Technical note on the paper “Residuals and goodness-of-fit tests for stationary marked Gibbs point processes”

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Abstract

The aim of this note is to prove a technical result concerning the positive definiteness of asymptotic covariance matrices of residuals for spatial point processes under some specific choices of test functions and Gibbs models. This note is a complement to the article Coeurjolly and Lavancier (2011).

We refer to Coeurjolly and Lavancier (2011) for the setting of this note, in particular for the notation.

The proposition below proves that for some particular choice of test functions $h$, the asymptotic covariance matrices involved in Proposition 3 and 4 of Coeurjolly and Lavancier (2011) are positive-definite for the 2-type marked Strauss point process and the area-interaction point process. As noticed in Section 5.2 in Coeurjolly and Lavancier (2011), this property is not true for the simple choice $h = 1$ (i.e. the raw residuals).

**Proposition 1.** For the 2-type marked Strauss point process and the area-interaction point process and under $\text{MPLE}$, the following results hold:

1. the matrix $\Sigma_1(\theta^\star)$ (in Proposition 3 of Coeurjolly and Lavancier (2011)) associated to the inverse residuals $h = e^V$,
2. the matrix $\Sigma_2(\theta^\star)$ (in Proposition 4 of Coeurjolly and Lavancier (2011)) associated to the empty space residuals,

are positive-definite.

From Proposition 11 in Coeurjolly and Lavancier (2011), the proof of this proposition mainly consists in checking the assumption [PD] below:

[PD] For some $\Delta := \cup_{|k| \leq 1} \Delta_k(D)$, where $D$ is not lower than the range of interaction, there exists $B \in \mathcal{F}$ and $A_0, \ldots, A_\ell, \ell \geq 1$ disjoint events of $\Omega_B := \{ \varphi : \varphi \Delta_k(D) \in B, 1 \leq |k| \leq 2 \}$ such that for $i = 0, \ldots, \ell$, $P_{\theta^\star}(A_i) > 0$ and for any $(\varphi_0, \ldots, \varphi_\ell) \in A_0 \times \cdots \times A_\ell$, the $(\ell, s)$ matrix with entries $(R_{\infty, \Delta}(\varphi; h, \theta^\star) - R_{\infty, \Delta}(\varphi_0; h, \theta^\star))_{j=1,\ldots,s}$, is injective, i.e. for any $y \in \mathbb{R}^s$,

\[
(\forall i \in \{1, \ldots, s\}, \; y^T \left( R_{\infty, \Delta}(\varphi_i; h, \theta^\star) - R_{\infty, \Delta}(\varphi_0; h, \theta^\star) \right) = 0) \implies y = 0,
\]

where $R_{\infty, \Delta}(\varphi; h, \theta^\star) = (R_{\infty, \Delta}(\varphi_0; h, \theta^\star))_{j=1,\ldots,s}$.

We recall that $R_{\infty, \Delta}(\varphi; h, \theta^\star) = I_A(\varphi; h, \theta^\star) - L_{PL}^{(1)}(\varphi; h, \theta^\star)W$, where $I_A$ is the innovation, $L_{PL}^{(1)}$ is the gradient vector of the log-pseudolikelihood, and $W$ is a deterministic vector, defined by $W = H(\theta^\star)^{-1}E(h; \theta)$, see Section 4.2 in Coeurjolly and Lavancier (2011).
The proof of Proposition 1 is splitted into two parts, according to the two models considered in its statement.

1. The two-type marked Strauss point process

We only deal with the inhibition case, that is \( \Theta = \mathbb{R}^2 \times \mathbb{R}_+^3 \). The following proofs could easily be extended to the hard-core case and to the multi-Strauss marked point process (see e.g. Billiot et al. (2008)). For any vector \( \mathbf{z} \) of length 5, we sometimes reparameterize it similarly as the parameter vector, that is \( \mathbf{z} = (z_1, z_2, z_2^{1,1}, z_2^{1,2}, z_2^{2,2})^T \).

Proof that \( \Sigma_\varepsilon(\varepsilon^*) \) is positive-definite for the two-type Strauss model

It is easy to check that \( \lambda_{\text{fin}} \) and \( \lambda_{\text{res}} \) are the two eigenvalues of \( \Sigma_\varepsilon(\varepsilon^*) \) with respective order \(|\mathcal{J}| - 1 \) and 1. The first eigenvalue is positive from Lemma 9 in Coeurjolly and Lavancier (2011).

Let us prove that \( \lambda_{\text{res}} > 0 \). We have to check \([\text{PD}]\) with \( h = e^V \) and \( s = 1 \).

We fix \( B = \emptyset \) in \([\text{PD}]\). Let \( \Omega := \Omega_\eta \). Without loss of generality, one may assume that \( \theta_2^{1,1} > 0 \).

Let us define, for \( \eta \geq 1 \) and \( \eta \leq 1 \). The first eigenvalue is positive from Lemma 9 in Coeurjolly and Lavancier (2011).

\[ \lambda_{\text{fin}}(\varepsilon^*) = 2 \] for some function \( \phi \) and \( \eta \leq 1 \).

Now, for some function \( f \)

\[ \Delta R_\infty(\varepsilon^*) := R_\infty(\varepsilon^*) - R_\infty(\varepsilon^*) \]

\[ = 2n \left( e^{\theta_1^1 + (n-1)\theta_2^{1,1}} - e^{\theta_1^1 + (2n-1)\theta_2^{1,1}} \right) + W_2^{1,1} (2n(2n-1) - n(n-1)) + f(\varepsilon^*) \]

\[ = -n e^{(2n-1)\theta_2^{1,1}} \left[ -2e^{\theta_1^1 + n\theta_2^{1,1}} + 2e^{\theta_1^1} - (3n-1)e^{-(2n-1)\theta_2^{1,1}} \right] + f(\varepsilon^*). \]
Fix $\varepsilon > 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $x_n < -\varepsilon$. Now by a continuity argument, there exists $\eta_0 = \eta_0(n_0)$ such that for all $\eta \leq \eta_0(n_0)$, $|f(\varphi_{n_0,-},\varphi_{n_0,+},W,\eta)| \leq \varepsilon/2$. Therefore by assuming that $\Delta R_{\infty,\varphi}(\varphi_{n_0,-},\varphi_{n_0,+}) = 0$, we obtain for $\eta \leq \eta_0$

$$0 = |\Delta R_{\infty,\varphi}(\varphi_{n_0,-},\varphi_{n_0,+})| \geq |x_{n_0}| - |f(\varphi_{n_0,-},\varphi_{n_0,+},W,\eta)| \geq \varepsilon/2 > 0$$

which leads to a contradiction and proves [PD].

**Proof that $\Sigma_\theta(\theta^*)$ is positive-definite for the two-type Strauss model**

We have to check Assumption [PD] where, for all $j = 1, \ldots, s$, $h_j$ is the test function given by $h_j(x^m, \varphi; \theta) = 1_{[0,r_j]}(d(x^m, \varphi))e^{V(x^m|\varphi; \theta)}$. We fix as before $B = 0$ in [PD].

Let $0 < r_1 < \ldots < r_s < +\infty$. We choose $D$ in [PD] large enough so that $D > r_s$. Define

$$A_{k,j}^1(\eta) = \left\{ \varphi \in \mathbb{P} : \varphi(\Delta_0(D)) = 2, \varphi\left( B\left( (0,0), \frac{D}{4} \right) \times \{1\} \right) = 1, \varphi\left( B\left( r_k - \frac{\eta}{2}, 0 \right), \frac{D}{4} \right) \times \{1\} = 1 \right\},$$

$$A_{k,j}^1(\eta) = \left\{ \varphi \in \mathbb{P} : \varphi(\Delta_0(D)) = 2, \varphi\left( B\left( (0,0), \frac{D}{4} \right) \times \{1\} \right) = 1, \varphi\left( B\left( r_k + \frac{\eta}{2}, 0 \right), \frac{D}{4} \right) \times \{1\} = 1 \right\},$$

Let $\varphi_{k,\bullet} \in A_{k,s}^1(\eta)$ for $\bullet = -, +$ and $k = 1, \ldots, s$. Let $c_k$ the constant given by

$$c_k = \begin{cases} 2e^{\theta_{l,1}^1 + \theta_{s,1}^1} & \text{if } r_k \leq R_{1,1}, \\ 2e^{\theta_{l,1}^1} & \text{otherwise.} \end{cases}$$

Then for $k, j = 1, \ldots, s$ and for $\eta$ small enough

$$I_{\mathbb{P}}(\varphi_{k,-}; h_j, \theta^*) = \int_{\mathbb{P} \times \mathbb{M}} h_j(x^m, \varphi_{k,-})e^{-V(x^m|\varphi_{k,-}; \theta^*)}\mu(dx^m) - \left\{ \begin{array}{ll} c_k & \text{if } k \leq j, \\ 0 & \text{otherwise.} \end{array} \right.$$
Indeed if for some $x \in \mathbb{R}^s$, the relation $x^T \left( R_{\infty, \Xi}(\varphi; h, \theta^*) - R_{\infty, \Xi}(\varphi_0; h, \theta^*) \right) = 0$ holds true for any $i = 1, \ldots, 2s$, then in particular the l.h.s. of (1) vanishes for all $k = 1, \ldots, s$, leading to $x = 0$.

2. The Area-interaction point process

We fix for simplicity $d = 2$, though the proofs may be extended easily to higher dimensions.

Proof that $\Sigma_1(\theta^*)$ is positive-definite for the area-interaction model

Similarly as for the Strauss process above, the proof reduces to check $[\text{PD}]$ with $h = e^V$ and $s = 1$. Let us fix $B = \emptyset$ and let us consider, for some $\eta, \omega > 0$, the two following events:

\[
\begin{align*}
A_1(\eta, \omega) &:= \{ \varphi \in \Omega : \varphi(\Delta_0(D)) = 2, \varphi(\mathcal{B}((0,0), \eta)) = 1, \varphi(\mathcal{B}((0, \omega), \eta)) = 1 \} \\
A_2(\eta, \omega) &:= \{ \varphi \in \Omega : \varphi(\Delta_0(D)) = 3, \varphi(\mathcal{B}((0,0), \eta)) = 1, \varphi(\mathcal{B}((0, \omega), \eta)) = 2 \}
\end{align*}
\]

Fix $\eta, \omega$, let $\varphi_j \in A_j(\eta, \omega)$ and denote by $S(\varphi) := \sum_{x \in \varphi} e^{V(x|\varphi \setminus x)}$ so that

\[
l(\varphi_j; e^V, \theta^*) = |\Sigma| - S(\varphi_j).
\]

When $\eta \to 0$,

\[
S(\varphi_1) \to 2e^{\theta_1^*} g(\omega) \quad \text{and} \quad S(\varphi_2) \to 2e^{\theta_1^*} + e^{\theta_1^* + \theta_2^*} g(\omega)
\]

where $g(\omega) := |\mathcal{B}((0,0), \Omega) \cup \mathcal{B}((0, \omega), \mathcal{R})| = -|\mathcal{B}((0,0), \Omega)|$. Moreover, denoting $\tilde{v}_2(\varphi) = \sum_{x \in \varphi} v_2(x|\varphi \setminus x)$

\[
\begin{align*}
\left( \text{LPL}_{\Sigma}^{(1)}(\varphi_j; \theta^*) \right)_1 &= \int_{\Sigma} e^{-V(x|\varphi_1; \theta^*)} dx - \left\{ \begin{array}{ll} 2 & \text{if } j = 1 \\ 3 & \text{if } j = 2 \end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\left( \text{LPL}_{\Sigma}^{(1)}(\varphi_j; \theta^*) \right)_2 &= \int_{\Sigma} v_2(x|\varphi_j) e^{-V(x|\varphi_j; \theta^*)} dx - \tilde{v}_2(\varphi_j).
\end{align*}
\]

Again, when $\eta \to 0$, one may note that for $k = 1, 2$

\[
\int_{\Sigma} v_k(x|\varphi_1) e^{-V(x|\varphi_1; \theta^*)} dx - \int_{\Sigma} v_k(x|\varphi_2) e^{-V(x|\varphi_2; \theta^*)} dx \to 0
\]

and

\[
\tilde{v}_2(\varphi_1) \to 2g(\omega) \quad \text{and} \quad \tilde{v}_2(\varphi_2) \to g(\omega).
\]

These computations lead to

\[
R_{\infty, \Xi}(\varphi_1; e^V, \theta^*) - R_{\infty, \Xi}(\varphi_2; e^V, \theta^*) = 2e^{\theta_1^*} - e^{\theta_1^* + \theta_2^*} g(\omega) - W_1 + g(\omega)W_2 + f(\varphi_1, \varphi_2, \mathcal{W}, \eta),
\]

where the function $f$ is such that for all $\varepsilon > 0$, there exists $\eta$ small enough such that $|f(\varphi_1, \varphi_2, \mathcal{W}, \eta)| \leq \varepsilon$. Let $\varphi_j \in A_j(\eta, \gamma)$, then, since $g(0) = 0$, assuming that the l.h.s. of the previous equation equals 0 leads to $W_1 = e^{\theta_1^*}$. Now, let $\omega > 0$ and again assume that $R_{\infty, \Xi}(\varphi_1; e^V, \theta^*) = R_{\infty, \Xi}(\varphi_2; e^V, \theta^*)$, we therefore obtain (by the continuity argument)

\[
W_2 = \frac{e^{\theta_1^* + \theta_2^*} g(\omega) - e^{\theta_1^*}}{g(\omega)}.
\]

But $W_2$ is a constant and so cannot depend on $\omega$. Therefore one of the assumptions made before is untrue, which proves $[\text{PD}]$.

Proof that $\Sigma_2(\theta^*)$ is positive-definite for the area-interaction model

We have to check Assumption $[\text{PD}]$ where $h_j$ is the test function given by $h_j(x^m, \varphi; \theta) = 1_{[0, r]}(d(x^m, \varphi)) e^{V(x^m|\varphi; \theta)}$, and where, again, we choose $B = \emptyset$. 

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Let 0 < r_1 < \ldots < r_s < +\infty. We choose D such that D \geq r_s and we set
\[ A_{i,-}(\eta) = \left\{ \varphi \in \Pi : \varphi(\Delta_0(D)) = 2, \varphi \left( B \left( (0,0), \frac{n}{4} \right) \right) = 1, \varphi \left( B \left( (r_i - \frac{n}{2}, 0), \frac{n}{4} \right) \right) = 1 \right\}, \]
\[ A_{i,+}(\eta) = \left\{ \varphi \in \Pi : \varphi(\Delta_0(D)) = 2, \varphi \left( B \left( (0,0), \frac{n}{4} \right) \right) = 1, \varphi \left( B \left( (r_i + \frac{n}{2}, 0), \frac{n}{4} \right) \right) = 1 \right\}. \]

Then, for i,j \in \{1, \ldots, s\} and k \in \{1, 2\}, let \varphi_{i,-} \in A_{i,-} and \varphi_{i,+} \in A_{i,+}, then
\[ I_\Delta(\varphi_{i,-}; h_j, \theta^*) = \int_{\Delta} h_j(x, \varphi_{i,-}; \theta^*) e^{-V(x|\varphi_{i,-}; \theta^*)} dx - \left\{ \begin{array}{ll} S(\varphi_{i,-}) & \text{if } i \leq j \\ 0 & \text{otherwise.} \end{array} \right. \]
\[ I_\Delta(\varphi_{i,+}; h_j, \theta^*) = \int_{\Delta} h_j(x, \varphi_{i,+}; \theta^*) e^{-V(x|\varphi_{i,+}; \theta^*)} dx - \left\{ \begin{array}{ll} S(\varphi_{i,+}) & \text{if } i < j \\ 0 & \text{otherwise.} \end{array} \right. \]
\[ \left( \text{LPL}_\Delta^{(1)}(\varphi_{i,\bullet}; \theta^*) \right)_k = \int_{\Delta} v_k(x|\varphi_{i,\bullet}) e^{-V(x|\varphi_{i,\bullet}; \theta^*)} dx - \sum_{x \in \varphi_{i,\bullet}} v_k(x|\varphi_{i,\bullet} \setminus x), \]
for \bullet = -, +. We can prove that for small \eta, \left( \text{LPL}_\Delta^{(1)}(\varphi_{i,-}; \theta^*) \right)_k \simeq \left( \text{LPL}_\Delta^{(1)}(\varphi_{i,+}; \theta^*) \right)_k and \[ S(\varphi_{i,-}) \simeq S(\varphi_{i,+}) \simeq \kappa_i := 2 \theta_{i,1}^* + \theta_{i,2}^* |B(0,R) \cap B(r_i, R)|. \]
Let \( x \in \mathbb{R}^s \setminus \{0\} \), then from previous computations, for some function \( f \),
\[ x^T \left( \text{R}_\Delta^{(1)}(\varphi_{i,+}; h, \theta^*) - \text{R}_\Delta^{(1)}(\varphi_{i,-}; h, \theta^*) \right) = \kappa_i x_i + f(x, \varphi_{i,+}, \varphi_{i,-}, h, \eta) \quad (2) \]
where for every \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that \( |f(x, \varphi_{i,+}, \varphi_{i,-}, h, \eta)| \leq \varepsilon \). Therefore, assuming that the l.h.s. of (2) equals 0 leads to \( x_i = 0 \) for \( i = 1, \ldots, s \). Assumption [PD] follows by the same argument as for the two-type marked Strauss point process.

References
