From optimal stopping to stochastic optimization

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Outline

1. The optimal stopping problem
2. An optimization point of view
3. How to effectively solve the optimization problem
4. Numerical experiments
**Framework**

Consider the optimal stopping problem with time—$t$ value

$$U_t = \text{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}[Z_{\tau} | \mathcal{F}_t]$$

- The non–negative process $Z$ is càdlàg and adapted to the natural filtration $\mathcal{F}$ of $d$—dimensional Brownian motion. Assume $\mathbb{E} \left[ \sup_t Z_t^2 \right] < \infty$.
- The set $\mathcal{T}_t$ is the set of all $\mathcal{F}$—stopping times with values in $[t, T]$.
- A typical example is the pricing of an American option with discounted payoff $Z$. 
Dual approach (1)

The Snell envelope process \((U_t)_{0 \leq t \leq T}\) admits a Doob–Meyer decomposition

\[
U_t = U_0 + M^*_t - A^*_t.
\]

[ Rogers, 2002]: \(U_0 = \inf_{M \in H^1_0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - M_t) \right] = \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - M^*_t) \right] \)

- This problem admits more than a single solution.
- For any stopping time \(\tau\) smaller than the largest optimal strategy,

\[
U_0 = \inf_{M \in H^1_0} \mathbb{E} \left[ \sup_{\tau \leq t \leq T} (Z_t - M_t) \right] = \mathbb{E} \left[ \sup_{\tau \leq t \leq T} (Z_t - M^*_t) \right].
\]
Dual approach (2)

- Some of the martingales $M$ attaining the infimum are surely optimal

$$U_0 = \sup_{0 \leq t \leq T} (Z_t - M_t) \quad a.s.$$  

- From [Schoenmakers et al., 2013], any martingale satisfying

$$\text{Var} \left( \sup_{0 \leq t \leq T} (Z_t - M_t) \right) = 0$$

is surely optimal.

- From [Jamshidian, 2007], for any optimal stopping time $\tau$ and any surely optimal martingale $M$,

$$\left( M_{t \wedge \tau} \right)_t = \left( M^*_{t \wedge \tau} \right)_t.$$
Dual approach (3)

With our square integrability assumption, we can rewrite the minimization problem as

\[
U_0 = \inf_{X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})} \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - \mathbb{E}[X|\mathcal{F}_t]) \right].
\]

s.t. \( \mathbb{E}[X] = 0 \)

How to approximate \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) by a finite dimensional vector space in which conditional expectations are tractable in a closed form?
Truncated Wiener chaos expansion ($d = 1$)

Let $H_i$ the $i$-th Hermite polynomial.

Take a regular grid $0 = t_0 < t_1 < \cdots < t_n = T$ and $G_i = \frac{B_{t_i} - B_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}$.

Define the truncated Wiener chaos space of order $p$

$$\mathcal{H}_p = \text{span} \left\{ \prod_{i=1}^n H_{\alpha_i}(G_i) : \alpha \in \mathbb{N}^n, \|\alpha\|_1 = p \right\}$$

For $F \in L^2(\Omega, \mathcal{F}_T)$, we introduce the truncated chaos expansion of order $p$

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}} \lambda_{\alpha} \prod_{i \geq 1} H_{\alpha_i}(G_i) = \sum_{\alpha \in A_{p,n}} \lambda_{\alpha} \hat{H}_{\alpha}(G_1, \ldots, G_n)$$

where $A_{p,n} = \{ \alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq p \}$ with $\|\alpha\|_1 = \sum_{i \geq 0} \alpha_i$. 
Key property of the truncated Wiener chaos expansion

For $k \leq n$,

$$
\mathbb{E}[C_{p,n}(F) | \mathcal{F}_{tk}] = \sum_{\alpha \in A_{p,n}^k} \lambda_\alpha \hat{H}_\alpha(G_1, \ldots, G_n)
$$

with $A_{p,n}^k = \{ \alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq p, \alpha_\ell = 0 \ \forall \ell > k \}$.

“Computing $\mathbb{E}[\cdot | \mathcal{F}_{tk}]$” $\Leftrightarrow$ “Dropping all non $\mathcal{F}_{tk}$ – measurable terms”
Extension to the multi–dimensional case

The truncated chaos expansion of order $p$ of $F \in L^2(\Omega, \mathcal{F}_T)$ is given by

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}} \lambda_{\alpha} \hat{H}_{\alpha} \otimes \lambda(G_1, \ldots, G_n) = C_{p,n}(\lambda)$$

where

$$\hat{H}_{\alpha} \otimes \lambda(G_1, \ldots, G_n) = \prod_{j=1}^{d} \hat{H}_{\alpha_j}(G_{1}^{j}, \ldots, G_{n}^{j}) \quad \forall \alpha \in (\mathbb{N}^n)^d,$$

$$A_{p,n} \otimes \alpha = \{ \alpha \in (\mathbb{N}^n)^d : \|\alpha\|_1 \leq p \}.$$
Return to the optimal stopping problem

We approximate the original problem

$$\inf_{X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})} \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - \mathbb{E}[X|\mathcal{F}_t]) \right]$$

s.t. $\mathbb{E}[X] = 0$

by

$$\inf_{\lambda \in \mathbb{R}^{A_{p,n}}} V_{p,n}(\lambda) \quad (1)$$

s.t. $\lambda_0 = 0$

with

$$V_{p,n}(\lambda) = \mathbb{E} \left[ \max_{0 \leq k \leq n} (Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda)|\mathcal{F}_{t_k}]) \right].$$
Properties of the minimization problem (1)

**Proposition 1**

The minimization problem (1) has at least one solution.

- The function $V_{p,n}$ is clearly convex (maximum of affine functions).
- Not strongly convex but,

$$V_{p,n}(\lambda) \geq \frac{|\lambda|}{2} \inf_{\mu \in \mathbb{R}^{A_p,n}, |\mu|=1} \mathbb{E} \left[ |C_{p,n}(\mu)| \right].$$
Properties of the minimization problem (2)

\[ \mathcal{I}(\lambda, Z, G) = \{0 \leq k \leq n : \text{the pathwise maximum is attained at time } k\} . \]

**Proposition 2**

*Let* \( p \geq 1 \). *Assume that*

\[
\forall 1 \leq r \leq k \leq n, \ \forall F \in \mathcal{C}_{p-1,n}, \ F \neq 0, \\
\exists 1 \leq q \leq d \text{ s.t. } \mathbb{P}\left( \forall t \in [t_{r-1}, t_r], \ D_t^q Z_{tk} + F = 0 \mid Z_{tk} > 0 \right) = 0.
\]

*Then, the function* \( V_{p,n} \) *is differentiable at all points* \( \lambda \in \mathbb{R}^{A_{p,n}} \) *with no zero component and its gradient* \( \nabla V_{p,n} \) *is given by*

\[
\nabla V_{p,n}(\lambda) = \mathbb{E} \left[ \mathbb{E} \left[ \hat{H}^d(G_1, \ldots, G_n) \mid \mathcal{F}_{t_i} \right] \mid i = \mathcal{I}(\lambda, Z, G) \right] .
\]
Properties of the minimization problem (3)

- Differentiability is ensured as soon as \( I(\lambda, Z, G) \) is a.s. reduced to a unique element: purpose of the blue condition.

- Let \( \lambda_{p,n}^\# \) be a solution, \( V_{p,n}(\lambda_{p,n}^\#) = \inf_\lambda V_{p,n}(\lambda) \). Then,

\[
\nabla V_{p,n}(\lambda_{p,n}^\#) = 0.
\]
Convergence to the true solution

Proposition 3

The solution of the minimization problem (1), $V_{p,n}(\lambda_{p,n}^\#)$, converges to the optimal stopping value $U_0$ when both $p$ and $n$ go to infinity and moreover

$$0 \leq V_{p,n}(\lambda_{p,n}^\#) - U_0 \leq 2 \|M_T^* - C_{p,n}(M_T^*)\|_2.$$
Practically solving the optimization problem (1)

We approximate the solution of

\[ V_{p,n}(\lambda_{p,n}^{\#}) = \inf_{\lambda \in A_{p,n}} V_{p,n}(\lambda) = \inf_{\lambda \in A_{p,n}} \mathbb{E} \left[ \max_{0 \leq k \leq n} (Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda)|F_{t_k}]) \right] \]

by introducing the well–known *Sample Average Approximation* (see [Rubinstein and Shapiro, 1993]) of \( V_{p,n} \) defined by

\[ V_{p,n}^{m}(\lambda) = \frac{1}{m} \sum_{i=1}^{m} \max_{0 \leq k \leq n} (Z_{t_k}^{(i)} - \mathbb{E}[C_{p,n}^{(i)}(\lambda)|F_{t_k}]). \]

Note that the conditional expectation boils down to truncating the chaos expansion and hence is tractable in a closed form.
Practically solving the optimization problem (2)

For large enough $m$, $V_{p,N}^m$ is convex, a.s. differentiable and tends to infinity at infinity. Then, there exists $\lambda_{p,n}^m$ such that

$$V_{p,n}^m(\lambda_{p,n}^m) = \inf_{\lambda \in \mathbb{R}^{A_{p,n}}} V_{p,n}^m(\lambda).$$

**Proposition 4**

$V_{p,n}^m(\lambda_{p,n}^m)$ converges a.s. to $V_{p,n}(\lambda_{p,N}^#)$ when $m \to \infty$. The distance from $\lambda_{p,n}^m$ to the set of minimizers of $V_{p,n}$ converges to zero as $m$ goes to infinity.
Practically solving the optimization problem (3)

Write $M_k(\lambda) = \mathbb{E}[C_{p,n}(\lambda)|\mathcal{F}_k]$ for $0 \leq k \leq n$.

**Proposition 5**

Assume $\lambda_{p,n}^\#$ is unique. Then,

$$\frac{1}{m} \sum_{i=1}^{m} \left( \max_{0 \leq k \leq n} Z_{tk}^{(i)} - M_k^{(i)}(\lambda_{p,n}^m) \right)^2 - \mathbb{V}_{p,n}(\lambda_{p,n}^m)^2$$

is a convergent estimator of $\text{Var}(\max_{0 \leq k \leq n} Z_{tk} - M_k(\lambda_{p,n}^\#))$ and moreover, if $\lambda_{p,n}^m$ is bounded,

$$\lim_{m \to \infty} m \text{Var} \left( \mathbb{V}_{p,n}(\lambda_{p,n}^m) \right) = \text{Var}(\max_{0 \leq k \leq n} Z_{tk} - M_k(\lambda_{p,n}^\#)).$$
The algorithm: bespoke martingales

Define the first time the option goes in the money by

\[ \tau_0 = \inf\{k \geq 0 : Z_{t_k} > 0\} \land n. \]

Consider martingales only starting once the option has been in the money

\[ N_k(\lambda) = M_k(\lambda) - M_{k \land \tau_0}(\lambda). \]

In the dual price, \( \max_{0 \leq k \leq n} \) can be shrunk to \( \max_{\tau_0 \leq k \leq n} \).

Using Doob’s stopping theorem, we have

\[
\mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - M_k(\lambda)) \right] = \mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - (M_k(\lambda) - M_{\tau_0}(\lambda))) \right]
\]

The martingales \( M(\lambda) \) or \( N(\lambda) \) lead to the same minimum value.

The set of martingales \( N^\lambda \) is far more efficient from a practical point of view.
The algorithm: a gradient descent with line search

\[
x_0 \leftarrow 0, \ k \leftarrow 0, \ \gamma \leftarrow 1, \ d_0 \leftarrow 0, \ v_0 \leftarrow \infty ;
\]

\[
\text{while True do}
\]

\[
\text{Compute } v_{k+1/2} \leftarrow V^m_{p,n}(x_k - \gamma \alpha_k d_k) ;
\]

\[
\text{if } v_{k+1/2} < v_k \text{ then}
\]

\[
x_{k+1} \leftarrow x_k - \gamma \alpha_k d_k ;
\]

\[
v_{k+1} \leftarrow v_{k+1/2} ;
\]

\[
d_{k+1} \leftarrow \nabla V^m_{p,n}(x_{k+1}) ;
\]

\[
\text{if } \left| \frac{v_{k+1} - v_k}{v_k} \right| \leq \varepsilon \text{ then return;}
\]

\[
\text{else}
\]

\[
\gamma \leftarrow \gamma / 2 ;
\]

\[
\text{end}
\]

\[
\text{end}
\]
The algorithm: a gradient descent with line search

\[ x_0 \leftarrow 0, \ k \leftarrow 0, \ \gamma \leftarrow 1, \ d_0 \leftarrow 0, \ v_0 \leftarrow \infty ; \]

while True do

Compute \( v_{k+1/2} \leftarrow V_{p,n}^m(x_k - \gamma \alpha_k d_k) \);

if \( v_{k+1/2} < v_k \) then

\[ x_{k+1} \leftarrow x_k - \gamma \alpha_k d_k ; \]

\[ v_{k+1} \leftarrow v_{k+1/2} ; \]

\[ d_{k+1} \leftarrow \nabla V_{p,n}^m(x_{k+1}) ; \]

if \( \frac{|v_{k+1} - v_k|}{v_k} \leq \varepsilon \) then return;

else

\[ \gamma \leftarrow \gamma / 2 ; \]

end

end

Take \( \alpha_\ell = \frac{V_{p,n}^m(x_\ell) - \mathbb{E}[Z_T]}{\| \nabla \tilde{V}_{p,n}^m(x_\ell) \|^2} \), see [Polyak, 1987].
Some remarks on the algorithm

- Given the expression of $V_{p,n}^m$, both the value function and its gradient are computed at the same time without extra cost.

\[
V_{p,n}(\lambda) = \mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} \left( Z_{t_k} - \mathbb{E}[\lambda \cdot H^\otimes_d (G_1, \cdots, G_n) | \mathcal{F}_{t_k}] \right) \right],
\]

\[
= \mathbb{E}[Z_{t_{\mathcal{I}(\lambda,z,G)}}] - \lambda \cdot \nabla \tilde{V}_{p,n}(\lambda).
\]

- Checking the admissibility of a step $\gamma$ costs as much as updating $x_k$.

- The algorithm is *almost* embarrassingly parallel:
  - Few iterations of the gradient descent are required ($\approx 10$).
  - Each iteration is fully parallel: each process treats its bunch of paths.
  - No demanding centralized computations
  - Very little communication: a few broadcasts only.
Parallel implementation

In parallel Generate \((G^{(1)}, Z^{(1)}), \ldots, (G^{(m)}, Z^{(m)})\) \(m \ x_0 \leftarrow 0 \in \mathbb{R}^{A_p, n} \);

while True do

  Broadcast \(x_\ell, d_\ell, \gamma, \alpha_\ell\);

  In parallel Compute \(\max_{\tau_0 \leq k \leq n} (Z_{t_k}^{(i)} - N_k^{(i)} (x_\ell - \gamma \alpha_\ell d_\ell))\);

  Make a reduction of the above contributions to obtain \(V_{p,n}^m (x_\ell+1/2)\) and

  \[
  \nabla V_{p,n}^m (x_\ell+1/2);
  \]

  \(v_{\ell+1/2} \leftarrow V_{p,n}^m (x_\ell - \gamma \alpha_\ell d_\ell)\);

  if \(v_{\ell+1/2} < v_\ell\) then

    \(x_{\ell+1} \leftarrow x_\ell - \gamma \alpha_\ell d_\ell\);

    \(v_{\ell+1} \leftarrow v_{\ell+1/2}; \quad d_{\ell+1} \leftarrow \nabla V_{p,n}^m (x_{\ell+1})\);

    if \(\frac{|v_{\ell+1} - v_\ell|}{v_\ell} \leq \varepsilon\) then return;
  
  else

    \(\gamma \leftarrow \gamma / 2\);

  end

end
Basket option in the BS model

<table>
<thead>
<tr>
<th>$p$</th>
<th>$n$</th>
<th>$S_0$</th>
<th>price</th>
<th>Stdev</th>
<th>time (sec.)</th>
<th>reference price</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>100</td>
<td>2.27</td>
<td>0.029</td>
<td>0.17</td>
<td>2.17</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>100</td>
<td>2.23</td>
<td>0.025</td>
<td>0.9</td>
<td>2.17</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>110</td>
<td>0.56</td>
<td>0.014</td>
<td>0.07</td>
<td>0.55</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>110</td>
<td>0.53</td>
<td>0.012</td>
<td>0.048</td>
<td>0.55</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>100</td>
<td>2.62</td>
<td>0.021</td>
<td>0.91</td>
<td>2.43</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>100</td>
<td>2.42</td>
<td>0.021</td>
<td>14</td>
<td>2.43</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>110</td>
<td>0.61</td>
<td>0.012</td>
<td>0.33</td>
<td>0.61</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>110</td>
<td>0.55</td>
<td>0.008</td>
<td>10</td>
<td>0.61</td>
</tr>
</tbody>
</table>

**Tab.:** Prices for the put basket option with parameters $T = 3$, $r = 0.05$, $K = 100$, $\rho = 0$, $\sigma^j = 0.2$, $\delta^j = 0$, $d = 5$, $\omega^j = 1/d$, $m = 20,000$. 
Scalability of the parallel algorithm

The tests were run on a BullX DLC supercomputer containing 3204 cores.

<table>
<thead>
<tr>
<th>#processes</th>
<th>time (sec.)</th>
<th>efficiency</th>
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<td>4365</td>
<td>1</td>
</tr>
<tr>
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<tr>
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<td>0.69</td>
</tr>
<tr>
<td>512</td>
<td>10.7</td>
<td>0.59</td>
</tr>
</tbody>
</table>

**TAB.**: Scalability of the parallel algorithm on the 40—dimensional geometric put option described above with $T = 1$, $r = 0.0488$, $K = 100$, $\sigma^j = 0.3$, $\rho = 0.1$, $\delta^j = 0$, $n = 9$, $p = 2$, $m = 200,000$. 
Conclusion

- Purely optimization approach. No need of an optimal strategy.
- The problem is in large dimension but convex.
- *Almost* embarrassingly parallel and scales very well.
- Can deal with path dependent options
Solving optimal stopping problems via empirical dual optimization.

The duality of optimal exercise and domineering claims: a Doob-Meyer decomposition approach to the Snell envelope.

Smooth minimization of non-smooth functions.

Introduction to optimization.
*Optimization Software*.

Monte Carlo valuation of American options.