Pricing American options using martingale bases

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Outline

1. American Options
2. An optimization point of view
3. How to effectively solve the optimization problem
4. Numerical experiments
Consider a multi–dimensional financial market driven by a $d$–dimensional Brownian motion $B$.

The process $(S_t)_{t \leq T}$ is the underlying asset with values in $\mathbb{R}^{d'}$, $d' \leq d$.

The discounted payoff process writes $(Z_t = e^{-\int_0^t r_s ds \phi(S_t)})_{t \leq T}$.

Assume $\mathbb{E} \left[ \sup_t Z_t^2 \right] < \infty$.

Consider an American option. Its discounted price at time $t$ of the Bermudan option is given by

$$U_t = \underset{\tau \in \mathcal{T}_t}{\text{esssup}} \mathbb{E}[Z_\tau | \mathcal{F}_t]$$

where $\mathcal{T}_t$ is the set of all $\mathcal{F}$– stopping times with values in $[t, T]$. 
Framework (2)

- The *Snell envelope* process \((U_t)_{0 \leq t \leq T}\) admits a Doob–Meyer decomposition

\[
U_t = U_0 + M_t^* - A_t^*
\]

where \(M^*\) is a martingale and \(A^*\) a predictable increasing process both vanishing at zero and square integrable.
Dual price (1)

We know from [Rogers, 2002] that

\[ U_0 = \inf_{M \in H_0^1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - M_t) \right] = \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - M_*^t) \right] \]

- This problem admits more than a single solution.
- Some of the martingales \( M \) attaining the infimum are surely optimal

\[ U_0 = \sup_{0 \leq t \leq T} (Z_t - M_t) \quad a.s. \]

- Let \( \tau \) be an optimal stopping time, then

\[ U_0 = \inf_{M \in H_0^1} \mathbb{E} \left[ \sup_{\tau \leq t \leq T} (Z_t - M_t) \right] = \mathbb{E} \left[ \sup_{\tau \leq t \leq T} (Z_t - M_*^t) \right]. \]
Dual price (2)

- From [Schoenmakers et al., 2013], any martingale satisfying

\[
\text{Var} \left( \sup_{0 \leq t \leq T} (Z_t - M_t) \right) = 0
\]

is surely optimal.

- Let \( \tau \) be an optimal stopping time, then for any surely optimal martingale \( M \),

\[
(M_{t \wedge \tau})_t = (M^*_t)_{t \wedge \tau}.
\]

See [Jamshidian, 2007].
Dual price (3)

With our square integrability assumption, we can rewrite the minimization problem as

$$U_0 = \inf_{X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})} \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - \mathbb{E}[X|\mathcal{F}_t]) \right].$$

s.t. $\mathbb{E}[X] = 0$

How to approximate $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ by a finite dimensional vector space in which conditional expectations are tractable in a closed form?
Wiener chaos expansion \((d = 1)\)

Let \(H_i\) be the \(i\)th Hermite polynomial defined by

\[
H_0(x) = 1; \quad H_i(x) = (-1)^i e^{x^2/2} \frac{d^i}{dx^i}(e^{-x^2/2}), \text{ for } i \geq 1.
\]

- \(H'_i = H_{i-1}\) with the convention \(H_{-1} = 0\).
- If \(X, Y \sim \mathcal{N}(0, 1)\) and form a Gaussian vector,

\[
\mathbb{E}[H_i(X)H_j(Y)] = i! (\mathbb{E}[XY])^i \mathbf{1}_{\{i=j\}}.
\]

For \(i \geq 0\), the \(L^2\) closure of the space

\[
\mathcal{H}_i = \left\{ H_i \left( \int_0^T f_t dB_t \right) : f \in L^2([0, T]) \right\}
\]

corresponds to the Wiener chaos of order \(i\).
Truncated Wiener chaos expansion ($d = 1$)

Take a regular grid $0 = t_0 < t_1 < \cdots < t_n$ with step $h$ and consider

$$f_i(t) = \frac{1\{t_{i-1}, t_i]\}(t)}{\sqrt{h}}; \quad \int_0^T f_i(t) dB_t = \frac{B_{t_i} - B_{t_{i-1}}}{\sqrt{h}} = G_i \sim \mathcal{N}(0, 1).$$

For $F \in L^2(\Omega, \mathcal{F}_T)$, we introduce the truncated chaos expansion of order $p$

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}} \lambda_\alpha \prod_{i \geq 1} H_{\alpha_i}(G_i)$$

where $A_{p,n} = \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq p\}$ with $\|\alpha\|_1 = \sum_{i \geq 0} \alpha_i$.

In the following we write,

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}} \lambda_\alpha \hat{H}_\alpha(G_1, \ldots, G_n)$$

with $\hat{H}_\alpha(x) = \prod_{i \geq 1} H_{\alpha_i}(x_i)$. 
Key properties of the truncated Wiener chaos expansion $(d = 1)$

- Since the Hermite polynomials are orthogonal

\[
\lambda_\alpha = \frac{\mathbb{E} \left[ F \hat{H}_\alpha(G_1, \cdots, G_n) \right]}{(\prod_{i \geq 1} \alpha_i!)}.
\]

- For $k \leq n$,

\[
\mathbb{E}[C_{p,n}(F)|\mathcal{F}_{tk}] = \sum_{\alpha \in A_{p,n}^k} \lambda_\alpha \hat{H}_\alpha(G_1, \cdots, G_n)
\]

with $A_{p,n}^k = \{ \alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq p, \alpha_\ell = 0 \ \forall \ell > k \}$.

“Computing $\mathbb{E}[\cdot|\mathcal{F}_{tk}]$” ⇔ “Dropping all non $\mathcal{F}_{tk}$ – measurable terms”
Extension to the multi–dimensional case (1)

Take

\[ h_t^i(t) = \frac{1_{\{t_{i-1}, t_i]\}}(t)}{\sqrt{h}} e_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, d \]

where \((e_1, \ldots, e_d)\) denotes the canonical basis of \(\mathbb{R}^d\). The truncated Wiener chaos of order \(p \geq 0\) is given by

\[ \left\{ \prod_{j=1}^d \hat{H}_{\alpha_j} (G_1^j, \ldots, G_n^j) : \alpha \in (\mathbb{N}^n)^d, \|\alpha\|_1 \leq p \right\} \]

where \(\|\alpha\|_1 = \sum_{i=1}^n \sum_{j=1}^d \alpha_i^j\).
Extension to the multi–dimensional case (2)

With the concise notation

$$\hat{H}^\otimes d(\alpha, \ldots, G_n) = \prod_{j=1}^d \hat{H}_{\alpha_j}(G_1^j, \ldots, G_n^j) \quad \forall \alpha \in (\mathbb{N}^n)^d.$$ 

We introduce the truncated chaos expansion of order $p$ of $F \in L^2(\Omega, \mathcal{F}_T)$

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}^d} \lambda_\alpha \hat{H}^\otimes d(\alpha, \ldots, G_n)$$

where $A_{p,n}^d = \{ \alpha \in (\mathbb{N}^n)^d : \|\alpha\|_1 \leq p \}$. With an obvious abuse of notation, we write, for $\lambda \in \mathbb{R}^{A_{p,n}^d}$,

$$C_{p,n}(\lambda) = \sum_{\alpha \in A_{p,n}^d} \lambda_\alpha \hat{H}^\otimes d(\alpha, \ldots, G_n).$$
Return to the American option price

We approximate the original problem

\[
\inf_{X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})} \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - \mathbb{E}[X | \mathcal{F}_t]) \right]
\]

s.t. \( \mathbb{E}[X] = 0 \)

by

\[
\inf_{\lambda \in \mathbb{R}^{A_{p,n}}} V_{p,n}(\lambda)
\]

s.t. \( \lambda_0 = 0 \)

with

\[
V_{p,n}(\lambda) = \mathbb{E} \left[ \max_{0 \leq k \leq n} (Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda) | \mathcal{F}_{t_k}]) \right]
\]
Properties of the minimization problem (1)

**Proposition 1**

*The minimization problem (1) has at least one solution.*

- The function $V_{p,n}$ is clearly convex (maximum of affine functions).
- Not strongly convex but,

\[
V_{p,n}(\lambda) \geq \mathbb{E} [(C_{p,n}(\lambda))_] \geq \frac{1}{2} \mathbb{E} [|C_{p,n}(\lambda)|],
\]

\[
\mathbb{E} [|C_{p,n}(\lambda)|] = |\lambda| \mathbb{E} [|C_{p,n}(\lambda/|\lambda|)|] \geq |\lambda| \inf_{\mu \in \mathbb{R}^{A \otimes d}_{p,n}, |\mu|=1} \mathbb{E} [|C_{p,n}(\mu)|].
\]
Properties of the minimization problem (2)

\[ \mathcal{I}(\lambda, Z, G) = \{0 \leq k \leq n : \text{the pathwise maximum is attained at time } k\} . \]

**Proposition 2**

Let \( p \geq 1 \). Assume that

\[
\forall 1 \leq k \leq n, \forall F, \mathcal{F}_{tk} - \text{measurable}, F \in \mathcal{C}_{p-1,n}, F \neq 0,
\]

\[
\exists 1 \leq q \leq d \text{ s.t. } \mathbb{P} \left( \forall t \in [t_{r-1}, t_r], D_q^i \phi(S_{tk}) + F = 0 \mid \phi(S_{tk}) > 0 \right) = 0.
\]

Then, the function \( V_{p,n} \) is differentiable at all points \( \lambda \in \mathbb{R}^{A_p \otimes d} \) with no zero component and its gradient \( \nabla V_{p,n} \) is given by

\[
\nabla V_{p,n}(\lambda) = \mathbb{E} \left[ \mathbb{E} \left[ \hat{H}^{\otimes d}(G_1, \ldots, G_n) \mid \mathcal{F}_{ti} \right] \mid i=\mathcal{I}(\lambda,Z,G) \right].
\]
Properties of the minimization problem (3)

\[ V_{p,n}(\lambda) = \mathbb{E} \left[ \max_{0 \leq k \leq n} (Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda)|\mathcal{F}_{t_k}]) \right] \]

- The maximum is pathwise sub–differentiable. From [Bertsekas, 1973], \( V_{p,n} \) is sub–differentiable with sub–differential given by

\[
\left\{ \mathbb{E} \left[ \sum_{i \in \mathcal{I}(\lambda,Z,G)} \beta_i \mathbb{E}[\hat{H} \otimes d(G_1,\ldots,G_n)|\mathcal{F}_{t_i}] \right] : \beta_i \geq 0, \beta_i \mathcal{F}_T \text{ measurable s.t.} \sum_{i \in \mathcal{I}(\lambda,Z,G)} \beta_i = 1 \right\}.
\]
Properties of the minimization problem (4)

- Differentiability is ensured as soon as $\mathcal{I}(\lambda, Z, G)$ is a.s. reduced to a unique element: purpose of the blue condition.

- Alternative approach by [Belomestny, 2013]: use smoothing techniques instead (see [Nesterov, 2004]). General idea:

  \[
  \text{Replace } \max_k a_k \text{ by } p^{-1} \log \left( \sum_k \exp(p \ a_k) \right).
  \]

- Let $\lambda^\#$ be a solution, $V_{p,n}(\lambda^\#_{p,n}) = \inf_{\lambda} V_{p,n}(\lambda)$. Then $\nabla V_{p,n}(\lambda^\#_{p,n}) = 0$.  


Convergence to the true solution (1)

**Proposition 3**

The solution of the minimization problem (1), $V_{p,n}(\lambda_{p,n}^\#)$, converges to the price of the American options when both $p$ and $n$ go to infinity and moreover

$$0 \leq V_{p,n}(\lambda_{p,n}^\#) - U_0 \leq 2 \|M_T^* - C_{p,n}(M_T^*)\|_2.$$
Convergence to the true solution (2)

\[ V_{p,n}(\lambda_{p,n}^\#) = \inf_{\lambda} V_{p,n}(\lambda), \quad \text{and} \quad C_{p,n}(M_T^*) = C_{p,n}(\lambda_{p,n}^*). \]

\[ 0 \leq V_{p,n}(\lambda_{p,n}^\#) - U_0 \leq V_{p,n}(\lambda_{p,n}^*) - U_0 \]

\[ = \mathbb{E} \left[ \max_k (Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda_{p,n}^*) | \mathcal{F}_{t_k}]) - \max_k (Z_{t_k} - M_{t_k}^*) \right] \]

\[ \leq \sqrt{\mathbb{E} \left[ \max_k \mathbb{E} \left[ \left| M_T^* - C_{p,n}(\lambda_{p,n}^*) \right| | \mathcal{F}_{t_k} \right]^2 \right]} \]

\[ \leq 2 \| M_T^* - C_{p,n}(M_T^*) \|_2 \]

where the last upper–bound ensues from Doob’s inequality.
Convergence to the true solution (3)

- Non uniform bounds are larger!

\[
0 \leq V_{p,n}(\lambda_{p,n}^\#) - U_0 \leq 2 \| M_T^* - C_{p,n}(\lambda_{p,n}^\#) \|_2
\]

\( C_{p,n}(M_T^*) \) minimizes the \( L^2 \) distance between \( M_T^* \) and \( C_{p,n} \).

- Consider a \textbf{Bermudan} option with exercising dates \( t_0, \cdots, t_n \) and discounted payoff \( (Z_{t_k})_k \) adapted to the discrete time filtration generated by the Brownian increments only. Then, \( V_{p,n}(\lambda_{p,n}^\#) \) converges to the price of the Bermudan option when \( p \) only goes to infinity.
Practically solving the optimization problem (1)

We approximate the solution of

\[
V_{p,n}(\lambda^\#_{p,n}) = \inf_{\lambda \in A_{p,n}^d} V_{p,n}(\lambda) = \inf_{\lambda \in A_{p,n}^d} \mathbb{E} \left[ \max_{0 \leq k \leq n} \left( Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda)|\mathcal{F}_{t_k}] \right) \right]
\]

by introducing the well–known *Sample Average Approximation* (see [Rubinstein and Shapiro, 1993]) of \( V_{p,n} \) defined by

\[
V_{p,n}^m(\lambda) = \frac{1}{m} \sum_{i=1}^{m} \max_{0 \leq k \leq n} \left( Z_{t_k}^{(i)} - \mathbb{E}[C_{p,n}^{(i)}(\lambda)|\mathcal{F}_{t_k}] \right).
\]

Note that the conditional expectation boils down to truncating the chaos expansion and hence is tractable in a closed form.
Practically solving the optimization problem (2)

For large enough $m$, $V_{p,N}^m$ is convex, a.s. differentiable and tends to infinity at infinity. Then, there exits $\lambda_{p,n}^m$ such that

$$V_{p,n}^m(\lambda_{p,n}^m) = \inf_{\lambda \in \mathbb{R}^{A_{p,n}}} V_{p,n}^m(\lambda).$$

**Proposition 4**

$V_{p,n}^m(\lambda_{p,n}^m)$ converges a.s. to $V_{p,n}(\lambda_{p,N}^\#)$ when $m \to \infty$. The distance from $\lambda_{p,n}^m$ to the set of minimizers of $V_{p,n}$ converges to zero as $m$ goes to infinity.
Practically solving the optimization problem (3)

Write $M_k(\lambda) = \mathbb{E}[C_{p,n}(\lambda)|\mathcal{F}_{t_k}]$ for $0 \leq k \leq n$.

**Proposition 5**

Assume $\lambda_{p,n}^\#$ is unique. Then,

$$\frac{1}{m} \sum_{i=1}^{m} \left( \max_{0 \leq k \leq n} Z_{t_k}^{(i)} - M_k^{(i)}(\lambda_{p,n}^m) \right)^2 - V_{p,n}^m(\lambda_{p,n}^m)^2$$

is a convergent estimator of $\text{Var}(\max_{k \leq 0 \leq n} Z_{t_k} - M_k(\lambda_{p,n}^\#))$ and moreover, if $\lambda_{p,n}^m$ is bounded,

$$\lim_{m \to \infty} m \text{Var} \left( V_{p,n}^m(\lambda_{p,n}^m) \right) = \text{Var}(\max_{k \leq 0 \leq n} Z_{t_k} - M_k(\lambda_{p,n}^\#)).$$
The algorithm: bespoke martingales

Define the first time the option goes in the money by

$$\tau_0 = \inf\{k \geq 0 : Z_{t_k} > 0\} \wedge n.$$ 

Consider martingales only starting once the option has been in the money

$$N_k(\lambda) = M_k(\lambda) - M_{k \wedge \tau_0}(\lambda).$$

In the dual price, max$_{0 \leq k \leq n}$ can be shrunk to max$_{\tau_0 \leq k \leq n}$.

Using Doob’s stopping theorem, we have, for any fixed $\lambda$,

$$\mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - M_k(\lambda)) \right] = \mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - (M_k(\lambda) - M_{\tau_0}(\lambda))) \right]$$

The martingales $M(\lambda)$ or $N(\lambda)$ lead to the same minimum value.

The set of martingales $N^\lambda$ is far more efficient from a practical point of view.
The algorithm: a gradient descent with line search

Take $\alpha_{\ell} = \frac{\tilde{V}^m_{p,n}(x_{\ell}) - v^\#}{\|\nabla \tilde{V}^m_{p,n}(x_{\ell})\|^2}$, see [Polyak, 1987], but with the European price instead of the American one for $v^\#$.

$x_0 \leftarrow 0, k \leftarrow 0, \gamma \leftarrow 1, d_0 \leftarrow 0, v_0 \leftarrow \infty$;

while True do

| Compute $v_{k+1/2} \leftarrow \tilde{V}^m_{p,n}(x_k - \gamma \alpha_k d_k)$; |
| if $v_{k+1/2} < v_k$ then |
| $x_{k+1} \leftarrow x_k - \gamma \alpha_k d_k$; |
| $v_{k+1} \leftarrow v_{k+1/2}$; |
| $d_{k+1} \leftarrow \nabla \tilde{V}^m_{p,n}(x_{k+1})$; |
| if $\left|\frac{v_{k+1} - v_k}{v_k}\right| \leq \varepsilon$ then return; |

else |
| $\gamma \leftarrow \gamma/2$; |

end
Some remarks on the algorithm

- Given the expression of $V_{p,n}^m$, both the value function and its gradient are computed at the same time without extra cost.

$$V_{p,n}(\lambda) = E \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - E[\lambda \cdot H^{\otimes d}(G_1, \ldots, G_n)|\mathcal{F}_{t_k}]) \right],$$

$$= E[Z_{t_\tau(\lambda,z,G)}] - \lambda \cdot \nabla \tilde{V}_{p,n}(\lambda).$$

- Checking the admissibility of a step $\gamma$ costs as much as updating $x_k$.

- The algorithm is almost embarrassingly parallel:
  - Few iterations of the gradient descent are required ($\approx 10$).
  - Each iteration is fully parallel: each process treats its bunch of paths.
  - No demanding centralized computations
  - Very little communication: a few broadcasts only.
Parallel implementation

In parallel Generate \((G^{(1)}, Z^{(1)}), \ldots, (G^{(m)}, Z^{(m)})\) \(m x_0 \leftarrow 0 \in \mathbb{R}^{A_p \times d}\);

while True do

Broadcast \(x_\ell, d_\ell, \gamma, \alpha_\ell\);

In parallel Compute \(\max_{\tau_0 \leq k \leq n}(Z^{(i)}_k - N^{(i)}_k(x_\ell - \gamma \alpha_\ell d_\ell))\);

Make a reduction of the above contributions to obtain \(\tilde{V}^m_{p,n}(x_{\ell+1}/2)\) and \(\nabla \tilde{V}^m_{p,n}(x_{\ell+1}/2)\);

\(v_{\ell+1/2} \leftarrow \tilde{V}^m_{p,n}(x_\ell - \gamma \alpha_\ell d_\ell)\);

if \(v_{\ell+1/2} < v_\ell\) then

\(x_{\ell+1} \leftarrow x_\ell - \gamma \alpha_\ell d_\ell\);

\(v_{\ell+1} \leftarrow v_{\ell+1/2}\);

\(d_{\ell+1} \leftarrow \nabla \tilde{V}^m_{p,n}(x_{\ell+1})\);

if \(|v_{\ell+1} - v_\ell| / v_\ell \leq \varepsilon\) then return;

else

\(\gamma \leftarrow \gamma / 2\);

end

end

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Basket option in the BS model

<table>
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<th>n</th>
<th>$S_0$</th>
<th>price</th>
<th>Stdev</th>
<th>time (sec.)</th>
<th>reference price</th>
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**Tab.**: Prices for the put basket option with parameters $T = 3$, $r = 0.05$, $K = 100$, $\rho = 0$, $\sigma^j = 0.2$, $\delta^j = 0$, $d = 5$, $\omega^j = 1/d$, $m = 20,000$. 
Call option on the maximum of a basket

<table>
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<th>$p$</th>
<th>$m$</th>
<th>$S_0$</th>
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</table>

**TAB.** Prices for the call option on the maximum of $d$ assets with parameters $T = 3$, $r = 0.05$, $K = 100$, $\rho = 0$, $\sigma^i = 0.2$, $\delta^i = 0.1$, $n = 9$. 
### Geometric basket option

<table>
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<th>$\rho$</th>
<th>$p$</th>
<th>$m$</th>
<th>price</th>
<th>Stdev</th>
<th>time(sec)</th>
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</tr>
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<td>0.018</td>
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<td>0.1</td>
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<td>3.61</td>
<td>0.02</td>
<td>170</td>
<td>3.69</td>
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</table>

**Tab.**: Prices for the geometric basket put option with parameters $T = 1$, $r = 0.0488$ (it corresponds to a 5% annual interest rate), $K = 100$, $\delta^j = 0$, $n = 9$. 
Scalability of the parallel algorithm

The tests were run on a BullX DLC supercomputer containing 3204 cores.

<table>
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<th>#processes</th>
<th>time (sec.)</th>
<th>efficiency</th>
</tr>
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<td>1</td>
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<tr>
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<tr>
<td>512</td>
<td>10.7</td>
<td>0.59</td>
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</table>

**Table:** Scalability of the parallel algorithm on the 40—dimensional geometric put option described above with $T = 1$, $r = 0.0488$, $K = 100$, $\sigma^j = 0.3$, $\rho = 0.1$, $\delta^j = 0$, $n = 9$, $p = 2$, $m = 200,000$. 
Conclusion

- Purely optimization approach. No need of an optimal strategy.
- The problem is in large dimension but convex.
- *Almost* embarrassingly parallel and scales very well.
- Can deal with path dependent options
Solving optimal stopping problems via empirical dual optimization.

Stochastic optimization problems with nondifferentiable cost functionals.

The duality of optimal exercise and domineering claims: a Doob-Meyer decomposition approach to the Snell envelope.

Smooth minimization of non-smooth functions.

Introduction to optimization.
*Optimization Software.*
