Travelling Breathers with Exponentially Small Tails in a Chain of Nonlinear Oscillators

Guillaume James, Yannick Sire

Mathématiques pour l’Industrie et la Physique, UMR CNRS 5640, and Département GMM, Institut National des Sciences Appliquées, 135 avenue de Rangueil, 31077 Toulouse Cedex 4, France.
E-mail: james@insa-toulouse.fr; sire@insa-toulouse.fr

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Abstract: We study the existence of travelling breathers in Klein-Gordon chains, which consist of one-dimensional networks of nonlinear oscillators in an anharmonic on-site potential, linearly coupled to their nearest neighbors. Travelling breathers are spatially localized solutions which appear time periodic in a referential in translation at constant velocity. Approximate solutions of this type have been constructed in the form of modulated plane waves, whose envelopes satisfy the nonlinear Schrödinger equation (M. Remoissenet, Phys. Rev. B 33, n.4, 2386 (1986), J. Giannoulis and A. Mielke, Nonlinearity 17, p. 551–565 (2004)). In the case of travelling waves (where the phase velocity of the plane wave equals the group velocity of the wave packet), the existence of nearby exact solutions has been proved by Iooss and Kirchgässner, who have obtained exact solitary wave solutions superposed on an exponentially small oscillatory tail (G. Iooss, K. Kirchgässner, Commun. Math. Phys. 211, 439–464 (2000)). However, a rigorous existence result has been lacking in the more general case when phase and group velocities are different. This situation is examined in the present paper, in a case when the breather period and the inverse of its velocity are commensurate. We show that the center manifold reduction method introduced by Iooss and Kirchgässner is still applicable when the problem is formulated in an appropriate way. This allows us to reduce the problem locally to a finite dimensional reversible system of ordinary differential equations, whose principal part admits homoclinic solutions to quasi-periodic orbits under general conditions on the potential. For an even potential, using the additional symmetry of the system, we obtain homoclinic orbits to small periodic ones for the full reduced system. For the oscillator chain, these orbits correspond to exact small amplitude travelling breather solutions superposed on an exponentially small oscillatory tail. Their principal part (excluding the tail) coincides at leading order with the nonlinear Schrödinger approximation.
1. Introduction

We consider a one-dimensional lattice of nonlinear oscillators described by the following system (Klein-Gordon system):

$$\frac{d^2 x_n}{d\tau^2} + V'(x_n) = \gamma (x_{n+1} + x_{n-1} - 2x_n), \quad n \in \mathbb{Z},$$

(1)

where $x_n$ is the displacement of the $n$th particle from an equilibrium position, the coupling constant $\gamma$ is strictly positive and the on-site potential $V$ is analytic in a neighborhood of $x = 0$ (with $V'(0) = 0$, $V''(0) > 0$). This system describes a chain of particles linearly coupled to their first neighbors, in the local anharmonic potential $V$.

In this paper, we consider solutions of (1) satisfying

$$x_n(\tau) = x_{n-p}(\tau - T),$$

(2)

for a fixed $T \in \mathbb{R}$ and $p \geq 1$. The case when $p = 1$ in (2) corresponds to travelling waves. Solutions satisfying (2) for $p \neq 1$ consist of pulsating travelling waves, which are exactly translated by $p$ sites after a fixed propagation time $T$ and are allowed to oscillate as they propagate on the lattice. In particular, solutions of (1) having the form $x_n(\tau) = x(n - c \tau, \tau)$ ($x$ being $T$-periodic in its second argument) satisfy (2) under the condition $c = p/T$. A different situation arises when $c$ and $1/T$ are incommensurate, since the solution is not exactly translated on the lattice after time $T$ but is modified by a spatial shift. Solutions of type (2) having the additional property of spatial localization ($x_n(\tau) \to 0$ as $n \to \pm \infty$) are known as exact travelling breathers (with commensurate velocity and frequency) and have been studied numerically in different systems.

Approximate travelling breather solutions propagating on the lattice at a non constant velocity $c$ have also drawn a lot of attention. They have been numerically observed in various one-dimensional nonlinear lattices such as Fermi-Pasta-Ulam lattices [43, 8, 37, 13], Klein-Gordon chains [9, 6] and the discrete nonlinear Schrödinger (DNLS) equation [12]. The last two models exhibit similar features in some regimes where the DNLS equation can be derived from the Klein-Gordon system using appropriate scalings [35]. Other references are available in the review paper [15].

One way of generating approximate travelling breathers consists of “kicking” static breathers consisting of spatially localized and time periodic oscillations (see the basic papers [44, 30, 25, 5] for more details on these solutions). Static breathers are put into motion by perturbation in the direction of a pinning mode [6]. The possible existence of an energy barrier that the breather has to overcome in order to become mobile has drawn a lot of attention, see e.g. [9, 6, 13, 26] and the review paper [39].

It is a more delicate task to examine the existence of exact travelling breathers using numerical computations. Indeed, these solutions might not exist without being superposed on a small nonvanishing oscillatory tail which violates the property of spatial localization. This phenomenon is likely to occur since the existence of a nonvanishing oscillatory tail has been previously observed in some parameter regimes for solitary waves (spatially localized travelling waves) in Klein-Gordon chains [6]. Numerical results indicate similar phenomena for the propagation of kinks [10, 38, 4]. Fine analysis of numerical convergence problems also suggests that different nonlinear lattices do not support exact solitary waves or travelling breathers in certain parameter regimes [42, 3].

Nevertheless, several formal analytical methods have been used to obtain travelling breather solutions. On the one hand, approximate travelling breathers can be formally obtained via effective Hamiltonians, which approximately describe the motion of the breather center on the lattice, at a nonconstant velocity [31, 26]. On the other hand,
multi-scale expansions provide evolution equations for the envelopes of well-prepared initial conditions corresponding to modulated plane waves. This approach has been used by Remoissenet for Klein-Gordon lattices [36] and yields the nonlinear Schrödinger (NLS) equation as a modulation equation. For good parameter values, the NLS equation admits solitons corresponding (at least formally) to travelling breather solutions of the original system, which propagate at a constant velocity (the group velocity of the wave packet). At the order of the NLS approximation, the linear dispersion is exactly balanced by the effect of nonlinear terms. The same approach has been used by Tsurui for the Fermi-Pasta-Ulam lattice [45]. For the Klein-Gordon system (and generalizations with anharmonic coupling), the validity of the nonlinear Schrödinger equation on large but finite time intervals has been proved recently by Giannoulis and Mielke [19].

It is a challenging problem to determine if these approximate solutions could constitute the principal part of exact travelling breather solutions of the Klein-Gordon system. This would imply that linear dispersion is balanced by nonlinear terms at any order in the above mentioned multi-scale expansion.

This problem has been solved by Iooss and Kirchgässner in the case of travelling waves [22], where the phase velocity of the plane wave equals the group velocity of the wave packet. Travelling wave solutions of (1) (with \( p = 1 \) in (2)) are determined by the scalar advance-delay differential equation

\[
\frac{d^2x_1}{d\tau^2} + V'(x_1) = \gamma(x_1(\tau - T) - 2x_1 + x_1(\tau + T)).
\]  

(3)

Iooss and Kirchgässner have studied small amplitude solutions of (3) in different parameter regimes and have obtained in particular “nanopterons” consisting of a solitary wave superposed on an exponentially small oscillatory tail. The leading order part of these solutions (excluding their tail) coincides with approximate solutions obtained via the NLS equation.

However, the more general case when phase and group velocities are different has remained open until now. More generally, different situations have been observed for the existence of exact travelling breathers in various simpler models. On the one hand, exact travelling breathers can be explicitly computed in the integrable Ablowitz-Ladik lattice [1], and other examples of nonlinear lattices supporting exact travelling breathers can be obtained using an inverse method [14]. On the other hand, travelling breather solutions of the Ablowitz-Ladik lattice are not robust under various non-Hamiltonian reversible perturbations as shown in [7].

The aim of our study is to clarify the existence question of exact travelling breather solutions in the Klein-Gordon lattice (1), in a case when the breather period and the inverse of its velocity are commensurate (we develop the results announced in [40]). For fixed \( p \geq 2 \), problem (1)–(2) reduces to the \( p \)-dimensional system of advance-delay differential equations

\[
\frac{d^2x_1}{d\tau^2} + \left[ \begin{array}{c}
V'(x_1) \\
\vdots \\
V'(x_n) \\
V'(x_p)
\end{array} \right] = \gamma \left[ \begin{array}{c}
x_2(\tau) - 2x_1(\tau) + x_p(\tau + T) \\
\vdots \\
x_{n+1}(\tau) - 2x_n(\tau) + x_{n-1}(\tau) \\
x_1(\tau - T) - 2x_p(\tau) + x_{p-1}(\tau)
\end{array} \right].
\]  

(4)

For the sake of simplicity we restrict ourselves to the case \( p = 2 \) in (4). The general case \( p \geq 2 \) is analyzed in a work in progress. The latter is technically more difficult but the approach used in our paper works as well.
We analyze small amplitude solutions of (4) (with $p = 2$) using the method developed by Iooss and Kirchgässner [22] in the context of travelling waves (see [20] for an application of this method to Fermi-Pasta-Ulam lattices). The method is based on a reduction to a center manifold in the infinite dimensional case as described in references [27, 33, 46]. System (4) is rewritten as a reversible evolution problem in a suitable functional space, and considered for parameter values $(T, \gamma)$ near a critical curve where the imaginary part of the spectrum consists of a pair of double eigenvalues and two pairs of simple ones. Close to this curve, the pair of double eigenvalues splits in two pairs of eigenvalues with opposite nonzero real parts, which opens the possibility of finding homoclinic solutions to 0.

Near these parameter values, the center manifold theorem reduces the problem locally to a reversible 8-dimensional system of differential equations. Thanks to an appropriate choice of variables, the reduction procedure is similar to the case analyzed by Iooss and Kirchgässner [22]. However, the simplest homoclinic bifurcation yields in our case a higher-dimensional reduced system, with a supplementary pair of simple imaginary eigenvalues.

The reduced system is put in a normal form which is integrable up to higher order terms. In some regions of the parameter space, the truncated normal form admits reversible homoclinic orbits to 0, which bifurcate from the trivial state and correspond to approximate solutions of (4). These approximate solutions coincide with spatially localized modulated plane waves obtained via the NLS equation. However, by analogy with results of Lombardi [28] we conjecture that these solutions do not generically persist when higher order terms are taken into account in the normal form. To make a more precise statement fix $V(x) = \frac{1}{2}x^2 + \alpha x^3 + \beta x^4$. We expect that a reversible solution of the reduced equation homoclinic to 0 and close to a small amplitude homoclinic orbit of the truncated normal form might only exist if $(T, \gamma, \alpha, \beta)$ is chosen on a discrete collection of codimension-$m$ submanifolds of $\mathbb{R}^4$ ($m > 0$). The codimension depends on the number of pairs of purely imaginary eigenvalues (i.e. the number of resonant phonons) in our parameter regime and symmetry assumptions. In our case (with two pairs of purely imaginary eigenvalues, in addition to hyperbolic ones), we expect $m = 2$ when homoclinic orbits to 0 correspond to travelling breather solutions of (1)–(2) (with $p = 2$), and $m = 1$ when homoclinic orbits to 0 correspond to solitary waves (homoclinic orbits to 0 possess an additional symmetry in that case).

For general parameter values, instead of homoclinic orbits to 0 one can expect the existence of reversible homoclinic orbits to exponentially small 2–dimensional tori, originating from the two additional pairs of simple purely imaginary eigenvalues. These solutions should constitute the principal part of exact travelling breather solutions of (1) superposed on a small quasi-periodic oscillatory tail. However, in order to obtain exact solutions one has to prove the persistence of the corresponding homoclinic orbits as higher order terms are taken into account in the normal form. This step is non-trivial and would require to generalize results of Lombardi [28] available when one pair of simple imaginary eigenvalues is removed. The most intricate part of the problem is to obtain a sharp (exponentially small) estimate of the minimal tail size of solutions. Another promising approach for obtaining such estimates is developed in the recent work of Iooss and Lombardi [23] on polynomial normal forms with exponentially small remainder for analytic vector fields. However the application of their theory to our situation would require several nontrivial extensions (to the $(i\omega_1)^2i\omega_1i\omega_2$ resonance and to systems with an additional infinite-dimensional hyperbolic part).
In this paper we prove the persistence of some homoclinic solutions in the case when the on-site potential $V$ is even. Indeed, due to the additional invariance $x_n \rightarrow -x_n$ one can find solutions of (1)–(2) (with $p = 2$) satisfying $x_n(\tau) = -x_{n-1}(\tau - \frac{T}{2})$. These solutions correspond to solutions of the normal form system possessing a particular symmetry. For the normal form restricted to the associated (6-dimensional) invariant subspace, results of Lombardi [28] are applicable since the linear part does not possess an extra pair of simple purely imaginary eigenvalues (the bifurcation corresponds to a pair of double eigenvalues and a pair of simple ones). As a result the full normal form admits homoclinic orbits to small periodic ones for near-critical parameter values $(T, \gamma)$. These solutions correspond to exact travelling breather solutions of (1) superposed on a small periodic oscillatory tail, which can be made exponentially small with respect to the central oscillation size. The minimal tail size should be generically nonzero for a given value of $(T, \gamma)$, but might vanish on a discrete collection of curves in the $(T, \gamma)$ parameter plane. As a consequence, in a given system (1) (with fixed coupling constant $\gamma$ and symmetric on-site potential $V$), exact travelling breather solutions decaying to 0 at infinity (and satisfying (2) for $p = 2$) might exist in the small amplitude regime, for isolated values of the breather velocity $2/T$.

We insist on the fact that our study is local, and analytical results for large amplitude solutions would be of interest. Results of this type exist for solitary waves or kinks in several one-dimensional nonlinear lattices (see [18, 17, 32, 41, 16]) but the problem is still open for large amplitude travelling breather solutions.

The paper is organized as follows. In Sect. 2 we formulate (1)-(2) as an evolution problem in an infinite-dimensional Banach space. Sections 3 and 4 are devoted to the linearized problem (spectral study, optimal regularity result) and the reduction to a center manifold. In Sect. 5 we study the reduced equation and describe its small amplitude homoclinic solutions when higher-order terms are neglected. These terms are taken into account in the even-potential case. Section 6 describes the corresponding leading-order travelling breather solutions of the Klein-Gordon system, and exact solutions (with small oscillatory tails) in the case of even potentials.

2. Formulation of the Problem

In this section, we formulate the initial problem (1)–(2) in an appropriate way.

The case $p = 2$ in (2) leads to the following system:

$$
\frac{d^2}{d\tau^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} V'(x_1) \\ V'(x_2) \end{bmatrix} = \gamma \begin{bmatrix} x_2(\tau) - 2x_1(\tau) + x_2(\tau + T) \\ x_1(\tau) - 2x_2(\tau) + x_1(\tau - T) \end{bmatrix}.
$$

(5)

Note that travelling wave solutions of (1) satisfying $x_n(\tau) = x_{n-1}(\tau - T/2)$ are particular solutions of (2) with $p = 2$. Consequently, the solutions considered in our case include those found by Iooss and Kirchgässner [22].

We shall analyze small amplitude solutions of (5) using the center manifold reduction method introduced by Iooss and Kirchgässner [22] in the context of reversible advance-delay differential equations. For this purpose, one has to make a convenient choice of variables which allows us to recover some essential estimates in their reduction process (optimal regularity result).

We rescale (5) using $t = \frac{\tau}{T}$ and consider the new variable $(u_1(t), u_2(t)) = (x_1(\tau), x_2(\tau + \frac{T}{2}))$. This yields
\[ x_n(\tau) = u_1\left(\frac{\tau}{T} - \frac{n - 1}{2}\right) \text{ if } n \text{ is odd}, \]
\[ x_n(\tau) = u_2\left(\frac{\tau}{T} - \frac{n - 1}{2}\right) \text{ if } n \text{ is even}. \quad (6) \]

With this change of variables, we have
\[ \frac{d^2}{dt^2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + T^2 \begin{bmatrix} V'(u_1) \\ V'(u_2) \end{bmatrix} = \gamma T^2 \begin{bmatrix} u_2(t - \frac{T}{2}) - 2u_1(t) + u_2(t + \frac{T}{2}) \\ u_1(t + \frac{T}{2}) - 2u_2(t) + u_1(t - \frac{T}{2}) \end{bmatrix}. \quad (7) \]

Note that solutions of (7) with \( u_1 = u_2 \) correspond to travelling wave solutions of (1) satisfying \( x_n(\tau) = x_{n-1}(\tau - \frac{T}{2}) \).

As in [22] we set \( U = (u_1, u_2, \dot{u}_1, \dot{u}_2, X_1(t, v), X_2(t, v))^T \), where \( v \in [-1/2, 1/2] \) and \( X_1(t, v) = u_1(t + v), X_2(t, v) = u_2(t + v) \). We define the following trace operators:
\[ \delta_{1/2} X_i(t, v) = X_i(t, 1/2), \quad \delta_{-1/2} X_i(t, v) = X_i(t, -1/2). \quad (8, 9) \]

Furthermore, we assume \( V \) analytic in a neighborhood of 0, with the following Taylor expansion at \( x = 0 \):
\[ V(x) = \frac{1}{2} x^2 - \frac{a}{3} x^3 - \frac{b}{4} x^4 + \text{h.o.t.} \quad (10) \]

We can write the system (7) as an evolution problem
\[ \frac{dU}{dt} = LU + F(U) \quad (11) \]
with \( L \) given by
\[ L = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & \alpha_2 & \alpha_2(\delta_{1/2} + \delta_{-1/2}) \\ 0 & \alpha_1 & 0 & 0 & \alpha_2(\delta_{-1/2} + \delta_{1/2}) & 0 \\ 0 & 0 & 0 & 0 & \partial_v & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_v \end{pmatrix}, \quad (12) \]
\[ \alpha_2 = T^2 \gamma \text{ and } \alpha_1 = -T^2(1 + 2\gamma). \]

The nonlinear operator \( F \) is given by
\[ F(U) = T^2(0, 0, f(u_1), f(u_2), 0, 0)^T \quad (13) \]
and
\[ f(u) = au^2 + bu^3 + \text{h.o.t.} \quad (14) \]

We now write (11) in appropriate function spaces. For this purpose we introduce the Banach spaces
\[ \mathbb{D} = \mathbb{R}^4 \times (C^0[-1/2, 1/2])^2. \quad (15) \]
\[ \mathbb{D} = \left\{ U \in \mathbb{R}^4 \times (C^1[-1/2, 1/2])^2 / X_1(0) = u_1, X_2(0) = u_2 \right\}. \quad (16) \]
The operator $L$ maps $D$ into $H$ continuously, $F : D \to D$ is $C^{k-1}$ with $F(U) = O(\|U\|^k_D)$.

We observe that the symmetry $R$ on $H$ defined by
\[
R(u_1, u_2, \xi_1, \xi_2, X_1(v), X_2(v)) = (u_1, u_2, -\xi_1, -\xi_2, X_1(-v), X_2(-v))
\]
satisfies $(L + F) \circ R = -R(L + F)$. Therefore, if $U$ is a solution of (11) then $RU(-t)$ is also a solution, i.e. the system (11) is reversible under $R$. This property is due to the invariance $t \to -t$ of (7). A solution $U$ of (11) is said to be reversible under $R$ if $RU(-t) = U(t)$ for all $t \in \mathbb{R}$. Reversible solutions under $R$ correspond to solutions of (1)–(2) satisfying $x_{-n}(-\tau - T) = x_n(\tau)$.

In addition, note that the permutational symmetry
\[
S(u_1, u_2, \xi_1, \xi_2, X_1, X_2) = (u_2, u_1, -\xi_1, \xi_2, -X_1, X_2)
\]
commutes with $L + F$. As we observed previously, travelling wave solutions (i.e. solutions of (1) satisfying $x_n(\tau) = x_{n-1}(\tau - T/2)$) appear as fixed points of $S$.

This additional invariance implies that $R_1 = RS = SR$ is also a reversibility symmetry for Eq. (11). Reversible solutions under $R_1$ correspond to solutions of (1)–(2) satisfying $x_{-2n}(-\tau - T/2) = x_{2n+1}(\tau)$.

The problem (11) is ill-posed as an initial value problem in $D$. Nevertheless, it is possible to construct bounded solutions for all $t \in \mathbb{R}$. Using the method developed in [22], we are able to reduce (11) locally to a finite dimensional system of ordinary differential equations. The dimension of this reduced system depends on the bifurcation parameters $\gamma$ and $T$ (we shall fix $T > 0$ since Eq. (11) is even in $T$). In the next section, we describe the spectrum of $L$ in various parameter regions.

3. Spectral Problem

The linear operator $L$ is closed in $H$ with domain $D$ and has a compact resolvent. It follows that its spectrum consists of isolated eigenvalues $\sigma$ with finite multiplicities.

Let us compute the eigenvalues of $L$. Solving $LU = \sigma U$ with
\[
U = (\hat{u}_1, \hat{u}_2, \xi_1, \xi_2, X_1, X_2)^T
\]
leads to the equation
\[
A(\hat{u}_1, \hat{u}_2)^T = 0,
\]
where
\[
A = \begin{pmatrix}
\sigma^2 + T^2(1 + 2\gamma) & -2T^2\gamma \cosh(\sigma/2) \\
-2T^2\gamma \cosh(\sigma/2) & \sigma^2 + T^2(1 + 2\gamma)
\end{pmatrix}.
\]
The dispersion relation $\det A = 0$ reads
\[
N(\sigma, T, \gamma) := (\sigma^2 + T^2(1 + 2\gamma))^2 - 4(\gamma T^2)^2 \cosh^2(\sigma/2) = 0.
\]

The spectrum of $L$ is then given by the roots of $N(\sigma, T, \gamma) = 0$. Since $L$ has real coefficients and due to the reversibility, the spectrum is invariant under the reflection on the real and the imaginary axis.

We need basic properties of the spectrum in order to apply the reduction method [22]. As in reference [22], $L$ is not bi-sectorial and the central part ($\sigma = iq$) of its spectrum is isolated from the hyperbolic part ($\sigma \neq iq$). More precisely, the following result can be obtained as in [22], p. 443.
Lemma 3.1. For all \((\gamma, T) \in \mathbb{R}^2_+\), there exists \(p_0\) such that all eigenvalues \(\sigma = p + iq\) of \(L\) with \(p \neq 0\) satisfy \(|p| \geq p_0\).

For the central part of the spectrum \((\sigma = iq)\), the dispersion relation reads
\[
(-q^2 + T^2(1 + 2\gamma))^2 = 4(\gamma T^2)^2 \cos^2(q/2).
\]
(19)
In what follows we study the solutions of (19). Since (19) is even in \(q\), we restrict ourselves to the case \(q \geq 0\).

3.1. Spectrum on the imaginary axis for \(\gamma T^2 < 4\). The spectrum of \(L\) on the imaginary axis has a particularly simple structure for \(\gamma T^2 < 4\). From the previous relation we deduce two cases:
\[
T^2(1 + 2\gamma) - q^2 = \pm 2\gamma T^2 \cos(q/2).
\]
(20)
Case + in (20). We consider the equation
\[
T^2(1 + 2\gamma) - q^2 = 2\gamma T^2 \cos(q/2).
\]
(21)
This equation can be written
\[
T^2 = q^2 - 4\gamma T^2 \sin^2(q/4).
\]
(22)
We now consider \((T^2, \alpha^2)\) as new parameters (recall \(\alpha^2 = \gamma T^2\)). Equation (22) reads
\[
T^2 = f_{\alpha^2}(q) = q^2 - 4\alpha^2 \sin^2(q/4).
\]
(23)
If \(\alpha^2 < 4\), \(f_{\alpha^2} : [0, +\infty[ \rightarrow \mathbb{R}^+\) is a strictly increasing function of \(q\) and Eq. (23) yields \(q = f_{\alpha^2}^{-1}(T^2)\). This proves the existence of a pair of simple eigenvalues \(\sigma = \pm i f_{\alpha^2}^{-1}(T^2)\) for \(\gamma T^2 < 4\). The corresponding eigenvectors \(V, \bar{V}\) read
\[
V = (1, 1, iq, iq, e^{iqv}, e^{iqv})^T.
\]
We observe that \(RV = \bar{V}\) and \(SV = V\).

Case – in (20). We consider the equation
\[
T^2(1 + 2\gamma) - q^2 = -2\gamma T^2 \cos(q/2).
\]
(24)
In this case, we have
\[
T^2 = g_{\alpha^2}(q) = q^2 - 4\alpha^2 \cos^2(q/4).
\]
If \(\alpha^2 < 4\), \(g_{\alpha^2} : [0, +\infty[ \rightarrow \mathbb{R}\) is a strictly increasing function of \(q\) and then \(q = g_{\alpha^2}^{-1}(T^2)\).

This proves the existence of another pair of simple eigenvalues \(\sigma = \pm ig_{\alpha^2}^{-1}(T^2)\) for \(\gamma T^2 < 4\). The corresponding eigenvectors \(V, \bar{V}\) read
\[
V = (-1, 1, -iq, iq, -e^{iqv}, e^{iqv})^T.
\]
We observe that \(RV = \bar{V}\) and \(SV = -V\).

Note that \(if_{\alpha^2}^{-1}(T^2) = ig_{\alpha^2}^{-1}(T^2) = i(2k + 1)\pi\) for \(T^2(1 + 2\gamma) = (2k + 1)^2\pi^2\) \((k \in \mathbb{N})\). In this case, the two pairs of eigenvalues collide, yielding a pair of double semi-simple eigenvalues (with eigenvectors having different symmetries).

In what follows we extend the spectral study to the whole parameter space. In particular we shall consider the occurrence of double and triple purely imaginary eigenvalues.
3.2. Double and triple eigenvalues on the imaginary axis. For having (at least) double purely imaginary eigenvalues, we have to verify (19) and \( \frac{dN(iq,T,\gamma)}{dq} = 0 \), i.e.

\[
2q(-q^2 + T^2(1+2\gamma)) = (\gamma T^2)^2 \sin(q). \tag{25}
\]

Moreover, \( iq \) is a triple eigenvalue when \( q \) satisfies (19),(25) and the following equation \( \left( \frac{d^2N(iq,\gamma,T)}{dq^2} \right) = 0 \):

\[
-6q^2 + 2T^2(1+2\gamma) = (\gamma T^2)^2 \cos(q). \tag{26}
\]

The following lemma gives a description of the set of double and triple eigenvalues on the imaginary axis, as a function of \((\gamma, T) \in \mathbb{R}^2_+\). These results are sketched in Fig. 1.

**Lemma 3.2.** Consider the curve \( \Gamma \) parametrized by \((T(q), \gamma(q)) \) with \( q \in \mathbb{R}^+ \) and \( T, \gamma \) defined by the system (19)–(25). This curve (which we call a bifurcation curve) is given by:

if \( q \in [4k\pi, (2k+1)2\pi] \) (for an integer \( k \geq 1 \)),

\[
T^2 = q^2 - 4q \tan(q/4), \tag{27}
\]

\[
\gamma = \frac{2q}{T^2 \sin(q/2)}. \tag{28}
\]

if \( q \in [(2k-1)2\pi, 4k\pi] \) (\( k \geq 1 \)),

---

**Fig. 1.** Bifurcation curves and purely imaginary eigenvalues of \( L \) (upper half complex plane). “TP” (respectively “TW”) stands for the curves corresponding mainly to pulsating travelling wave (respectively travelling wave) bifurcations. The bold line corresponds to the subset \( \Lambda \).
\[ T^2 = q^2 + \frac{4q}{\tan(q/4)}. \]  

(29)

\[ \gamma = -\frac{2q}{T^2 \sin(q/2)}. \]  

(30)

The range of \( q \) is determined by the condition \( T^2 > 0 \). We denote by \( \Gamma_k \) the restriction of \( \Gamma \) to the interval \( q \in [2k\pi, 2(k+1)\pi] \). The curve \( \Gamma \) lies in the parameter region where \( \gamma T^2 > 4 \).

For \( (T, \gamma) \in \Gamma \) (except on a countable set of points \( \Omega \)), the spectrum of \( L \) on the imaginary axis consists of a pair of double non-semi-simple eigenvalues \( \pm iq \) and at least two distinct pairs of simple eigenvalues.

The set of exceptional parameter values \( \Omega \) consists of the following types of points:

- **Cusps on \( \Gamma \)** correspond to the existence of a pair of triple eigenvalues \( \pm i\bar{q} \) (Jordan block of index 3) satisfying \( \tan(\bar{q}/2) = \bar{q}/2 \) and a pair of simple eigenvalues.
- **The point of tangent intersection between \( \Gamma_k \) and the curve** \( T^2(1+2\gamma) = (2k+1)^2\pi^2 \) leads to the existence of a pair of triple eigenvalues (with a two-dimensional eigenspace) and a pair of simple eigenvalues.
- **A point of transverse intersection between \( \Gamma_m \) and a curve** \( T^2(1+2\gamma) = (2k+1)^2\pi^2 \) \( (k \in \mathbb{N}) \) leads to the existence of two pairs of double eigenvalues (one being semi-simple and the other non-semi-simple), and at least one pair of simple eigenvalues if \( m \neq k \).
- **Double points on \( \Gamma \)** correspond to the existence of two pairs of double non-semi-simple eigenvalues, and pairs of simple eigenvalues, depending on the parameter region.

**Proof.** First, we divide (19) by (25) to obtain the following equation:

\[ T^2(1+2\gamma) = q^2 + \frac{4q}{\tan(q/2)}. \]  

(31)

Substituting the expression for \( T^2(1+2\gamma) \) in (25), we obtain

\[ \gamma = \frac{2q}{T^2 |\sin(q/2)|}. \]  

(32)

We have to consider two cases: \( \sin(q/2) > 0 \) and \( \sin(q/2) < 0 \).

Fixing \( \gamma = \frac{2q}{T^2 \sin(q/2)} \) in (31) yields

\[ T^2 = q^2 - 4q \tan(q/4). \]  

(33)

In the same way, fixing \( \gamma = -\frac{2q}{T^2 \sin(q/2)} \) in (31) leads to

\[ T^2 = q^2 + \frac{4q}{\tan(q/4)}. \]  

(34)

Furthermore, Eq. (32) shows that \( \gamma T^2 > 4 \).

The spectrum of \( L \) on the imaginary axis as a function of \( \gamma \), \( T \) is sketched in Fig. 1. The spectrum outside \( \Gamma \) is obtained by continuity arguments.
We note that for $T^2(1 + 2\gamma) = (2k + 1)^2\pi^2$, $k \in \mathbb{N}$, $q_* = (2k + 1)\pi$ is a solution of (20) for both cases $+$ and $-$. Therefore, $\pm iq_* = \pm i(2k + 1)\pi$ is a pair of at least double eigenvalues.

One can check that $\Gamma_k$ has a tangent intersection with the curve $T^2(1 + 2\gamma) = (2k + 1)^2\pi^2$ at the point $(T, \gamma) = (T(q_*), \gamma(q_*))$. Moreover, Eq. (26) is satisfied at this point and consequently $iq_*$ is a triple eigenvalue of $L$ (one can check that the associated eigenspace is two-dimensional). The existence of another pair of simple eigenvalues follows by a continuity argument.

Moreover, one can show that $\Gamma_k$ has only one other (transverse) intersection with the curve $T^2(1 + 2\gamma) = (2k + 1)^2\pi^2$, at a point $(T, \gamma) = (T(q_0), \gamma(q_0))$ with $q_0 \neq q_*$. In this case one has two pairs of double eigenvalues ($iq_*$ being semi-simple and $i q_0$ non-semi-simple). Similar intersections between $\Gamma_m (m \neq k)$ and $T^2(1 + 2\gamma) = (2k + 1)^2\pi^2$ lead to extra pairs of simple eigenvalues.

Finally, for $q \neq (2k + 1)\pi$ Eqs. (31),(32) and (26) lead to

$$\tan(q/2) = (q/2).$$

(35)

In any fixed interval $[2k\pi, (2k + 1)\pi) (k \geq 1)$ this equation has a unique solution $\tilde{q}$ (which determines $\gamma, T$ uniquely). This solution corresponds to a triple eigenvalue $i\tilde{q}$ (and one has a Jordan block of index 3). Such triple eigenvalues appear as cusp points of the bifurcation curve ($(dq_{\gamma})$ and $(dq_T)$ vanish at $q = \tilde{q}$).

**Remark.** Since our bifurcating solutions include the travelling waves found by Iooss and Kirchgässner [22], it is interesting to compare our bifurcation diagram with the one of reference [22].

More precisely, there exist travelling wave solutions of (1)–(2) (with $p = 2$) satisfying

$$x_{n-1}(\tau - \frac{T}{2}) = x_n(\tau).$$

(36)

In order to establish a comparison of Lemma 3.2 with reference [22], we replace $q$ by $2q$ in the parametrization of $\Gamma$. This yields

$$\gamma = \frac{4q}{T^2|\sin(q)|},$$

(37)

and if $q \in [2k\pi, (2k + 1)\pi]$

$$T^2 = 4q^2 - 8q \tan(q/2),$$

(38)

otherwise

$$T^2 = 4q^2 + \frac{8q}{\tan(q/2)}.$$  

(39)

Now replacing $T$ by $2T$ in (38) yields exactly the parametrization of the bifurcation curve given on p. 443 in [22]. Consequently, small amplitude solutions which bifurcate in the neighborhood of $\Gamma_q$ include travelling wave solutions of reference [22]. These solutions can be combined with an additional mode corresponding to an extra pair of simple eigenvalues on the imaginary axis.
On the contrary, small amplitude solutions which bifurcate in the neighborhood of \( \Gamma_{2k+1} \) mainly consist (apart from spatially periodic travelling waves) of pulsating travelling waves not described in reference [22].

In what follows, we define \( \Delta \) as the subset of \( \Gamma \) such that the central part of the spectrum is \( \Sigma_0 = \{ \pm iq_1, \pm iq_2, \pm iq_0 \} \), where \( \pm iq_0 \) is a pair of non semi-simple double eigenvalues and \( \pm iq_1, \pm iq_2 \) two pairs of simple ones (\( \Delta \) corresponds to the bold line in Fig. 1). One can check the following properties.

Lemma 3.3. Fix \((T, \gamma) \in \Delta \) and let \( V_0, V_1, V_2 \) be the eigenvectors associated to \( iq_0 \), \( iq_1 \), \( iq_2 \) respectively. Denote by \( \hat{V}_0 \) the generalized eigenvector associated to \( iq_0 \). The eigenvectors can be chosen in the following way:

\[
V_1 = (-1, 1, -1, iq_1, iq_1, -e^{iq_1 v}, e^{iq_1 v})^T, \quad V_2 = (1, 1, iq_2, iq_2, e^{iq_2 v}, e^{iq_2 v})^T, \\
V_0 = (\epsilon, 1, 1, iq_0, iq_0, e^{iq_0 v}, e^{iq_0 v})^T, \quad \hat{V}_0 = (0, 0, 0, 0, 0, 0, 0)^T,
\]

where \( \epsilon = -1 \) if \( q_0 \in [(2k-1)2\pi, 4k\pi] \) and \( \epsilon = 1 \) if \( q_0 \in [4k\pi, (2k+1)2\pi] \) \( (k \geq 1) \). Moreover these eigenvectors satisfy

\[
RV_0 = \hat{V}_0, \quad RV_1 = \hat{V}_1, \quad RV_2 = \hat{V}_2, \quad R\hat{V}_0 = -\hat{V}_0, \\
SV_0 = \epsilon V_0, \quad SV_1 = -V_1, \quad SV_2 = V_2, \quad S\hat{V}_0 = \epsilon \hat{V}_0.
\]

4. Optimal Regularity Problem and Reduction on a Center Manifold

In this section we fix \((T, \gamma) \in \Delta \), compute the spectral projection on the hyperbolic subspace (invariant subspace under \( L \) corresponding to the hyperbolic spectral part) and prove an optimal regularity result for the associated inhomogeneous linearized equation. This result is a crucial assumption for applying center manifold reduction theory [46]. Our proof closely follows the method given in [22].

We call \( P_0, P_1, P_2 \) respectively the spectral projection on the 4-dimensional invariant subspace associated to \( \pm iq_0 \), on the 2-dimensional subspace corresponding to \( \pm iq_1 \), on the 2-dimensional subspace corresponding to \( \pm iq_2 \). We also define \( P = P_0 + P_1 + P_2 \) (spectral projection on the 8-dimensional central subspace) and use the notations \( \mathbb{D}_h \equiv (I - P)\mathbb{D}, \mathbb{H}_h = (I - P)\mathbb{H}, \mathbb{D}_c = P\mathbb{D}, U_h = (I - P)U \). The affine linearized system on \( \mathbb{H}_h \) reads

\[
\frac{dU_h}{dt} = LU_h + F_h(t),
\]

where \( F(t) = (0, 0, f_1(t), f_2(t), 0, 0)^T \) lies in the range of the nonlinear operator (13).

We shall note \( U_h = (u_1^h, u_2^h, x_1^h, x_2^h, X_1^h(v), X_2^h(v))^T \).

Our aim is to check the optimal regularity property of Eq. (40) (see [46], property (ii) p.127). This property can be stated as follows. We introduce the following Banach space, for a given Banach space \( Z \) and \( \alpha \in \mathbb{R}^+ \):

\[
E_j^\alpha(Z) = \left\{ f \in C^j(\mathbb{R}, Z) \parallel f \parallel_J = \max_{0 \leq k \leq j} \sup_{t \in \mathbb{R}} e^{-\alpha |t|} \| D^k f(t) \| < \infty \right\}.
\]

We need to check that system (40) admits a unique solution \( U_h \) in \( E_j^\alpha(\mathbb{D}_h) \cap E_j^\alpha(\mathbb{H}_h) \) for \( 0 \leq \alpha < \alpha_0 \) (for some \( \alpha_0 > 0 \)), the operator \( K_j : E_j^\alpha(\mathbb{R}^2) \to E_j^\alpha(\mathbb{D}_h), (f_1, f_2) \mapsto U_h \) being bounded.

As the linear operator \( L \) is not bi-sectorial, we do not have classical estimates on its resolvent and have to compute \( U_h \) explicitly.
4.1. Computation of the spectral projection on the hyperbolic subspace. The spectral projection on the central subspace is defined by the Dunford integral

\[ P = \frac{1}{2\pi i} \int_C (\sigma I - L)^{-1} dC, \]

where \( C \) is a regular curve surrounding \( \pm iq_1, \pm iq_2, \pm iq_0 \). The spectral projection on the hyperbolic subspace is \( P_h = I - P \).

We shall use the following result for computing \( P_h \).

**Lemma 4.1.** Let \( h(z) = \frac{f(z)}{g(z)} \) be a function of \( z \in \mathbb{C} \). Assume the function \( f(z) \) is entire and the function \( g(z) \) admits a double pole at \( z = z_0 \). Then the residue of \( h \) at \( z = z_0 \) is given by

\[ \text{Res}(h, z_0) = \frac{2 f'(z_0)g''(z_0) - f(z_0)g''(z_0)}{g''(z_0)^2}. \]

In the following lemma, we compute the spectral projection on the hyperbolic subspace of a vector \( F \) lying in the range of the nonlinear operator (13).

**Lemma 4.2.** Let \( F \in \mathbb{D} \) be a vector of the type \( F = (0, 0, f_1, f_2, 0, 0)^T \). Then the projection of \( F \) on the hyperbolic subspace reads

\[ F_h = (0, 0, k_3f_1 + k_4f_2, k_5f_1 + k_6f_2, k_7(v)f_1 + k_8(v)f_2, k_9(v)f_1 + k_{10}(v)f_2)^T, \]

where \( k_3, k_4, k_5, k_6 \in \mathbb{R} \) and \( k_7, k_8, k_9, k_{10} \in C^\infty([-1/2, 1/2]) \) depend on \( \gamma, T \).

**Proof.** We first compute the resolvent of \( L \). One has to solve \((\sigma \mathbb{1} - L)U = F\), which yields the system

\[ \begin{align*}
\xi_1 &= \sigma u_1, \\
(\sigma^2 - \alpha_1)u_1 - 2\alpha_2 \cosh(\sigma/2)u_2 &= f_1, \\
(\sigma^2 - \alpha_1)u_2 - 2\alpha_2 \cosh(\sigma/2)u_1 &= f_2, \end{align*} \]

where \( u = (u_1, u_2, \xi_1, \xi_2, X_1(v), X_2(v))^T \). We have then

\[ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{N(\sigma, \gamma, T)} \begin{pmatrix} (\sigma^2 - \alpha_1)f_1 + 2\alpha_2 \cosh(\sigma/2)f_2 \\ (\sigma^2 - \alpha_1)f_2 + 2\alpha_2 \cosh(\sigma/2)f_1 \end{pmatrix}. \]

Now we compute the spectral projection \( P_1 \). Since \( \sigma = iq_1 \) is a simple root of (18), one has

\[ \text{Res}(u_1, iq_1) = \frac{i(-q_1^2 + \alpha_1)f_1 + 2\alpha_2 \cos(q_1/2)f_2}{4q_1(q_1^2 + \alpha_1) + 2\alpha_2^2 \sin(q_1)}, \]

\[ \text{Res}(u_2, iq_1) = \frac{i(-q_1^2 + \alpha_1)f_2 + 2\alpha_2 \cos(q_1/2)f_1}{4q_1(q_1^2 + \alpha_1) + 2\alpha_2^2 \sin(q_1)}. \]

Denoting \( (P_1 F)_i \), the \( i \)-th component of \( P_1 F \), we get consequently

\[ (P_1 F)_1 = \text{Res}(u_1, iq_1) + \text{Res}(u_1, -iq_1) = 0, \]

\[ (P_1 F)_2 = \text{Res}(u_2, iq_1) + \text{Res}(u_2, -iq_1) = 0. \]
In the same spirit

\[(P_1 F)_3 = \frac{-2q_1(-q_1^2 + \alpha_1)f_1 + 2\alpha_2 \cos(q_1/2)f_2}{4q_1(q_1^2 + \alpha_1) + 2\alpha_2^2 \sin(q_1)},\]

\[(P_1 F)_4 = \frac{-2q_1(-q_1^2 + \alpha_1)f_2 + 2\alpha_2 \cos(q_1/2)f_1}{4q_1(q_1^2 + \alpha_1) + 2\alpha_2^2 \sin(q_1)},\]

\[(P_1 F)_5 = \frac{-2\sin(q_1v)(-q_1^2 + \alpha_1)f_1 + 2\alpha_2 \cos(q_1/2)f_2}{4q_1(q_1^2 + \alpha_1) + 2\alpha_2^2 \sin(q_1)},\]

\[(P_1 F)_6 = \frac{-2\sin(q_1v)(-q_1^2 + \alpha_1)f_2 + 2\alpha_2 \cos(q_1/2)f_1}{4q_1(q_1^2 + \alpha_1) + 2\alpha_2^2 \sin(q_1)},\]

which completes the computation of \(P_1 F\). The computations are identical for the spectral projection \(P_2\) associated to \(\pm i\xi_2\). For computing the spectral projection \(P_0\) associated to the double eigenvalues \(\pm iq_0\), we use formula (43). These computations lead to Eq. (44). \(\Box\)

4.2. Resolution of the affine equation for bounded functions of \(t\). We first solve (40) in the spaces \(E_j^\alpha\) with \(\alpha = 0\), i.e. we consider bounded functions of \(t\) (note that \(E_j^0(H) = C_b^\alpha(H)\)). Fixing \(\alpha = 0\) will allow us to take the Fourier transform in time of the system in the tempered distributional space \(S'(\mathbb{R})\).

From (40), we directly deduce

\[X_1^h(t, v) = u_1^h(t + v) + \int_0^v (k_7(s)f_1(t + v - s) + k_{10}(s)f_2(t + v - s))ds \quad (47)\]

\[= u_1^h(t + v) + \int_{t+v}^{t} (k_7(t + v - s)f_1(s) + k_{10}(t + v - s)f_2(s))ds, \quad (48)\]

\[X_2^h(t, v) = u_2^h(t + v) + \int_0^v (k_9(s)f_1(t + v - s) + k_{10}(s)f_2(t + v - s))ds \quad (49)\]

\[= u_2^h(t + v) + \int_{t+v}^{t} (k_9(t + v - s)f_1(s) + k_{10}(t + v - s)f_2(s))ds \quad (50)\]

(this expression comes from the two last equations of the affine linear system and from conditions \(X_1(0, t) = u_1(t), X_2(0, t) = u_2(t)\)).

From the previous equations and the fact that \((k_i)_{i=7,10}\) and their derivatives are bounded functions of \(t\), we deduce that

\[\|X_1^h\|_{E_0^h(C^{[t-1, t+2]/2})} \leq \|u_1^h\|_{E_0^h} + C(\|f_1\|_{E_0^h} + \|f_2\|_{E_0^h}), \quad (51)\]

\[\|X_2^h\|_{E_0^h(C^{[t-1, t+2]/2})} \leq \|u_2^h\|_{E_0^h} + C(\|f_1\|_{E_0^h} + \|f_2\|_{E_0^h}). \quad (52)\]

We now have to estimate \(u_1^h, u_2^h, \xi_1^h, \xi_2^h\). Taking the Fourier transform in time of the system (40) in the tempered distributional space \(S'(\mathbb{R})\), we have

\[(ik - L)\hat{U}_h = \hat{F}_h. \quad (53)\]
We deduce
\[ \xi_1^h = iku_1^h, \]
\[ \xi_2^h = iku_2^h, \]
\[ X_1^h = e^{iku_1^h} + \hat{f}_1 \int_0^v e^{iku(s)}k_7(s)ds + \hat{f}_2 \int_0^v e^{iku(s)}k_8(s)ds, \]
\[ X_2^h = e^{iku_2^h} + \hat{f}_1 \int_0^v e^{iku(s)}k_9(s)ds + \hat{f}_2 \int_0^v e^{iku(s)}k_{10}(s)ds. \]

For \( u_1^h, u_2^h \), we have
\[ -\left(k^2 + \alpha_1 \right) u_1^h - 2\alpha_2 \cos(k/2)u_2^h = (\hat{F}_h)_3 + C_1(k) \hat{f}_1 + C_2(k) \hat{f}_2, \tag{54} \]
\[ -\left(k^2 + \alpha_1 \right) u_2^h - 2\alpha_2 \cos(k/2)u_1^h = (\hat{F}_h)_4 + D_1(k) \hat{f}_1 + D_2(k) \hat{f}_2, \]
where
\[ (\hat{F}_h)_3 = k_3 \hat{f}_1 + k_4 \hat{f}_2, \]
\[ (\hat{F}_h)_4 = k_5 \hat{f}_1 + k_6 \hat{f}_2, \]
and \( C_i, D_i \) are \( C^\infty \) functions of \( k \), being \( O(1/|k|) \) as \( k \to \pm \infty \). Solving the system (54) leads to
\[ N(ik, \gamma, T) \begin{pmatrix} u_1^h \\ u_2^h \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{55} \]
where
\[ h_1 = -\left(k^2 + \alpha_1 \right)[(k_3 + C_1(k)) \hat{f}_1 + (k_4 + C_2(k)) \hat{f}_2] + 2\alpha_2 \cos(k/2)[(k_5 + D_1(k)) \hat{f}_1 + (k_6 + D_2(k)) \hat{f}_2], \]
\[ h_2 = 2\alpha_2 \cos(k/2)[(k_3 + C_1(k)) \hat{f}_1 + (k_4 + C_2(k)) \hat{f}_2] - \left(k^2 + \alpha_1 \right)[(k_5 + D_1(k)) \hat{f}_1 + (k_6 + D_2(k)) \hat{f}_2]. \]

Equation (55) can be written
\[ N(ik, \gamma, T) \begin{pmatrix} \hat{u}_1^h + \hat{H}_1 \hat{f}_1 + \hat{H}_2 \hat{f}_2 \\ \hat{u}_2^h + \hat{G}_1 \hat{f}_1 + \hat{G}_2 \hat{f}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{56} \]

As the operator \((ik - L_h)^{-1}\) is analytic in a strip around the real axis, we deduce that \( \hat{H}_1, \hat{H}_2, \hat{G}_1, \hat{G}_2 \) are analytic functions in this strip. Moreover, \( \hat{H}_1, \hat{H}_2, \hat{G}_1, \hat{G}_2 \) are \( O(1/|k|) \) as \( k \to \pm \infty \) due to the fact that \( N(ik, \gamma, T) = O(k^4) \) and \( h_1, h_2 \) are \( O(k^2) \) as \( k \to \pm \infty \). Since \( N(iq_j, \gamma, T) = 0, N'(iq_0, \gamma, T) = 0 \) and \( N'(iq_j, \gamma, T), N'(iq_0, \gamma, T) \) do not vanish, Eq. (56) yields
\[ \hat{u}_1^h + \hat{H}_1 \hat{f}_1 + \hat{H}_2 \hat{f}_2 = a_1^+ \delta_{i+q_1} + a_1^- \delta_{i-q_1} + a_2^+ \delta_{iq_2} + a_2^- \delta_{-iq_2} + a_0^+ \delta_{iq_0} + a_0^- \delta_{-iq_0} + b_0^+ \delta_{iq_0} + b_0^- \delta_{-iq_0}, \tag{57} \]
\begin{align*}
\hat{u}_2^h + \hat{G}_1 f_1 + \hat{G}_2 f_2 &= c_1^+ \delta_{iq1} + c_1^- \delta_{-iq1} + c_2^+ \delta_{iq2} + c_2^- \delta_{-iq2} \\
&\quad + c_0^+ \delta_{iq0} + c_0^- \delta_{-iq0} + d_0^+ \delta_{iq0} + d_0^- \delta_{-iq0}.
\end{align*}

(58)

Furthermore, \( k \to (1 + |k|^2)^{1/2} \hat{H}_i \) and \( k \to (1 + |k|^2)^{1/2} \hat{G}_i \) belong to \( L^2(\mathbb{R}) \). Therefore, using the inverse Fourier Transform and Lemma 3, p.448 of [22], there exist \( G_i, H_i \in H^1(\mathbb{R}) \) (i.e. \( e^{itH_i} \in H^1(\mathbb{R}) \), \( e^{itG_i} \in H^1(\mathbb{R}) \), \( \delta > 0 \) small enough) such that \( \hat{G}_i, \hat{H}_i \) are the unique Fourier transforms of \( G_i, H_i \). We have the following estimates

\[
\| \frac{dH_1}{dt} \ast f_1 \|_{C^0_b} = \sup_{t \in \mathbb{R}} | \int \frac{dH_1}{dt}(t - s)f_1(s)ds | \leq C(\delta) \| f_1 \|_{C^1_b} \| H_1 \|_{H^1(\mathbb{R})}.
\]

(59)

The same estimate is valid for \( \frac{dH_2}{dt} \ast f_2, \frac{dG_1}{dt} \ast f_1, \frac{dG_2}{dt} \ast f_2 \).

Now we make the solution of (40) explicit. We set \( \hat{U}_h = (\hat{u}_1^h, \hat{u}_2^h, \hat{\xi}_1^h, \hat{\xi}_2^h, \hat{X}_1^h, \hat{X}_2^h)^T \) and

\[
\begin{align*}
\hat{u}_1^h &= -H_1 \ast f_1 - H_2 \ast f_2, \\
\hat{u}_2^h &= -G_1 \ast f_1 - G_2 \ast f_2, \\
\hat{\xi}_1^h &= \frac{d\hat{u}_1^h}{dt}, \\
\hat{\xi}_2^h &= \frac{d\hat{u}_2^h}{dt}, \\
\hat{X}_1^h(t, v) &= \hat{u}_1^h(t + v) + \int_0^v (k_1(s)f_1(t + v - s) + k_2(s)f_2(t + v - s))ds, \\
\hat{X}_2^h(t, v) &= \hat{u}_2^h(t + v) + \int_0^v (k_3(s)f_1(t + v - s) + k_4(s)f_2(t + v - s))ds.
\end{align*}
\]

By construction, \( \hat{u}_h \) satisfies (40) and \( P \hat{U}_h = 0 \) (hence \( P \hat{U}_h = 0 \)) for \( (f_1, f_2) \in E_0^0(\mathbb{R}^2) \) with \( \alpha < 0 \) (\( \hat{f}_i \) are analytic functions in a strip around the real axis). Since the computations are formally the same for \( \alpha = 0 \), we have \( P \hat{U}_h = 0 \) for \( \alpha = 0 \), hence \( P \hat{U}_h = 0 \) for \( \alpha = 0 \).

Moreover, we have

\[
\| \hat{U}_h \|_{C^1_b(D_{\alpha})} \leq C(\| f_1 \|_{C^0_b(\mathbb{R})} + \| f_2 \|_{C^0_b(\mathbb{R})})
\]

(61)

due to estimates (51), (52), (59) (with analogous estimates on \( H_2, G_1 \)). For \( \alpha = 0 \), we obtain \( \hat{u}_1^h, \hat{u}_2^h \) by adding to \( \hat{u}_1^h, \hat{u}_2^h \) the inverse Fourier transforms of Dirac measures, i.e.

\begin{align*}
\hat{u}_1^h &= \hat{u}_1^h + a_1^+ e^{iqt} + a_1^- e^{-iqt} + a_2^+ e^{iqt} + a_2^- e^{-iqt} \\
&\quad + (a_0^+ + itb_0^+) e^{iqt} + (a_0^- - itb_0^-) e^{-iqt},
\end{align*}

(62)

\begin{align*}
\hat{u}_2^h &= \hat{u}_2^h + c_1^+ e^{iqt} + c_1^- e^{-iqt} + c_2^+ e^{iqt} + c_2^- e^{-iqt} \\
&\quad + (c_0^+ + itd_0^+) e^{iqt} + (c_0^- - itd_0^-) e^{-iqt}.
\end{align*}

(63)
Since $P\tilde{U}_h = 0$, we have $PU_h = 0$ if and only if
\[
a_1^+ = a_2^+ = c_1^+ = c_2^+ = b_0^+ = a_0^+ = d_0^+ = 0. \tag{64}
\]
It follows that $U_h = \tilde{U}_h$. Finally, we have proved the following

**Lemma 4.3.** Assume $F = (0, 0, f_1, f_2, 0, 0)^T$ and $f_1, f_2 \in C^0_b(\mathbb{R})$. Then the affine linear system (40) has a unique bounded solution $U_h \in C^0_b(\mathbb{H}_{h}) \cap C^1_b(\mathbb{H}_{h})$ and the operator $K_h : C^0_b(\mathbb{R}^2) \rightarrow C^0_b(\mathbb{H}_{h})$, $(f_1, f_2) \mapsto U_h$, is bounded.

**Remark.** The first and second components of (44) vanish due to our choice of variables $(u_1, u_2)$ in (7). This would not be the case using $(x_1, x_2)$ and the proof of optimal regularity results would require additional work (in this case $H_l, G_l$ are only $O(1/|k|)$ as $k \rightarrow \pm \infty$).

### 4.3. Affine equation in exponentially weighted spaces.

The problem now is to extend Lemma 4.3 to the case $(f_1, f_2) \in E^a_\beta(\mathbb{R}^2)$, with $\alpha > 0$ sufficiently close to 0. This has been done in [22] by constructing a suitable distribution space, but the following lemma gives an alternative proof (see [34]).

**Lemma 4.4.** Consider Banach spaces $\mathcal{D}, \mathcal{Y}$ and $\mathcal{X}$ such that: $\mathcal{D} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{X}$. Let $L$ be a closed linear operator in $\mathcal{X}$ of domain $\mathcal{D}$, such that the equation
\[
\frac{dU}{dt} = LU + f \tag{65}
\]
admits for any fixed $f \in C^0_b(\mathcal{Y})$ a unique solution $U = Kf$ in $C^0_b(\mathcal{D}) \cap C^1_b(\mathcal{X})$, with in addition $K \in \mathcal{L}(C^0_b(\mathcal{Y}), C^0_b(\mathcal{D}))$. Then there exists $\alpha_0 > 0$ such that if $0 \leq \alpha < \alpha_0$, for all $f \in E^a_\beta(\mathcal{Y})$ the system (65) admits a unique solution in $E^a_\beta(\mathcal{D}) \cap E^1_\beta(\mathcal{X})$ with
\[
\|U\|_{E^a_\beta(\mathcal{D})} \leq C(\alpha)\|f\|_{E^a_\beta(\mathcal{Y})}. \tag{66}
\]

**Proof.** Let $f \in E^a_\beta(\mathcal{Y})$. We set $\tilde{f}(t) = \frac{f(t)}{\cosh(\alpha t)} \in C^0_b(\mathcal{Y})$ and $\tilde{U}(t) = \frac{U(t)}{\cosh(\alpha t)}$. The property $U \in E^a_\beta(\mathcal{D}) \cap E^1_b(\mathcal{X})$ is equivalent to $\tilde{U} \in C^0_b(\mathcal{D}) \cap C^1_b(\mathcal{X})$. Furthermore, we have
\[
\frac{d\tilde{U}}{dt} = L\tilde{U} + \tilde{f} - \alpha \tanh(\alpha t)\tilde{U}.
\]
This equation is equivalent to
\[
\tilde{U} + \alpha K(\tanh(\alpha t)\tilde{U}) = K\tilde{f}. \tag{67}
\]
Equation (67) can be written
\[
(I + \alpha T)\tilde{U} = K\tilde{f},
\]
where $T\tilde{U} = K(\tanh(\alpha t)\tilde{U})$. We have then $T \in \mathcal{L}(C^0_b(\mathcal{D}))$ and $\|T\| \leq \|K\|$. If $0 \leq \alpha < \frac{1}{\|K\|}$, $I + \alpha T$ is invertible in $C^0_b(\mathcal{D})$ and we have $\|(I + \alpha T)^{-1}\| \leq \frac{1}{1 - \alpha \|K\|}$. Therefore, (67) is equivalent to
\[
\tilde{U} = (I + \alpha T)^{-1} K \tilde{f} \in C^0_b(\mathcal{D})
\]
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∥U∥_{C_0^b(D)} \leq \frac{∥K∥}{1-α∥K∥} ∥\tilde{f}∥_{C_0^b(\Omega)}. Then, we have

U = \cosh(αt) \tilde{U} ∈ E_{α}^0(D) and

∥U∥_{E_{α}^0(D)} \leq ∥\tilde{U}∥_{C_0^b(D)} \leq \frac{2∥K∥}{1-α∥K∥} ∥\tilde{f}∥_{E_{α}^0(\Omega)}. (68)

This ends the proof. ⊓⊔

Applying this result to our problem yields the following.

Proposition 4.5. There exists α_0 > 0 such that for all F = (0, f_1, f_2, 0, 0)^T with f_1, f_2 ∈ E_{α}^0(R) and α ∈ [0, α_0], the affine linear system (40) has a unique solution

Uh ∈ E_{α}^0(D) \cap E_{α}^1(H) such that for all t ∈ R and Uc ∈ C_0^b(D).

• If U : R → D solves (11) and U(t) ∈ U ∀t ∈ R then Uh(t) = ψ(Uc(t), γ, T) for all t ∈ R and Uc is a solution of

\frac{dU_c}{dt} = LU_c + PF(U_c + ψ(Uc, γ, T)). (69)

• If U_c : R → D is a solution of (69) with U_c ∈ U_c = PL \forall t ∈ R, then U = U_c + ψ(Uc, γ, T) is a solution of (11).

• The map ψ(., γ, T) commutes with R and S. Moreover, the reduced system (69) is reversible under R and equivariant under S.

5. Study of the Reduced Equation

According to normal form theory (see e.g. [21]), one can perform a polynomial change of variables \( U_c = \tilde{U}_c + \vec{P}_{γ,T}(\tilde{U}_c) \) close to the identity which simplifies the reduced Eq. (69) and preserves its symmetries. In this section, we compute this normal form at order 3 and give an explicit expression of a particular coefficient, which sign is essential for the bifurcation of small amplitude homoclinic orbits.

5.1. Normal form computation. The linear operator \( L \) restricted to the eight-dimensional subspace D_c (denoted as \( L_c \)) has the following structure in the basis \( (V_0, \vec{V}_0, V_1, \vec{V}_1, \bar{V}_0, \bar{\vec{V}}_0, \bar{V}_1, \bar{\vec{V}}_1) \):

\[
L_c = \begin{pmatrix}
  iq_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & iq_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & iq_1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & iq_2 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -iq_0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -iq_0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -iq_1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -iq_2 & 0
\end{pmatrix}.
\]
Moreover, the reversibility symmetry $R$ and the symmetry $S$ have the following structure.

One has

\[
R = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]

Moreover, if $(\tilde{T}_0, \gamma_0) \in \Gamma_{2k}$ we have

\[
S = \text{diag}(1, 1, -1, 1, 1, -1, 1)
\]

and $(\tilde{T}_0, \gamma_0) \in \Gamma_{2k+1}$ yields

\[
S = \text{diag}(-1, -1, -1, -1, -1, -1, 1).
\]

Consequently, our reduced equation has many similarities with the one considered in [22], Sect. 6 (case of a $(iq_0)^2$ resonance). The only differences are an extra pair of simple purely imaginary eigenvalues $\pm iq_1$ for the linearized operator, and the additional symmetry $S$. More precisely, the truncated normal form considered in [22] has a symmetry similar to $S$ (which follows in fact from a phase invariance), but this symmetry is broken for the full system. In our case, Theorem 4.1 ensures that the full reduced system is equivariant under $S$.

It follows that the normal form has a structure similar to the one obtained in [22].

To compute the normal form, we exclude points of $\Delta$ which are close to points where $s_0 + rq_1 + r'q_2 = 0$ for $s, r, r' \in \mathbb{Z}$ and $0 < |s| + |r| + |r'| \leq 4$ (such values correspond to strong resonances), and denote this new set as $\Delta_0$. The normal form computation is very similar to [22] (Sect. 6 and Appendix 2), to which we refer for details.

In what follows we set $\tilde{U}_c = AV_0 + B\tilde{V}_0 + CV_1 + DV_2 + \tilde{A}V_0 + \tilde{B}\tilde{V}_0 + \tilde{C}V_1 + \tilde{D}V_2$.

The normal form of (69) at order 3 is given in the following lemma.

**Lemma 5.1.** The normal form of (69) at order 3 reads

\[
\begin{align*}
\frac{dA}{dt} &= iq_0A + B + iA\mathcal{P}(u_1, u_2, u_3, u_4) \\
&\quad + O((|A| + |B| + |C| + |D|)^4), \\
\frac{dB}{dt} &= iq_0B + iB\mathcal{P}(u_1, u_2, u_3, u_4) + AS(u_1, u_2, u_3, u_4) \\
&\quad + O((|A| + |B| + |C| + |D|)^4), \\
\frac{dC}{dt} &= iq_1C + iC\mathcal{Q}(u_1, u_2, u_3, u_4) + O((|A| + |B| + |C| + |D|)^4), \\
\frac{dD}{dt} &= iq_2D + iD\mathcal{T}(u_1, u_2, u_3, u_4) + O((|A| + |B| + |C| + |D|)^4),
\end{align*}
\]

where

\[ u_1 = \tilde{A}A, u_2 = C\tilde{C}, u_3 = D\tilde{D}, u_4 = i(\tilde{A}B - \tilde{A}B) \]
and $P, S, Q, T$ are polynomials with smooth parameter dependent real coefficients, for
$(T, γ)$ in the neighborhood of $Δ_0$. We have

$$P(u_1, u_2, u_3, u_4) = p_1(γ, T) + p_2 u_1 + p_3 u_2 + p_4 u_3 + p_5 u_4,$$
$$S(u_1, u_2, u_3, u_4) = s_1(γ, T) + s_2 u_1 + s_3 u_2 + s_4 u_3 + s_5 u_4,$$
$$Q(u_1, u_2, u_3, u_4) = q_1(γ, T) + q_2 u_1 + q_3 u_2 + q_4 u_3 + q_5 u_4,$$
$$T(u_1, u_2, u_3, u_4) = t_1(γ, T) + t_2 u_1 + t_3 u_2 + t_4 u_3 + t_5 u_4,$$

where $p_1, q_1, q_2, t_1$ vanish on $Δ_0$.

The truncated normal form (obtained by neglecting terms of orders 4 and higher) is
integrable with the following first integrals:

$$A\bar{B} - \bar{A}B, |B|^2 - \int_0^{[A]^2} S(x, |C|^2, |D|^2, i(A\bar{B} - \bar{A}B))dx, |C|^2, |D|^2. \tag{72}$$

Note that if one fixes $|C| = 0$ and $|D| = 0$, the truncated normal form yields the
classical 1:1 resonance [24].

In what follows we describe some solutions of the truncated normal form. We shall
concentrate on the description of homoclinic solutions to the equilibrium 0, to a periodic
or a quasi-periodic orbit, which may exist when $L_c$ has 4 eigenvalues with nonzero real
parts (perturbation of $±iq_0$). The existence of these homoclinic orbits is linked to the
sign of the coefficient $s_2$ in the polynomial $S$. The following section is devoted to its
computation.

### 5.2. Computation of the coefficient $s_2$

We choose $(T_0, γ_0) ∈ Δ_0$. Equation (11) can be expanded as

$$\frac{dU}{dt} = L_0 U + (γ - γ_0)L(1)U + (T - T_0)L(2)U + M_2(U, U) + M_3(U, U, U) + ..., \tag{73}$$

where $L_0$ is the linear operator for $(T_0, γ_0) ∈ Δ_0$ and $L^{(i)}$ are linear operators. Moreover,
$M_j$ is a $j$-linear symmetric map satisfying

$$M_2(U, U) = aT_0^2(0, 0, u_1^2, u_2^2, 0, 0)^T, \tag{74}$$
$$M_3(U, U, U) = bT_0^2(0, 0, u_1^3, u_2^3, 0, 0)^T. \tag{75}$$

Using the Taylor expansion of the center manifold at $(0, γ_0, T_0)$ we find

$$U = AV_0 + BV_0 + CV_1 + DV_2 + ABV_0 + BCV_0 + BDV_2 + + \sum (γ - γ_0)^m(T - T_0)^n A^m B^n C^n D^n \bar{A}^m \bar{B}^n \bar{C}^n \bar{D}^n \phi^{(m,n)}_{r_0/1r_2/00/01/2}. \tag{76}$$

Using this expression and the normal form in Eq. (73), we find by identification at orders
$A^2, |A|^2, A|A|^2$ (we omit the index $(m, n) = (0, 0)$ in the notations)

$$(2iq_0 I - L)\phi_{20000000} = M_2(V_0, V_0), \nonumber$$
$$-L\phi_{10001000} = 2M_2(V_0, \bar{V}_0), \nonumber$$
$$ip_2 V_0 + s_2 \bar{V}_0 + (iq_0 I - L)\phi_{20001000} = 2M_2(\bar{V}_0, \phi_{20000000}) + 2M_2(V_0, \phi_{10001000}) + 3M_3(V_0, V_0, \bar{V}_0).$$
The first two equations have a unique solution given by expressions (45), (46). The last equation yields the following compatibility condition (expression (45) reduces the problem to a two-dimensional system)

\[(2 - \frac{q_0}{\tan(q_0/2)})s_2 = T_0^2 (6b + 8a^2 - \frac{4a^2T_0^2}{2\gamma_0T_0^2 \cos(q_0) - T_0^2(1 + 2\gamma_0) + 4q_0^2}).\]  

(77)

The other coefficients in (71) could be computed by identification in a similar way.

5.3. Description of small amplitude solutions for the normal form system. This section describes some reversible homoclinic solutions of the truncated normal form given in Lemma 5.1. The problem of their persistence for the full system is discussed in different cases.

We choose \((\gamma, T) \approx (\gamma_0, T_0)\) \((T_0, \gamma_0) \in \Delta_0\), in such a way that the linearized operator \(L\) has four symmetric eigenvalues close to \( \pm iq_0\) and having non-zero real parts \(s_1(\gamma, T) > 0\) in (71)). We shall distinguish \(S\)-invariant and non-\(S\)-invariant solutions, where \(S\) is the permutational symmetry (17). We recall that \(S\)-invariant solutions correspond to travelling waves.

5.3.1. Solutions bifurcating at \((T_0, \gamma_0) \in \Gamma_{2\square}\).

- \(S\)-invariant homoclinic solutions and persistence problems

We consider the normal form system (70) restricted to the invariant subspace \(\text{Fix}(S)\). In this case we have \(C = 0\) and recover the \((iq_0)^2(iq_2)\) resonance case as in [22]. The subspace \(\text{Fix}(S)\) contains in particular the stable and unstable manifolds of 0. Provided \(s_2(\gamma_0, T_0) < 0\) and \((\gamma, T) \approx (\gamma_0, T_0)\), the truncated normal form system admits homoclinic orbits to 0 with \(D = 0\). In addition there exist homoclinic solutions to small periodic orbits with \(D \neq 0\). These solutions are given by \((\alpha \approx 0)\),

\[
A(t) = r_0(t)e^{i(q_0t + \psi(t) + \theta)},\quad B(t) = r_1(t)e^{i(q_0t + \psi(t) + \theta)},\quad D(t) = \alpha e^{i(q_2t + \varphi_2(t) + \theta_2)},
\]

where

\[
r_0(t) = \left(\frac{2(s_1 + s_4\alpha^2)}{-s_2}\right)^{1/2} (\cosh(t(s_1 + s_4\alpha^2)^{1/2}))^{-1},
\]

\[
r_1(t) = \frac{dr_0}{dt}(t),
\]

\[
\psi(t) = (p_1 + p_4\alpha^2)t + 2\frac{p_2}{s_2}(s_1 + s_4\alpha^2)^{1/2} \tanh(t(s_1 + s_4\alpha^2)^{1/2}),
\]

\[
\varphi_2(t) = (t_1 + t_4\alpha^2)t + t_2 \int_0^t r_0^2(\tau) d\tau,
\]

and \(\theta, \theta_2 \in \mathbb{R}\).

These orbits are reversible under \(R\) if one chooses \(\theta\) and \(\theta_2\) equal to 0 or \(\pi\). In this case, the problem of their persistence for the full vector field (with additional nonresonance conditions on the eigenvalues) has been treated by Lombardi in [28]. Reversible homoclinic solutions to periodic orbits persist above a critical tail size \(\alpha = \alpha_c\), which is exponentially small with respect to \(|A(0)|\) (size of “central” oscillations). This yields exact travelling wave solutions of the Klein-Gordon system [22], which converge towards

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The first two equations have a unique solution given by expressions (45), (46). The last equation yields the following compatibility condition (expression (45) reduces the problem to a two-dimensional system)
periodic waves at infinity and have a larger amplitude at the center of the chain. On the contrary, reversible homoclinic orbits to 0 should not persist generically for the full normal form (70) when higher order terms are taken into account [28]. In what follows we explain this statement in more detail and give a brief account of persistence and nonpersistence results obtained in [28].

Consider the normal form (70) restricted to the invariant subspace \( C = 0 \). We fix \((T_0, \gamma_0) \in \Delta_0 \cap \Gamma_{2k}, \) with additional nonresonance conditions on the eigenvalues (see [28], p. 359) which are generically realized. We assume \( s_2(\gamma_0, T_0) < 0 \) and \( s_1(\gamma, T) > 0. \)

For simplicity we fix \( \gamma = \gamma_0 \) and let \( T \approx T_0 \) vary. In the linearized system, 4 hyperbolic eigenvalues have small real parts \( \pm \nu = O(1) \) with \( \nu > 0 \) (we shall use \( \nu \) instead of \( T - T_0 \) as a small parameter), \( O(1) \) imaginary parts \( \pm i\omega_0(\nu) \) \( (\omega_0(0) = q_0) \), and there is in addition one pair of \( O(1) \) purely imaginary eigenvalues \( \pm i\omega_2(\nu) \) \( (\omega_2(0) = q_2) \).

Using the following scaling (see [28], p. 364)

\[
A(t) = \sigma \nu \tilde{A}(\nu t), \quad B(t) = \sigma \nu^2 \tilde{B}(\nu t), \quad C(t) = \nu^{3/2} \tilde{C}(\nu t), \quad D(t) = \nu^{3/2} \tilde{D}(\nu t)
\]

with \( \sigma = \left( -\frac{2}{s_2} \right)^{1/2} \), the normal form (70) can be written

\[
\frac{dy}{dt} = N(Y, \nu) + R(Y, \nu),
\]

where \( Y = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})^T \). The linearized system has the eigenvalues \( \pm 1 \pm i\omega_0/\nu \) and \( \pm i\omega_2/\nu \) (a slow hyperbolic part coincides with fast oscillatory parts in this system). Moreover \( N \) is a cubic polynomial in \( Y \), \( R \) contains higher order terms in \( Y \) and is \( O(1) \) as \( \nu \to 0 \). The truncated system (with \( R = 0 \)) has explicit reversible solutions \( \pm h \) homoclinic to 0, being \( O(1) \) as \( \nu \to 0 \) thanks to the scaling (the unscaled solutions have been given above). The rescaled solution \( h \) has simple poles \( z = \pm i\pi/2 \) in the complex plane (one has \( r_0(t) = 1/\cosh t \) in the above notations).

We start with some comments on the generic nonpersistence of reversible homoclinic orbits to 0 [28]. Setting \( Y = h + y \), where the perturbation \( y \) is assumed reversible under \( R \) and homoclinic to 0, (70) can be rewritten in the form

\[
\frac{dy}{dt} = D_N(h(t), \nu) y = f(y, h(t), \nu).
\]

Applying the Fredholm alternative, one obtains a compatibility condition (linked to the eigenvalues \( \pm i\omega_2/\nu \) and reversibility) having the form

\[
\int_0^{+\infty} \langle y^*(t), f(y(t), h(t), \nu) \rangle dt = 0,
\]

where the dual vector \( y^* \) reads

\[
y^*(t) = (0, 0, -ie^{i\psi_r(t)}, 0, 0, ie^{-i\psi_r(t)}),
\]

and \( \psi_r \) has the form \( \psi_r(t) = \omega_2 t/\nu + \nu n(v) \tanh t \). This yields a condition of the type

\[
I(v) = \text{Im} \int_0^{+\infty} e^{-i\omega_2 t/\nu} g(y(t), h(t), t, \nu) dt = 0,
\]

consisting of a bi-oscillatory integral in which the approximate homoclinic solution \( h \) also rotates at the high frequency \( \omega_0/\nu \).
The usual way to check if \( I(\nu) \) vanishes is to split the integral in two parts \( I(\nu) = M(\nu) + J(\nu) \), where the Melnikov function

\[
M(\nu) = \text{Im} \int_0^{+\infty} e^{-i\omega t/\nu} g(0, h(t), t, \nu) \, dt
\]

depends on the explicitly known function \( h \) and is usually expected (at least in classical perturbation theory) to be the leading part of \( I(\nu) \). One finds as \( \nu \to 0 \) (see [28], p. 397)

\[
M(\nu) = \nu^{-3/2} e^{-c/\nu} (A_1 + O(\nu))
\]

(\( c > 0 \)), hence \( M(\nu) \) is exponentially small. However, \( M(\nu) \) is not the leading part of \( I(\nu) \) in our case. Indeed, fine estimation techniques [28] yield

\[
I(\nu) = \nu^{-3/2} e^{-c/\nu} (\Lambda + O(\nu^{1/4}))
\]

with \( \Lambda \neq A_1 \) in general. The reason is that \( h \) is the leading part of \( Y \) on \( \mathbb{R} \), but not near the poles of \( Y \) (close to \( z = \pm i\pi/2 \)), and the leading part of \( Y \) near the poles is precisely the relevant part for computing \( I(\nu) \).

More precisely, the coefficient \( \Lambda \) in (83) is given by a complex integral, involving a (not explicitly known) solution on the stable manifold of \( Y = 0 \), extended in the complex plane and approximated near the poles \( \pm i\pi/2 \) at leading order (see [29, 28]). As a consequence, analytical computations of \( \Lambda \) seem very difficult but numerical ones might be achieved. Moreover, an additional difficulty for obtaining estimate (83) has to be pointed out. Since center manifolds are not analytic in general (not even \( C^\infty \)), one cannot work with the (a priori) non-analytic reduced Eq. (70). In order to preserve analyticity, one works directly with the evolution problem (11), splitted into an infinite-dimensional hyperbolic part coupled with the normal form (70), whose principal part remains unchanged (see [28], p. 331). The same techniques as in the finite-dimensional case apply, because Lemma 4.3 and Eq. (60) give the necessary optimal regularity properties for the hyperbolic part of the linearized system (see [28], Sect. 8).

According to expression (83), if \( \Lambda \neq 0 \) (which should be satisfied except for exceptional choices of \((T_0, \gamma_0)\) and \( V \) and \( T - T_0 \) is sufficiently small, reversible homoclinic orbit to 0 close to \( \pm h \) do not exist. Consequently, reversible homoclinic orbits to 0 should not persist generally for the full normal form. This result needs several comments.

Firstly, it might happen that \( \Lambda = 0 \) for isolated values of \((T_0, \gamma_0) \in \Gamma_{2k} \). In that case, one might expect the existence of a curve \( I(T, \gamma) = 0 \) in the parameter plane (with \( (T, \gamma) \approx (T_0, \gamma_0) \)) on which the compatibility condition (81) is satisfied and reversible homoclinic orbits to 0 exist. However this situation is non-generic in the parameter plane.

Moreover, the above analysis only concerns reversible homoclinic orbits, and non-reversible homoclinic orbits to 0 might exist. In addition, homoclinic solutions are searched in a small neighborhood of \( h \) in \( L^\infty(\mathbb{R}) \), and reversible homoclinic orbits with several loops (which do not satisfy this criteria) might exist as it is mentioned in [28]. Consequently, \( \Lambda \neq 0 \) only implies the nonexistence of homoclinic orbits to 0 of a certain type when \( \nu \) is small enough.

We end with some precisions about persistence of reversible solutions homoclinic to periodic orbits. One can show [28] that for \( \nu \) small enough and \( \alpha \) in an interval of the type

\[
\alpha \in (K_1 e^{-\pi/\nu}, K_2),
\]

(84)
\( a > 0 \), Eq. (78) admits reversible solutions of the form
\[
Y_{\alpha,\nu}(t) = y(t) + h(t) + X_{\alpha,\nu}(t + \varphi \tanh(\lambda t)),
\] (85)
where \( y \) is homoclinic to 0 and \( X_{\alpha,\nu} \) is a reversible time-periodic solution of (78) with amplitude \( \alpha \). The frequency of \( X_{\alpha,\nu} \) is close to \( \omega_2/\nu \) and its principal part (in the unscaled form) has been given above (case \( A = B = C = 0 \) in the truncated normal form (70)).

Very roughly speaking, looking for a solution of the form (85) yields a compatibility condition of the type
\[
\alpha \sin \varphi = \text{Im} \int_0^{+\infty} e^{-i\omega_2t/\nu} G(y(t), h(t), t, \alpha, \nu, \varphi) \, dt,
\] (86)
which holds for a suitable choice of the phase \( \varphi = \varphi(\alpha, \nu) \) provided (84) is satisfied, due to the exponential smallness of the right side of (86) (see [28], Sect. 9.3 for more details).

**Non \( S \)-invariant solutions**

Provided \( s_2(\gamma_0, T_0) < 0 \) and \( (\gamma, T) \approx (\gamma_0, T_0) \), the truncated normal form admits homoclinic solutions to small quasi-periodic orbits, which are not invariant under \( S \) due to the additional component \( C(t) \). These solutions are given by \( (\alpha, \beta \approx 0, \beta \neq 0) \)
\[
A(t) = r_0(t)e^{i(q_0 t + \psi(t) + \theta)}, \quad B(t) = r_1(t)e^{i(q_0 t + \psi(t) + \theta_1)}, \quad C(t) = \beta e^{i(q_2 t + \psi_2(t) + \theta_2)}, \quad D(t) = \alpha e^{i(q_2 t + \psi_2(t) + \theta_2)},
\] (87)
where \( s_1 + s_4 \alpha^2 + s_3 \beta^2 \)
\[
\begin{align*}
r_0(t) &= \left( \frac{2 \tilde{s}}{s_2^2} \right)^{1/2} (\cosh(t \tilde{s}^{1/2}))^{-1}, \\
r_1(t) &= \frac{d r_0(t)}{dt}, \\
\psi(t) &= (p_1 + p_4 \alpha^2 + p_3 \beta^2) t + 2 \frac{p_3}{\tilde{s}_2} \tilde{s}^{1/2} \tanh(t \tilde{s}^{1/2}), \\
\psi_1(t) &= (\tilde{q}_1 + \tilde{q}_4 \alpha^2 + \tilde{q}_3 \beta^2) t + \tilde{q}_2 \int_0^t r_0^2(\tau) \, d\tau, \\
\psi_2(t) &= (t_1 + t_4 \alpha^2 + t_3 \beta^2) t + t_2 \int_0^t r_0^2(\tau) \, d\tau,
\end{align*}
\]
and \( \theta, \theta_1, \theta_2 \in \mathbb{R} \). This family of solutions does not include homoclinic orbits to 0, since the latter are \( S \)-invariant. These orbits are reversible under \( R \) if one chooses \( \theta, \theta_1 \) and \( \theta_2 \) equal to 0 or \( \pi \), and reversible under \( R S \) if one chooses \( \theta_1 = \pm \pi/2 \) and \( \theta, \theta_2 \) equal to 0 or \( \pi \).

The persistence of these orbits for the full vector field is still an open problem. In the reversible cases this may be analyzed using techniques developed by Lombardi [28] for the \((i q_0)^2 i q_2\) resonance (see the above paragraph on \( S \)-invariant solutions), but the extra pair of eigenvalues \( \pm i q_1 \) makes the problem more difficult.

For \( \beta \ll |A(0)| \), solutions (87) of the truncated normal form correspond to approximate solutions of the Klein-Gordon system, consisting of a travelling wave superposed on a small oscillatory mode (mainly visible at the tail).
5.3.2. Solutions bifurcating at \((T_0, \gamma_0) \in \Gamma_{2k+1}\).

- **S-invariant solutions**
  
  We consider solutions of the truncated normal form \((70)\) on the invariant subspace \(\text{Fix}(S)\). These solutions satisfy \(A = B = C = 0\). They are periodic, given by \(D(t) = \alpha e^{i\omega_1 t + t_2}\) with \(\omega^* = q_2 + t_1 + t_4 \alpha^2\). Their persistence for the full vector field (restricted to \(\text{Fix}(S)\)) follows from the Devaney-Lyapunov theorem. These solutions correspond to spatially periodic travelling waves of the Klein-Gordon system, which have been obtained in [22].

- **Non-\(S\)-invariant solutions**
  
  For \(s_2(\gamma_0, T_0) < 0\) and \((\gamma, T) \approx (\gamma_0, T_0)\), the truncated normal form admits homoclinic solutions to small quasi-periodic orbits, given by Eq. \((87)\). Their persistence for the full vector field is still an open problem. For reversible solutions this problem may be treated using the techniques developed by Lombardi [28], but in the present case an extra pair of purely imaginary eigenvalues makes the problem more difficult.

- **The existence of homoclinic orbits to \(0\) reversible under \(R\)** would be only possible with two compatibility conditions satisfied. The situation is similar to Sect. 5.3.1 \(((q_0)^2 t q_2\text{ resonance for }S\text{-invariant solutions})\), except one obtains in the present case one compatibility condition for each pair of simple purely imaginary eigenvalues. Here the linearized system has 4 hyperbolic eigenvalues with small real parts \(\pm \nu\) (we shall use \(\nu\) as a small parameter) and \(O(1)\) imaginary parts \(\pm i\omega_{0,1}(\nu)\) \((\omega_{0,1}(0) = q_j)\). Using the same notations as in Sect. 5.3.1, compatibility conditions take the form of oscillatory integrals

\[
I_1(\nu) = \text{Im} \int_0^{+\infty} e^{-i\omega t/\nu} g_1(y(t), h(t), t, \nu) \, dt = 0, \quad (88)
\]

\[
I_2(\nu) = \text{Im} \int_0^{+\infty} e^{-i\omega t/\nu} g_2(y(t), h(t), t, \nu) \, dt = 0. \quad (89)
\]

As in Sect. 5.3.1, \(h(t) + y(t)\) denotes a reversible homoclinic orbit to 0 of the rescaled reduced equation. Its principal part \(h(t)\) is explicit and given (in the unscaled form) by \((87)\) (with \(C = D = 0\) and \(\theta\) equal to 0 or \(\pi\)). The existence of homoclinic orbits to 0 reversible under \(R_1 = R S\) would imply two compatibility conditions similar to \((88)\)–\((89)\) (one has \(\theta = \pm \pi/2\) in \((87)\) and one takes the real part of the integral in \((88)\)).

Instead of homoclinic orbits to 0, we conjecture the persistence of reversible homoclinic orbits to exponentially small \(2\)-dimensional tori, originating from the two additional pairs of simple imaginary eigenvalues.

As we shall see, solutions \((87)\) of the truncated normal form correspond to approximate solutions of the Klein-Gordon system, consisting of a pulsating travelling wave with oscillations of size \(|A(0)|\) at the center.

5.3.3. Persistence result in a particular case. We consider the case when the potential \(V\) in \((1)\) is even (case \(a = 0\) in \((10)\)). Due to the additional invariance \(x_n \rightarrow -x_n\) of \((1)\), Eq. \((11)\) is also invariant under \(-S\). Fixed points of \(-S\) correspond to solutions of \((1)\) satisfying

\[x_{n+1}(\tau) = -x_n(\tau - \frac{T}{2}).\]
In this case we have $x_n(\tau) = (-1)^{n+1}x_1(\tau - \frac{(n-1)T}{2})$ and $x_1$ satisfies a simpler scalar advance-delay differential equation

$$\frac{d^2x_1(\tau)}{d\tau^2} + V'(x_1(\tau)) = -\gamma(x_1(\tau + T/2) + 2x_1(\tau) + x_1(\tau - T/2)).$$

(90)

For $(T_0, \gamma_0) \in \Gamma_{2k+1}$, the symmetry $-S$ has the following structure on the central subspace

$$-S = \text{diag}(1, 1, 1, -1, 1, 1, 1).$$

We consider the normal form (70) on the invariant subspace $\text{Fix}(-S)$, which corresponds to fixing $D = 0$. In particular, the stable and unstable manifolds of 0 are included in $\text{Fix}(-S)$.

By considering the flow on $\text{Fix}(-S)$, we recover the $(iq_0)^2(iq_1)$ resonance case treated in [28] and summarized in Sect. 5.3.1. Under non-resonance assumptions ($\frac{q_1}{q_0} \not\in \mathbb{Z}$ and $\frac{q_1}{q_0} \not\in \mathbb{N}$), reversible homoclinic solutions to periodic orbits given by (87) (with $D = 0$) persist for the full vector field above a critical tail size $\beta = \beta_c$, which is exponentially small with respect to $|A(0)|$ (size of “central” oscillations). These solutions are either reversible under $R$ (for $\theta, \theta_1$ equal to $0$ or $\pi$ in (87)) or $R_1 = RS = -R$ (for $\theta, \theta_1$ equal to $\pm \pi/2$). As we shall see in Sect. 6, these orbits yield exact travelling breather solutions of the Klein-Gordon system, superposed on an exponentially small oscillatory tail.

Homoclinic orbits to 0 reversible under $R$ do not persist for the full normal form (70) if the compatibility condition (88) (corresponding to the pair of eigenvalues $\pm iq_1$) is not satisfied [28]. A similar condition holds for reversible solutions under $-R$. Note that the compatibility condition (89) is automatically satisfied by fixing $a = 0$ in $V$, thanks to the symmetry $-S$ of (70). Indeed, the stable manifold of 0 has no $D$-component and the $D$-component of the full normal form (70) vanishes for $D = 0$, which implies the vanishing of $g_2$ in (89).

As in Sect. 5.3.1 for $S$-invariant solutions, there might be a discrete collection of curves $I_1(T, \gamma) = 0$ in the parameter plane (with $(T, \gamma) \approx \Gamma_{2k+1}$) on which the relevant compatibility condition would be satisfied and reversible homoclinic orbits to 0 would exist.

In the next section, we study the sign of the crucial normal form coefficient $s_2$ for $(\gamma_0, T_0) \in \Delta_0$ (homoclinic orbits are found for $s_2 < 0$).

### 5.4. Sign of the bifurcation coefficient $s_2$

In the following, we determine the sign of the coefficient $s_2(\gamma_0, T_0)$ as a function of the parameters $(T_0, \gamma_0) \in \Gamma_k$ and parameters $a,b$ in the potential (see (10)). We recall that the homoclinic solutions (87) exist for $s_2(\gamma_0, T_0) < 0$ and $(T_0, \gamma_0) \in \Delta_0$.

#### 5.4.1. Case of an even potential ($a = 0$)

For $a = 0$ we have

$$s_2 = \frac{3T_0^2b}{1 - \frac{q_0}{2\tan(q_0/2)}}.$$

Let us define

$$Z(q_0) = 1 - \frac{q_0}{2\tan(q_0/2)}.$$  

(91)
We have then

\[ \text{sign}(s_2) = \text{sign}(b)\text{sign}(Z(q_0)). \quad (92) \]

By Lemma 3.2, one has \( Z = 0 \) at cusp points of the bifurcation curve \( \Gamma \) (these points have been removed from the parameter set \( \Delta_0 \)). Consequently, the sign of \( s_2 \) depends on the parameter position with respect to the cusps. More precisely, \( Z < 0 \) on the right branch of \( \Gamma_k \) and \( Z > 0 \) on the left one. It follows that \( s_2 \) has the sign of \( b \) on the left branch of \( \Gamma_k \), and the sign of \( -b \) on the right branch.

As a conclusion, if the potential \( V \) is hard (\( b < 0 \)) the homoclinic solutions of the truncated normal form described above exist for parameter values near the left branch of each “tongue” \( \Gamma_k \) restricted to \( \Delta_0 \). If \( V \) is soft (\( b > 0 \)), homoclinic solutions exist for parameter values near the right branch. We sum up the situation in Fig. 2.

5.4.2. General case \((a \neq 0)\). We now consider the general case \( a \neq 0 \). We introduce the parameter \( \eta = \frac{b}{a^2} \) and recall the expression of \( s_2 \),

\[ (1 - \frac{q_0}{2\tan(q_0/2)})s_2 = T_0^2 a^2 (3\eta + 4 - \frac{2T_0^2}{2\gamma_0 T_0^2 \cos(q_0)} - T_0^2 (1 + 2\gamma_0) + 4q_0^2). \quad (93) \]

One can obtain a simpler expression for \( s_2 \). Indeed, one can prove the identity

\[ q_0^2 = T_0^2 (1 + 2\gamma_0) - 2 \cos (q_0/2) (-1)^m T_0^2 \gamma_0 \quad (94) \]

using successively

---

Fig. 2. Regions in the parameter space where small amplitude homoclinic orbits exist in the case \( a = 0 \) (even potentials)
\[ q_0^2 = -4 \frac{q_0}{\tan(q_0/2)} + T_0^2 (1 + 2\gamma_0) \] (see Eq. (31)) and

\[ \frac{2q_0}{\sin(q_0/2)} = (-1)^m T_0^2 \gamma_0 \] (see Eqs. (28)–(30)).

Identity (94) allows us to simplify the right side of (93). Indeed, we obtain by substitution

\[ 2\gamma_0 T_0^2 \cos(q_0) - T_0^2 (1 + 2\gamma_0) + 4q_0^2 = T_0^2 (3 + 6\gamma_0 + 2\gamma_0 \cos(q_0) - 8\gamma_0 (-1)^m \cos(q_0/2)), \]

which simplifies in

\[ 2\gamma_0 T_0^2 \cos(q_0) - T_0^2 (1 + 2\gamma_0) + 4q_0^2 = T_0^2 (3 + 16\gamma_0 \sin^4(\frac{q_0}{4} - \frac{m\pi}{2})). \]

Consequently, one can write \( s_2 \) in the form

\[ (1 - \frac{q_0}{2\tan(q_0/2)}) s_2 = T_0^2 a^2 (3\eta + 4 - \frac{2}{3 + 16\gamma_0 \sin^4(\frac{q_0}{4} - \frac{m\pi}{2})}) \text{ for } (\gamma_0, T_0) \in \Gamma_m. \]

We study the sign of \( s_2 \) when \((T_0, \gamma_0)\) covers the left or the right branch of the “tongue” \( \Gamma_m \). To this end, we fix \( m \geq 1 \) and introduce the subset \( \Gamma_m^l \) of \( \Gamma_m \) such that \( Z(q_0) > 0 \) (left branch) and the subset \( \Gamma_m^r \) of \( \Gamma_m \) such that \( Z(q_0) < 0 \) (right branch). Note that \((T_0, \gamma_0) \in \Gamma_m^l \) is equivalent to \( q_0 \in \tilde{q}, q_{\max} \), where \( \tilde{q} \in (2m\pi, 2(m + 1)\pi) \) denotes the point satisfying \( Z(\tilde{q}) = 0 \) (corresponding to the cusp of \( \Gamma_m \)) and \( q_{\max} \in (2m\pi, 2(m + 1)\pi) \) is obtained by fixing \( T = 0 \) in Eq. (27) or (29) (\( \gamma \) goes to infinity and \( T = 0 \) at this value of \( q_0 \)). Similarly, having \((T_0, \gamma_0) \in \Gamma_m^r \) is equivalent to fixing \( q_0 \in (2m\pi, \tilde{q}) \) (see Fig. 3).

![Fig. 3. Definition of \( \Gamma_m^l \) and \( \Gamma_m^r \)](image)
We denote by $F_m$ the quantity

$$F_m(q_0) = 4 - \frac{2}{3 + 16\gamma_0(q_0) \sin^3(\frac{q_0}{4} - \frac{m\pi}{2})},$$

and $s_2$ writes

$$\left(1 - \frac{q_0}{2\tan(q_0/2)}\right)s_2 = T_0^2a^2(3\eta + F_m(q_0)).$$

Note that $F_m$ is a strictly increasing function of $q_0$ for $q_0 \in (2m\pi, q_{\text{max}})$. Moreover, we have $F_m(\frac{2}{3}m\pi) = \frac{10}{3}$, $F_m(q_{\text{max}}) = 4$ and $\frac{10}{3} < F_m(\bar{q}) < 4$. We deduce the following results.

- **Case** $(T_0, \gamma_0) \in \Gamma^l_1 (m \geq 1)$
  In this case, we have
  $$\text{sign}(s_2) = \text{sign}(3\eta + F_m(q_0)).$$
  If $\eta > -\frac{F_m(\bar{q})}{3}$, we have $s_2 > 0$ on $\Gamma^l_1$ (this is the case in particular for $b \geq 0$). For $-\frac{4}{3} < \eta < -\frac{F_m(\bar{q})}{3}$, $s_2$ is negative only on a piece of $\Gamma^l_1$. Finally, if $\eta < -\frac{4}{3}$ then $s_2$ is negative on $\Gamma^l_1$.

- **Case** $(T_0, \gamma_0) \in \Gamma^r_1 (m \geq 1)$
  In this case, we have
  $$\text{sign}(s_2) = -\text{sign}(3\eta + F_m(q_0)).$$
  If $\eta > -\frac{10}{3}$, we have $s_2 < 0$ on $\Gamma^r_1$ (this is the case in particular for $b \geq 0$). For $-\frac{F_m(\bar{q})}{3} < \eta < -\frac{10}{3}$, $s_2$ is negative only on a piece of $\Gamma^r_1$. Finally, if $\eta < -\frac{F_m(\bar{q})}{3}$ then $s_2$ is positive on $\Gamma^r_1$.

We illustrate our analysis for the particular curve $\Gamma_1$ (the other curves yield qualitatively similar results). Figure 4 describes the sign of $s_2$ depending on $q_0$ and $\eta$. Figure 5 indicates the regions on $\Gamma_1$ where $s_2 < 0$. We recall that the homoclinic solutions (87) exist for $s_2(\gamma_0, T_0) < 0$, $(T_0, \gamma_0) \in \Delta_0$ and $(T, \gamma) \approx (T_0, \gamma_0)$ outside of the “tongue” $\Gamma_k$.

## 6. Homoclinic Solutions for the Klein-Gordon System

In this section we construct approximate (leading order) travelling breather solutions of the Klein-Gordon system with reversible homoclinic solutions of the truncated normal form. In addition we obtain exact solutions in the case of even potentials.

We choose $(\gamma, T) \approx (\gamma_0, T_0)$ ($(T_0, \gamma_0) \in \Delta_0$), in such a way that the linearized operator $L$ has four symmetric eigenvalues close to $\pm iq_0$ and having non-zero real parts. In addition we require $s_2(\gamma_0, T_0) < 0$. In this case, the truncated normal form admits different types of homoclinic solutions $(A, B, C, D)$ described in Sect. 5.3. In the sequel we restrict our attention to reversible solutions under $R$ or $R_1 = R S$, for which a persistence theory has been developed [28].

According to (76), reversible approximate solutions of (11) are given by

$$U \approx AV_0 + BV_0 + CV_1 + DV_2 + \text{c.c.},$$

(101)
where $A, B, C, D$ have the form (87). One fixes $\theta, \theta_1, \theta_2$ equal to 0 or $\pi$ if $U$ is reversible under $R$. If $(T_0, \gamma_0) \in \Gamma_{2k+1}$ and $U$ is reversible under $R_1$, one has $\theta, \theta_1 = \pm \pi/2$ and $\theta_2$ equal to 0 or $\pi$. For $(T_0, \gamma_0)$ in $\Gamma_m \cap \Delta_0$, (101) yields the approximate solutions of (7),
\[
\begin{pmatrix}
u_1(t) \\
u_2(t) 
\end{pmatrix} \approx A(t) \begin{pmatrix} (-1)^m & 1 \\
1 & 1 
\end{pmatrix} + C(t) \begin{pmatrix} -1 \\
1 
\end{pmatrix} + D(t) \begin{pmatrix} 1 \\
1 
\end{pmatrix} + \mathrm{c.c.}
\]

Coming back to the original variables (using Eq. (6)), we obtain
\[
x_\eta(\tau) \approx \left[ (-1)^m A + (-1)^n C + D \right] \left( \frac{\tau}{T} - \frac{n-1}{2} \right) + \mathrm{c.c.}
\]

(102)

As \( \xi = \frac{\tau}{T} - \frac{n-1}{2} \rightarrow \pm \infty \) one has
\[
A(\xi) \sim A_0 e^{a|\xi|} e^{i(\hat{q}_0 \xi + \phi_0)}, \quad C(\xi) \sim \beta e^{i(\hat{q}_1 \xi + \phi_1 + \delta_1)}
\]

D(\xi) \sim \alpha e^{i(\hat{q}_2 \xi + \phi_2 + \delta_2)}

with \( a > 0 \). Approximate solutions given by (102) converge for \( C, D \neq 0 \) towards quasiperiodic solutions as \( \xi \rightarrow \pm \infty \), and have larger oscillations at the center for \( \alpha, \beta \ll |A(0)| \).

Homoclinic solutions bifurcating in the neighborhood of \( \Gamma_{2m} \cap \Delta_0 \) can be seen as superpositions of a travelling wave of permanent form \( x_{TW}(\tau) = (A + D)(\frac{\tau}{T} - \frac{n-1}{2}) \) and a pulsating travelling wave \( x_{TP}(\tau) = (-1)^n (A + C)(\frac{\tau}{T} - \frac{n-1}{2}) \). If \( \beta \ll |A(0)| \), the pulsating part \( x_{TP} \) is mainly visible at the wave tail. Note that pure travelling waves (with \( C = 0, D \neq 0 \)) exist in the full system (1) \([22]\).

In addition, homoclinic solutions bifurcating in the neighborhood of \( \Gamma_{2m+1} \cap \Delta_0 \) can be seen as superpositions of a pulsating travelling wave \( x_{TP}(\tau) = (-1)^n (A + C)(\frac{\tau}{T} - \frac{n-1}{2}) \) and a travelling wave of permanent form \( x_{TW}(\tau) = D(\frac{\tau}{T} - \frac{n-1}{2}) \). For \( \alpha, \beta \ll |A(0)| \), the wave mainly consists of a \( O(|A(0)|) \) pulsating part localized at the center and a small quasiperiodic tail.

If we fix \( \gamma = \gamma_0 \) and expand (102) for \( \delta = |T - T_0| \approx 0, \alpha \approx 0, \beta \approx 0 \) we obtain for bounded values of \( \tau, n \),
\[
x_{\eta+1}(\tau) \approx \delta^{1/2} \hat{A}(\delta^{1/2} (n - v_g \tau)) e^{i(k_0 \eta - \omega_0 \tau)} + \beta e^{i(k_1 \eta - \omega_1 \tau)} + \alpha e^{i(k_2 \eta - \omega_2 \tau)} + \mathrm{c.c.}
\]

(103)

where \( k_0 = \frac{\omega_0}{v_g} - m \pi \), \( \omega_0 = \frac{\omega_1}{v_g} \), \( k_1 = \frac{\omega_1}{v_g} - \pi \), \( k_2 = \frac{\omega_1}{v_g} \), \( \omega_1 = \frac{\omega_2}{v_g} \), \( v_g = \frac{2}{T_0} \), \( \hat{A} \) has the form \( \hat{A}(\xi) = c_1(\cosh(\xi \xi))^{-1} \) and phase shifts have been included in \( c_1, \alpha, \beta \) for notational simplicity. We note that \( (\omega_1, k_1) \) satisfies the equation
\[
\omega_1^2 = 1 + 4\gamma_0 \sin^2 \frac{k_1}{2}
\]

(104)

due to the fact that \( q_1 \) satisfies the dispersion relation (19). One recognizes in Eq. (104) the usual form of the dispersion relation of Eq. (1) linearized at \( x_\eta = 0 \). Moreover Eq. (103) shows that our approximate solutions can be seen as superpositions of modulated plane waves, and one can check that \( v_g \) is the group velocity \( \omega/\langle k_0 \rangle \) (see Eqs. (28), (30)). Note that, due to condition (2) (\( p = 2 \)), only specific wave vectors \( k_1, k_2 \) in the oscillatory tail are selected among the whole set of possible ones.

Without further symmetry assumptions (evenness of \( V \), or restriction to travelling wave solutions with \( C = 0 \) as in \([22]\)), the persistence of solutions (102) for Eq. (1) is still an open problem, which should be tackled using the finite dimensional reduced system (70).

From the analysis of Sect. 5.3, we conjecture that the particular reversible solutions decaying to 0 at infinity \( C = D = 0 \) should not persist generically in the Klein-Gordon system (1). To make a more precise statement, fix \( V(x) = \frac{1}{2}x^2 - \frac{d}{3}x^3 - \frac{b}{4}x^4 \) and assume
(T, γ) close to Δ0. We conjecture that a solution of (11) reversible under R or R1, homoclinic to 0 and close to an approximate solution (101) with C = D = 0 might only exist if \( (T, γ, a, b) \) is chosen on a discrete collection of codimension-l submanifolds of \( \mathbb{R}^4 \) \((l > 0)\). The codimension depends on the number of pairs of purely imaginary eigenvalues (i.e. the number of resonant phonons) in our parameter regime and symmetry assumptions. In the present case (with two pairs of purely imaginary eigenvalues, in addition to weakly hyperbolic ones), we expect \( l = 2 \) if \( (T_0, γ_0) \in \Gamma_{2m+1} \cap Δ_0 \) (case of travelling breather solutions) and \( l = 1 \) when \( (T_0, γ_0) \in \Gamma_{2m} \cap Δ_0 \) (case of solitary wave solutions, which have the additional invariance under \( S \)). The codimension is equal to the number of compatibility conditions obtained with the normal form (70) for each type of homoclinic bifurcation (see Sect. 5.3).

Instead of solutions decaying to 0 at infinity, we conjecture for \( (T_0, γ_0) \in \Gamma_{2m+1} \cap Δ_0 \) the persistence of reversible solutions homoclinic to quasi-periodic waves (since we conjecture the persistence of reversible homoclinic orbits to 2-dimensional tori in the normal form (70)). Reversible approximate solutions (102) should constitute the principal part of travelling breather solutions of (1) superposed on a small quasi-periodic oscillatory tail.

The following theorem summarizes the above results in the case of travelling breather solutions.

**Theorem 6.1.** Assume \( a_2(γ_0, T_0) < 0 \) defined by Eq. (77) for a fixed \( (T_0, γ_0) \in Δ_0 \cap \Gamma_{2k+1} \) and consider \( (γ, T) \approx (γ_0, T_0) \) such that the linear operator \( L \) in (11) has four symmetric eigenvalues close to \( ±iq_0 \) and having non-zero real parts. Then the reduced Eq. (69) written in the normal form (70) and truncated at order 4 admits small amplitude reversible solutions (under R or R S) homoclinic to 2-tori. Such solutions should correspond to the principal part of travelling breather solutions of system (1), superposed at infinity on an oscillatory (quasiperiodic) tail, and given at leading order by the expression

\[
x_n(τ) \approx \left[ (-1)^a A + (-1)^b C + D \right] \left( \frac{τ}{T} - \frac{n - 1}{2} \right) + c.c.,
\]

where \( A, C, D \) are defined in Eq. (87) (with \( θ_2 \) equal to 0 or \( π, θ, θ_1 = ±π/2 \) for R S-reversible solutions, and \( θ, θ_1 \) equal to 0 or \( π \) for R-reversible solutions).

In addition to leading order approximate solutions, we obtain exact travelling breather solutions superposed on a small oscillatory tail in the case of even potentials. This result follows directly from the center manifold reduction theorem (Theorem 4.1) and the analysis of the reduced equation (see Sect. 5.3.3).

**Theorem 6.2.** Assume \( a_2(γ_0, T_0) < 0 \) defined by Eq. (77) for a fixed \( (T_0, γ_0) \in Δ_0 \cap \Gamma_{2k+1} \) and consider \( (γ, T) \approx (γ_0, T_0) \) such that the linear operator \( L \) in (11) has four symmetric eigenvalues close to \( ±iq_0 \) and having non-zero real parts. Moreover assume that the potential \( V \) is even.

Equation (11) is invariant under the symmetry \(-S\) defined in (17). If \( (T_0, γ_0) \) lies outside some subset of \( Δ_0 \cap Γ_{2k+1} \) having zero Lebesgue measure (corresponding to resonant cases), the full reduced Eq. (69) restricted to Fix(\(-S\)) admits small amplitude reversible solutions (under \( ±R \)) homoclinic to periodic orbits. These solutions correspond to exact travelling breather solutions of system (1) superposed at infinity on an oscillatory (periodic) tail. Their principal part is given by

\[
x_n(τ) = (-1)^a[A + C] \left( \frac{τ}{T} - \frac{n - 1}{2} \right) + c.c. + h.o.t.,
\]

where \( A, C, D \) are defined in Eq. (87) (with \( θ_2 \) equal to 0 or \( π, θ, θ_1 = ±π/2 \) for R S-reversible solutions, and \( θ, θ_1 \) equal to 0 or \( π \) for R-reversible solutions).
where $A, C$ are given by Eq. (87) (with $\theta, \theta_1 = \pm \pi/2$ for reversible solutions under $-R$, and $\theta, \theta_1$ equal to 0 or $\pi$ for reversible solutions under $R$). For a fixed value of $(\gamma, T)$ (and up to a time shift), these solutions occur in a one-parameter family parametrized by the amplitude $\beta$ of oscillations at infinity. The lower bound of these amplitudes is $O(e^{-c/\mu^2})$, where $\mu = |T - T_0| + |\gamma - \gamma_0|$, $c > 0$.

**Remark.** The lower bound of the amplitudes should be generically nonzero, but may vanish on a discrete collection of curves in the parameter plane $(T, \gamma)$. As a consequence, in a given system (1) (with fixed coupling constant $\gamma$ and symmetric on-site potential $V$), exact travelling breather solutions decaying to 0 at infinity (and satisfying (2) for $p = 2$) may exist in the small amplitude regime, for isolated values of the breather velocity $2/T$.

We conclude by comparing our findings to a previous work. The existence of modulated plane waves in Klein-Gordon chains has been studied by Remoissenet [36] using formal multiscale expansions. Under this approximation, the wave envelope satisfies the nonlinear Schrödinger (NLS) equation. In this problem a rigorous analysis of the validity of NLS equation (on large but finite time intervals) has been performed in [19].

The condition obtained by Remoissenet for the existence of NLS solitons (for the specific wave number $k = k_0 = \frac{q_0}{T_0} - m\pi$) is exactly the condition $s_2 < 0$ derived in Sect. 5.3. Indeed, the condition obtained by Remoissenet is $PQ > 0$, where

$$Q = \frac{T_0}{2q_0}(4a^2 - \frac{2a^2}{3 + 16\gamma_0 \sin^4(\frac{k_0}{2})} + 3b),$$  \hspace{1cm} (107)

$$P = \frac{\gamma_0 T_0}{2q_0}(\cos(k_0) - \frac{\gamma_0 T_0^2}{q_0^2} \sin^2(k_0)).$$  \hspace{1cm} (108)

Using the same equations as in Sect. 5.4.2 one can express $P$ and $Q$ as a function of $\gamma_0, T_0, q_0$.

$$Q = \frac{T_0}{2q_0}(4a^2 - \frac{2a^2 T_0^2}{-T_0^2(1 + 2\gamma_0) + 2\gamma_0 T_0^2 \cos(q_0) + 4q_0^2} + 3b),$$  \hspace{1cm} (109)

$$P = \frac{1}{2q_0 T_0}(-4 + \gamma_0 T_0^2 (-1)^m \cos(q_0/2)).$$  \hspace{1cm} (110)

The coefficient $P$ is $Z(q_0)$ (defined in (91)) multiplied by a negative constant (use Eq. (96)). Similarly, the expression into brackets in $Q$ is exactly the same as the one in the normal form coefficient $s_2$. Consequently, the product $PQ$ differs from $s_2$ by a negative multiplicative factor, and thus $PQ > 0$ is equivalent to $s_2 < 0$.

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References


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