilde{h}_i(x), i = 0, 1, \ldots, n \} \) of \( \mathbb{R}_{2n+1}[x] \) satisfying the constraints

\[
\begin{aligned}
\{ \quad & h_i(x_j) = \delta_{ij} \\
& h'_i(x_j) = 0 \quad \text{and} \quad \tilde{h}_i(x_j) = 0 \\
& \tilde{h}'_i(x_j) = \delta_{ij} \quad \text{for} \quad 0 \leq i \leq n \\
& 0 \leq j \leq n
\end{aligned}
\]

(1.2)

Such a basis will then make it possible to write the Hermite interpolation polynomial in the form

\[
p_{2n+1}(x) = \sum_{i=0}^{n} y_i \ h_i(x) + \sum_{i=0}^{n} y'_i \ \tilde{h}_i(x).
\]

We make explicit constraints (1.2) in the following table where the constraints on each polynomial \( h_i(x) \) and \( \tilde{h}_i(x) \) are specified on the associate column (i.e., labelled by \( h_i \) or \( \tilde{h}_i \)).

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<tr>
<th>( x_0 )</th>
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</table>

| \( x_0 \) | 0 | 0 | \( \ldots \) | 0 | \( \ldots \) | 0 | 1 | 0 | \( \ldots \) | 0 | \( \ldots \) | 0 |
| \( x_1 \) | 0 | 0 | \( \ldots \) | 0 | \( \ldots \) | 0 | 0 | 1 | \( \ldots \) | 0 | \( \ldots \) | 0 |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \ddots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( x_i \) | 0 | 0 | \( \ldots \) | 0 | 0 | 1 | 0 | \( \ddots \) | \( \vdots \) | \( \vdots \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \ddots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( x_n \) | 0 | 0 | \( \ldots \) | 0 | 0 | 0 | 0 | \( \ldots \) | 0 | 1 |

We now have to prove that such a base exists. Construction of polynomials \( h_i(x) \) and \( \tilde{h}_i(x) \) is based on Lagrange basis polynomials \( L_i(x) \) associated with data points \( x_j \) and introduced in section ?? of chapter ??.

- Construction of \( h_i(x) \).

By relations (1.2) we have \( h_i(x_j) = h'_i(x_j) = 0 \) for \( 0 \leq j \leq n, j \neq i \), so that polynomial \( h_i(x) \) admits a double root at each point \( x_j \neq x_i \), and thus is on the form

\[
h_i(x) = L^2_i(x) \ r_i(x)
\]

where \( r_i(x) \) is a polynomial of degree less than or equal to 1. The two additional constraints that must satisfy \( h_i(x) \) leads to

\[
\begin{aligned}
1 &= h_i(x_i) &= L^2_i(x_i) \ r_i(x_i) \\
0 &= h'_i(x_i) &= L^2_i(x_i) \ r'_i(x_i) + 2L_i(x_i) \ L'_i(x_i) \ r_i(x_i) &= r'_i(x_i) + 2 L'_i(x_i)
\end{aligned}
\]

so that

\[
r_i(x) = 1 - 2 (x - x_i) L'_i(x_i).
\]

Finally, with the derivative formula (??) of \( L_i(x) \), given in chapter ??, we get

\[
h_i(x) = L^2_i(x) \left( 1 - 2 (x - x_i) \sum_{j=0}^{n} \frac{1}{x_i - x_j} \right)
\]

(1.3)
• Construction of $\bar{h}_i(x)$.

By relations (1.2) we have $\bar{h}_i(x_j) = \bar{h}_i'(x_j) = 0$ for $0 \leq j \leq n$, $j \neq i$, so that polynomial $\bar{h}_i(x)$ admits a double root at each point $x_j \neq x_i$, and thus is on the form

$$\bar{h}_i(x) = L_i^2(x) s_i(x)$$

where $s_i(x)$ is a polynomial of degree less than or equal to 1. The two additional constraints that must satisfy $\bar{h}_i(x)$ leads to

$$\begin{align*}
0 &= \bar{h}_i(x_i) = L_i^2(x_i) s_i(x_i) \\
1 &= \bar{h}_i'(x_i) = L_i^2(x_i) s_i'(x_i) + 2 L_i(x_i) L_i'(x_i) s_i(x_i) = s_i'(x_i)
\end{align*}$$

so that

$$s_i(x) = x - x_i$$

and finally

$$\bar{h}_i(x) = (x - x_i) L_i^2(x) \quad (1.4)$$

Proposition 1

The set of polynomials $\{h_i(x), \bar{h}_i(x), i = 0, 1, \ldots, n\}$ form a basis of the vector space $\mathbb{R}_{2n+1}[x]$. This basis is called the polynomial Hermite interpolation basis relative to data points $x_i$ and polynomials $h_i(x)$ and $\bar{h}_i(x)$ are named Hermite interpolation basis polynomials.

Proof: This family contains $2n + 2$ polynomials, each of them being of degree $2n + 1$ by construction, so that we just need to verify that this family is linearly independent. So, consider real values $\alpha_i$ and $\tilde{\alpha}_i$ ($0 \leq i \leq n$), such that

$$\alpha_0 h_0(x) + \cdots + \alpha_n h_n(x) + \tilde{\alpha}_0 \bar{h}_0(x) + \cdots + \tilde{\alpha}_n \bar{h}_n(x) = 0$$

for all real number $x$. This relation and its derivative applied in particular to each of the data point $x_i$ leads to the nullity of each of the $\alpha_i$ and $\tilde{\alpha}_i$. Which gives the result.

1.2 Hermite interpolating polynomial

We are now in situation to exhibit a solution of the Hermite interpolation problem as characterized above by constraints (1.1).

Proposition 2

There exists a unique polynomial in $\mathbb{R}_{2n+1}[x]$ satisfying the Hermite constraints (1.1). This polynomial is defined by

$$p_{2n+1}(x) = \sum_{i=0}^{n} y_i h_i(x) + \sum_{i=0}^{n} y'_i \bar{h}_i(x),$$

$$= \sum_{i=0}^{n} y_i L_i^2(x) \left( 1 - 2 \frac{x - x_i}{x_i - x_j} \sum_{j=0}^{n} \frac{1}{x_i - x_j} \right) + \sum_{i=0}^{n} y'_i (x - x_i) L_i^2(x). \quad (1.5)$$

This polynomial is the Hermite interpolating polynomial of the data $(x_i, y_i, y'_i)$.

Proof: One can easily check that the polynomial defined by (1.5) satisfy all the constraints (1.1).

Now, assume there exist two polynomials $p_{2n+1}(x)$ and $q_{2n+1}(x)$ in $\mathbb{R}_{2n+1}[x]$ satisfying these constraints. Then, polynomial $p_{2n+1}(x) - q_{2n+1}(x) \in \mathbb{R}_{2n+1}[x]$ admits $n + 1$ distinct double roots and is thus zero, which gives the result.
1.3 Examples

- Uniform and Chebyshev distribution of data points $x_i$.

As in the Lagrange interpolation, the choice of evenly spaced points is not recommended and can lead to oscillations on the edges of the interval. Below, in figure ??, we compare the uniform and Chebyshev distributions of data points.

![Figure 1.2: Hermite interpolation of order one of a function $f$ with uniform distribution of points.](image1)

- Comparison with Lagrange interpolation.

Since Hermite interpolation of order one involves two data $y_i$ and $y_i'$ at each point $x_i$, we multiply by two the number of data points in Lagrange interpolation. The comparison is performed with the same function and with uniform and Chebyshev distributions of data points.

![Figure 1.3: Hermite interpolation of order one of a function $f$ with Chebyshev distribution of points.](image2)

![Figure 1.4: Lagrange interpolation of a function $f$ (in blue) with uniform distribution of points.](image3)

The degree of the Hermite interpolating polynomial increases quickly with the number of data points. In case of the Hermite interpolation of a function, this process imposes the calculation of the derivatives. Again, one should prefer spline interpolation. However, in some situations where the
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1.5

precise satisfaction of Hermite constraints (on derivatives) are necessary, Hermite interpolation is an option.

We now specify a particular case of Hermite interpolation that will be useful for spline interpolation introduced in chapter ??

2 Cubic Hermite Interpolation over 2 points

We consider in this section the simple case of two interpolation data points \( \alpha \) and \( \beta \) \( (\alpha < \beta) \). Precisely, given the two Hermite data \((\alpha, y_\alpha, y'_\alpha)\) and \((\beta, y_\beta, y'_\beta)\), we know by the previous section that there exits a unique cubic polynomial \( p(x) \) interpolating these data, that is satisfying

\[
p(\alpha) = y_\alpha, \quad p'(\alpha) = y'_\alpha, \quad p'(\beta) = y'_\beta, \quad p(\beta) = y_\beta.
\]

This cubic Hermite interpolating polynomial can be written as follows

\[
p(x) = y_\alpha h_\alpha(x) + y_\beta h_\beta(x) + y'_\alpha \bar{h}_\alpha(x) + y'_\beta \bar{h}_\beta(x)
\]

with the cubic Hermite interpolation basis \( h_\alpha, h_\beta, \bar{h}_\alpha, \bar{h}_\beta \) relative to data points \( \alpha, \beta \).

However, it is usual to reduce this Hermite process to a standard Hermite process relative to the two points 0 and 1 (or sometimes the points \(-1\) and 1), which makes it possible in particular to apply this process more simply to \( n \) points taken 2 by 2 in the situation of cubic splines.

2.1 Cubic Hermite basis on \([0, 1]\)

The Hermite interpolating polynomial \( p(x) \) can be rewritten as follows

\[
p(x) = y_\alpha H_0 \left( \frac{x - \alpha}{\beta - \alpha} \right) + y'_\alpha \left( \frac{x - \alpha}{\beta - \alpha} \right) H_1 \left( \frac{x - \alpha}{\beta - \alpha} \right) + y'_\beta \left( \frac{x - \alpha}{\beta - \alpha} \right) H_2 \left( \frac{x - \alpha}{\beta - \alpha} \right) + y_\beta H_3 \left( \frac{x - \alpha}{\beta - \alpha} \right)
\]

where \( H_0, H_1, H_2, H_3 \) are four cubic polynomials forming the standard cubic Hermite basis on \([0, 1]\) and characterized by the following table.

<table>
<thead>
<tr>
<th>( H_i(0) )</th>
<th>( H_i(1) )</th>
<th>( H_i'(0) )</th>
<th>( H_i'(1) )</th>
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<tbody>
<tr>
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</table>

Hermite interpolation on the interval \([\alpha, \beta]\) is deduced from the Hermite interpolation on \([0, 1]\) by the affine transformation \( x \in [\alpha, \beta] \mapsto t = \frac{x - \alpha}{\beta - \alpha} \in [0, 1] \).
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Figure 1.6: Cubic Hermite polynomials basis $H_i$ on the interval $[0,1]$.

**Exercise 1**
Consider the three following functions which are plotted below respectively in blue, green and red.

\[
\begin{align*}
    f_1(x) &= \frac{\exp(x)}{2} - 1, \quad x \in [-1,0.5], \\
    f_2(x) &= \sin(x^2) - 1, \quad x \in [2,3], \\
    f_3(x) &= -1 + 2 \frac{\sin(2x)}{x}, \quad x \in [5,8].
\end{align*}
\]

Determine two functions $p_1$ and $p_2$ respectively defined on intervals $[0.5,2]$ and $[3,5]$, and plot all these functions, such that the concatenation of the five functions $f_1, p_1, f_2, p_2, f_3$ provides a $C^1$ function over the interval $[-1,8]$.

The following technical result will be essential for the construction of $C^2$ cubic interpolation splines in chapter ??

**Lemma 1**
Let $p(x)$ be the cubic Hermite interpolating polynomial relative to data $(\alpha, y_\alpha, y'_\alpha)$ and $(\beta, y_\beta, y'_\beta)$ and let $h = \beta - \alpha$. Then, the second derivatives of $p(x)$ at points $\alpha$ and $\beta$ can be expressed with respect to the interpolation data as follows.

\[
p''(\alpha) = \frac{2}{h^2} \left( 3y_\beta - 3y_\alpha - 2hy'_\alpha - hy'_\beta \right) \quad \text{and} \quad p''(\beta) = \frac{2}{h^2} \left( 3y_\alpha - 3y_\beta + 2hy'_\beta + hy'_\alpha \right).
\]

**Proof:** Notice that the two formulas are identical up to a data permutation and by replacing $h$ with $-h$. So we just need to prove the first one. Then, since the Hermite interpolating polynomial $p(x)$ and its derivative $p'(x)$ are respectively of degree 3 and 2, they coincide with their Taylor expansion respectively.
of order 3 and 2 at point $\alpha$.

$$p(x) = y_\alpha + (x - \alpha)y'_\alpha + \frac{(x - \alpha)^2}{2} p''(\alpha) + \frac{(x - \alpha)^3}{6} p'''(\alpha),$$

$$p'(x) = y'_\alpha + (x - \alpha)p''(\alpha) + \frac{(x - \alpha)^2}{2} p'''(\alpha).$$

For $x = \beta$, we get

$$p(\beta) = y_\alpha + h y'_\alpha + \frac{h^2}{2} p''(\alpha) + \frac{h^3}{6} p'''(\alpha) \quad \text{and} \quad p'(\beta) = y'_\alpha + h p''(\alpha) + \frac{h^2}{2} p'''(\alpha),$$

from which we deduce the result after eliminating the term $p'''(\alpha)$:

$$p''(\alpha) = \frac{2}{h^2}(3y_\beta - 3y_\alpha - 2hy'_\alpha - hy'_\beta).$$

2.2 Error bound for cubic Hermite interpolation over 2 points

We propose to estimate the error associated with the cubic Hermite interpolation over two points in the form of a problem.

Let $f \in C^4[\alpha, \beta]$ and let $p(x) = P_H(x, f)$ be the cubic Hermite interpolating polynomial of the function $f$ at points $\alpha$ and $\beta$.

Considering a fixed value $x$ in $[\alpha, \beta]$, we introduce the function $\phi$ defined by

$$u \in [\alpha, \beta] \quad \mapsto \quad \phi(u) = f(u) - p(u) - \frac{(u - \alpha)^2 (u - \beta)^2}{(x - \alpha)^2 (x - \beta)^2} \left( f(x) - p(x) \right).$$

1. Prove that $\phi$ cancels at points $\alpha$, $\beta$ and $x$. Deduce that $\phi'$ cancels at two distinct points in $[\alpha, \beta]$.
2. Prove that $\phi'(\alpha) = \phi'(\beta) = 0$.
3. Deduce that there exists $\zeta_x \in [\alpha, \beta]$ such that $\phi^{(4)}(\zeta_x) = 0$.
4. Prove that

$$f(x) - p(x) = \frac{(x - \alpha)^2 (x - \beta)^2}{24} f^{(4)}(\zeta_x).$$

5. Finally, deduce that for all $x \in [\alpha, \beta]$ we have

$$|f(x) - p(x)| \leq \frac{(\beta - \alpha)^4}{24} \max_{\zeta \in [\alpha, \beta]} |f^{(4)}(\zeta)|,$$

from which we get the upper bound for the error

$$\left| |f - P_H(\cdot, f)| \right| \leq \frac{(\beta - \alpha)^4}{24} \left| |f^{(4)}| \right|.$$
1. **Hermite interpolation of order 2 on \([0,1]\) : existence of a solution.**
   Prove that there exists a unique quintic polynomial \(q(x)\) such that
   \[
   q(0) = y_0 \\
   q(1) = y_1 \\
   q'(0) = y'_0 \\
   q'(1) = y'_1 \\
   q''(0) = y''_0 \\
   q''(1) = y''_1
   \]
   where \(y_0, y'_0, y''_0, y_1, y'_1, y''_1\) are prescribed real numbers.

2. **Hermite interpolation of order 2 on \([\alpha, \beta]\) : existence of a solution.**
   Deduce there exists a unique quintic polynomial \(p(x)\) such that
   \[
   p(\alpha) = y_\alpha \\
   p'\beta) = y'_\beta \\
   p''(\alpha) = y''_\alpha \\
   p''(\beta) = y''_\beta
   \]
   where \(y_\alpha, y'_\alpha, y''_\alpha, y_\beta, y'_\beta, y''_\beta\) are prescribed real numbers.

3. **Quintic Hermite basis on \([0,1]\).**
   Consider the following quintic polynomials, which are plotted below on figure 1.7
   \[
   Q_0(x) = -6x^5 + 15x^4 - 10x^3 + 1 \\
   Q_1(x) = -3x^5 + 8x^4 - 6x^3 + x \\
   Q_2(x) = \frac{1}{2}(-x^5 + 3x^4 - 3x^3 + x^2) \\
   Q_3(x) = \frac{1}{2}(x^5 - 2x^4 + x^3) \\
   Q_4(x) = -3x^5 + 7x^4 - 4x^3 \\
   Q_5(x) = 6x^5 - 15x^4 + 10x^3
   \]
   Calculate the following matrix which gathers the values of the polynomials \(Q_i(x), Q'_i(x), Q''_i(x)\) at points 0 and 1.
   \[
   \begin{bmatrix}
   Q_0(0) & Q'_0(0) & Q''_0(0) & Q'_0(1) & Q''_0(1) \\
   Q_1(0) & Q'_1(0) & Q''_1(0) & Q'_1(1) & Q''_1(1) \\
   Q_2(0) & Q'_2(0) & Q''_2(0) & Q'_2(1) & Q''_2(1) \\
   Q_3(0) & Q'_3(0) & Q''_3(0) & Q'_3(1) & Q''_3(1) \\
   Q_4(0) & Q'_4(0) & Q''_4(0) & Q'_4(1) & Q''_4(1) \\
   Q_5(0) & Q'_5(0) & Q''_5(0) & Q'_5(1) & Q''_5(1)
   \end{bmatrix}
   \]

4. Determine the unique solution \(q(x)\) to the Hermite interpolation problem of order 2 on the interval \([0,1]\), as a function of the polynomials \(Q_i\).

5. Determine the unique solution \(p(x)\) to the Hermite interpolation problem of order 2 on the interval \([\alpha, \beta]\), as a function of the polynomials \(Q_i\).

**Solution:** The unique solution \(p(x)\) to the Hermite interpolation problem of order 2 on the interval \([\alpha, \beta]\) is
   \[
   p(x) = \begin{pmatrix}
   Q_0(t) \\
   Q_1(t) \\
   Q_2(t) \\
   Q_3(t) \\
   Q_4(t) \\
   Q_5(t)
   \end{pmatrix}
   \begin{pmatrix}
   y_\alpha \\
   (\beta - \alpha) y'_\alpha \\
   (\beta - \alpha)^2 y''_\alpha \\
   y_\beta \\
   (\beta - \alpha) y'_\beta \\
   (\beta - \alpha)^2 y''_\beta
   \end{pmatrix}
   \]
   with \(t = \frac{x - \alpha}{\beta - \alpha}\)

**Exercise 2**

Comparison with the cubic Hermite interpolation.
Consider again the three functions given in exercise \footnote{1} and determine the two quintic polynomials.
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Figure 1.7: Quintic Hermite polynomials basis $Q_i$ on the interval $[0, 1]$ & zoom on the right figure.

$q_1(x)$ and $q_2(x)$ respectively defined on intervals $[0.5, 2]$ and $[3, 5]$, such that the concatenation of the five functions $f_1, q_1, f_2, q_2, f_3$ provides a $C^2$ function over the interval $[-1, 8]$.

Solution:

Figure 1.8: Comparison between cubic and quintic Hermite interpolation.

4 Exercises

Exercise 3

Hermite interpolation of order $n$ at one point. Let $n \in \mathbb{N}$, $a$ a fixed real number as well as $n+1$ real numbers $y_0, y_1, \ldots, y_n$.

Prove that there exists a unique polynomial $p(x)$ of degree $n$ such that

$p^{(k)}(a) = y_k, \quad k = 0, 1, \ldots, n.$

Exercise 4

An instructive example. Let $y_0, y'_1, y_2$ three real numbers. Determine the set of polynomials $p(x) = a_0 + a_1 x + a_2 x^2 \in \mathbb{R}_2[x]$ satisfying the constraints

$p(0) = y_0, \quad p'(1) = y'_1, \quad p(2) = y_2.$
according values of parameters $y_0$, $y'_1$ and $y_2$.

Exercise 5

From Lagrange to Hermite

Let $h \in [0, 1]$.

1. Write the quadratic Lagrange polynomials relative to data points $x_0 = 0$, $x_1 = h$, $x_2 = 1$.

2. Given real values $y_0$, $\alpha$, $y_2$, determine the unique polynomial $p(x)$ of degree less than or equal $2$ satisfying the constraints

$$p(x_0) = y_0, \quad p(x_1) = y_0 + \alpha h, \quad p(x_2) = y_2.$$ 

3. Write the previous polynomial $p(x)$ on the form

$$p(x) = y_0 p_0^h(x) + \alpha p_1^h(x) + y_2 p_2^h(x),$$

and specify the polynomials $p_0^h(x)$, $p_1^h(x)$, $p_2^h(x)$.

4. Prove that polynomials $p_0^h(x)$, $p_1^h(x)$, $p_2^h(x)$ converge, when $h$ tends to $0$, to three polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$ which satisfy

$$p_0(0) = 1, \quad p_0'(0) = 0, \quad p_0(1) = 0,$$

$$p_1(0) = 0, \quad p_1'(0) = 1, \quad p_1(1) = 0,$$

$$p_2(0) = 0, \quad p_2'(0) = 0, \quad p_2(1) = 1.$$ 

5. Comment the result of the previous question and develop a similar process leading to the standard cubic Hermite basis on $[0, 1]$.

Exercise 6

Alternative construction of the Hermite interpolating polynomial of order one over $n$ data points.

Let $n + 1$ distinct interpolation points $x_0 < x_1 < \cdots < x_n$ as well as two set of real values $y_0$, $y_1$, $\ldots$, $y_n$, and $y'_0$, $y'_1$, $\ldots$, $y'_n$.

We intend to determine a polynomial $P_{2n+1}(x)$ of degree $2n + 1$ such that

$$P_{2n+1}(x_i) = y_i, \quad 0 \leq i \leq n \quad \text{and} \quad P'_{2n+1}(x_i) = y'_i, \quad 0 \leq i \leq n.$$ 

We propose an iterative construction which consists in solving the constraints successively in the following order $y_0$, $y'_0$, $y_1$, $y'_1$, $\ldots$, $y_n$, $y'_n$, that is :

- find a polynomial $P_0$ of degree $0$ such that $P_0(x_0) = y_0$,

- find a polynomial $P_1$ of degree $1$ such that $P_1(x_0) = y_0$, $P_1'(x_0) = y'_0$,

- find a polynomial $P_2$ of degree $2$ such that $P_2(x_0) = y_0$, $P_2'(x_0) = y'_0$, $P_2(x_1) = y_1$,

- $\ldots$ 

- find a polynomial $P_{2k}$ of degree $2k$ satisfying the previous constraints $y_0, y'_0, \ldots, y_{k-1}, y'_{k-1}$ and $P_{2k}(x_k) = y_k$,

- find a polynomial $P_{2k+1}$ of degree $2k + 1$ satisfying the previous constraints $y_0, y'_0, \ldots, y_{k-1}, y'_{k-1}, y_k$ and $P_{2k+1}(x_k) = y'_k$,

- $\ldots$
1. We propose to construct these polynomials iteratively on the following form
\[ P_0(x) = \alpha_0, \]
\[ P_1(x) = P_0(x) + \alpha_1 (x - x_0), \]
\[ P_2(x) = P_1(x) + \alpha_2 (x - x_0)^2, \]
\[ P_3(x) = P_2(x) + \alpha_3 (x - x_0)^2(x - x_1), \]
\[ P_4(x) = P_3(x) + \alpha_4 (x - x_0)^2(x - x_1)^2, \]
\[ P_5(x) = P_4(x) + \alpha_5 (x - x_0)^2(x - x_1)^2(x - x_2), \]
Determine coefficients \( \alpha_i \) allowing to satisfy the required constraints set out above.

2. Determine precisely polynomials \( P_{2k}(x) \) and \( P_{2k+1}(x) \) according interpolation data.

3. Numerical application. Determine the Hermite interpolating polynomial of order one of the function \( f(x) = \sin(2\pi x) \):
   a) at data points \( x_k = 1/4 + k, \ k = 0, 1, \ldots, 5 \) (without any calculation...),
   b) at data points \( x_k = k/2, \ k = 0, 1, 2 \).