Breathers on diatomic Fermi–Pasta–Ulam lattices

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Abstract

We prove the existence of breathers (spatially localized and time-periodic oscillations) in diatomic Fermi–Pasta–Ulam (FPU) chains with arbitrary mass ratio. This completes an existence result by Livi, Spicci and MacKay valid for large mass ratio. The problem is formulated as a mapping in a loop space and analyzed via a discrete spatial centre manifold reduction. Plane wave solutions of the linearized system have frequencies in a higher “optic” band or a lower “acoustic” band. For frequencies close to band edges, all small amplitude solutions of the nonlinear system lie on a finite-dimensional centre manifold, which reduces the problem locally to the study of a finite-dimensional mapping. For good parameter values, the map admits homoclinic orbits to 0 corresponding to discrete breathers. When the FPU interaction potential satisfies a hardening condition, we find breathers with frequencies slightly above the optic band, or in the gap slightly above the acoustic band. For a potential satisfying the opposite softening condition, we obtain breathers with frequencies in the gap slightly below the optic band.

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1. Introduction

A Fermi–Pasta–Ulam (FPU) chain consists in a 1-D chain of masses connected by anharmonic springs. In this paper, we consider an infinite diatomic FPU chain with two alternating different masses $m_1$, $m_2$ ($m_1 < m_2$). This system is described by the set of equations

$$m_n \frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

(1)

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where $m_{2n+1} = m_1$, $m_{2n} = m_2$, $x_n$ represents mass displacements from their reference position and $V$ denotes a smooth interaction potential with $V'(0) = 0$. For the sake of simplicity we assume $V$ analytic in the neighbourhood of 0.

This paper deals with the existence of discrete breathers (DB), i.e. time-periodic solutions of (1) with spatially localized oscillations. More precisely, a solution of (1) is called discrete breather if $x_n(t+T) = x_n(t)$ for some $T > 0$ and if there exist constants $c_± ∈ \mathbb{R}$ such that $\lim_{n → ±∞} \|x_n - c_±\|_{L^∞(0,T)} = 0$. We also require that $dx_n/dt$ is not identically 0 (this eliminates equilibria and in particular simple shifts).

Discrete breather solutions are sustained by many nonlinear lattices and appear in different physical models, concerning e.g. DNA denaturation [29], dynamical properties of crystals [33] or Josephson junction arrays [11]. We refer the reader to [12,18,30,32] for reviews on the subject.

In diatomic FPU chains, DB have been considered in connection with the vibrational dynamics of ionic crystals. Their existence has been extensively studied numerically, see e.g. [26,2,13,8,14,15]. These works explore various types of mass ratio ($m_2/m_1$ large or not) and interaction potentials (hard or soft ones). They are based on direct numerical integrations or rotating-wave approximations in which only the first frequency components are kept in the time dependence.

Mathematical results concerning DB in diatomic FPU chains have been obtained in several contexts.

On the one hand, Arioli et al. [3–5] have used variational techniques. They have considered a class of potentials $V$ superquadratic at infinity, corresponding to interaction forces anti-restoring at small amplitudes ($V''(0) < 0$) and restoring at large amplitudes ($x V'(x) > 0$ for $|x|$ large enough). In this case, the linearization of (1) at $x_n = 0$ does not possess plane wave solutions. The authors have shown that DB of period $T$ exist provided $T$ is large enough. Moreover, if one modifies the above assumptions by assuming $V''(0) = 0$ (purely restoring forces), then there exist DB of any period $T > 0$ [3]. Note that their results apply to more general lattices with $m_{n+p} = m_n$ in (1).

On the other hand, Livi, Spicci and MacKay [27] have studied the more classical situation when nearest neighbours interactions act locally as restoring forces ($κ = V''(0) > 0$). In this case, the linearization of (1) at $x_n = 0$ has plane wave solutions with frequencies $ω$ in two separated bands, namely an “optic band” ($ω^2 ∈ [2κ/m_1, 2κ(1/m_1 + 1/m_2)]$) and an “acoustic band” ($ω^2 ∈ [0, 2κ/m_2]$). Livi et al consider hard interaction potentials having the form

$$V(x) = k_2 x^2 + k_4 x^4,$$

where $k_2, k_4 > 0$. For a large mass ratio $m_2/m_1$, they prove the existence of DB with frequencies above the optic band by continuation from the limit $m_2/m_1 = +∞$. In this uncoupled limit (also called anticontinuous limit), heavy masses do not move and trivial DB can be constructed by exciting only one light mass and keeping the others at rest. Note that numerical continuation results up to $m_1 = m_2$ have been obtained in [8].

In this paper, we assume $κ > 0$ and prove the existence of small amplitude DB for arbitrary mass ratio $m_2/m_1$. Moreover, we both treat the case of hard and soft interaction potentials. When $V$ satisfies the hardening condition $(κ/2)V''(0) - (V^{(5)}(0))^2 > 0$, we find DB with frequencies slightly above the optic band, or in the gap slightly above the acoustic band. When $V$ satisfies the opposite softening condition $(κ/2)V''(0) - (V^{(5)}(0))^2 < 0$, we obtain DB with frequencies in the gap slightly below the optic band. It is interesting to note that these conditions are independent of the mass ratio $m_2/m_1$. We state these results more precisely in the following theorem.

**Theorem 1.** Fix values of $B = (κ/2)V''(0) - (V^{(5)}(0))^2$ and $m_1, m_2$ with $m_2 > m_1$. Problem (1) has the following families of small amplitude discrete breather solutions, parametrized by their frequency $ω$.

(i) For $B < 0$, $ω ≈ ω_c = (2κ/m_1)^{1/2}$ and $ω < ω_c$, there exist DB solutions $x_n^1, x_n^2$ having the symmetries

$$x_{-n}^1(t) = -x_{n-2}^1 \left(t + \frac{π}{ω}\right), \quad x_{-n}^2(t) = -x_{n}^2(t).$$
(ii) For $B > 0$, $\omega \approx \omega_c = \left[ 2\kappa (1/m_1 + 1/m_2) \right]^{1/2}$ and $\omega > \omega_c$, there exist DB solutions $x_{n}^{1}, x_{n}^{2}$ having the symmetries

$$x_{n}^{1}(t) = -x_{n-2}^{1} \left( t + \frac{\pi}{\omega} \right), \quad x_{n}^{2}(t) = -x_{n}^{2} \left( t + \frac{\pi}{\omega} \right).$$

(iii) If $B > 0$, $m_2/m_1 \in (k^2, k(k + 2))$ (for some integer $k \geq 1$), $\omega \approx \omega_c = (2\kappa/m_1)^{1/2}$ and $\omega > \omega_c$, there exist DB solutions $x_{n}^{1}, x_{n}^{2}$ having the symmetries

$$x_{n}^{1}(t) = -x_{n-2}^{1}(t), \quad x_{n}^{2}(t) = -x_{n}^{2} \left( t + \frac{\pi}{\omega} \right).$$

In each case, these solutions have the form

$$x_{n}^{i}(t) = d_n + X_{n}^{i}(t)$$

where $X_{n}^{i}$ has 0 time-average and $\|X_{n}^{i}\|_{L^\infty}$ decays exponentially as $n \to \pm \infty$. The stationary term $d_n$ satisfies $d_n = O(\omega - \omega_c)$ for any fixed $n$, $\lim_{n \to \pm \infty} d_n = O(\omega - \omega_c)^{1/2}$ and has a kink shape if $V^{(3)}(0) \neq 0$. The oscillatory parts $X_{n}^{i}$ have the following form. In case (i)

$$X_{2n}^{i}(t) = O(\omega - \omega_c), \quad X_{2n-1}^{i}(t) = A_n \cos(\omega t) + O(\omega - \omega_c),$$

in case (ii)

$$X_{2n}^{i}(t) = -\frac{m_1}{m_2} A_n \cos(\omega t) + O(\omega - \omega_c), \quad X_{2n-1}^{i}(t) = A_n \cos(\omega t) + O(\omega - \omega_c),$$

in case (iii)

$$X_{2n}^{i}(t) = A_n \cos(\omega t) + O(\omega - \omega_c), \quad X_{2n-1}^{i}(t) = O(\omega - \omega_c),$$

where $0 < A_n \leq C |\omega - \omega_c|^{1/2}v^{\nu}, v = 1 + O(\omega - \omega_c)^{1/2} > 1$.

Note that an additional condition on $m_2/m_1$ is required in the case when $\omega$ lies slightly above the acoustic band (case (iii)). This condition can be interpreted as a non-resonance condition with plane waves. More precisely, if this condition is not fulfilled (i.e. if $m_2/m_1 \in [k(k + 2), (k + 1)^2], k \geq 1$) and $\omega$ is sufficiently close to $(2\kappa/m_2)^{1/2}$, then $(k + 1)\omega$ belongs to the optic band.

Our proof also provides explicit expressions of DB solutions at leading order in $\omega - \omega_c$. Their oscillatory parts decay exponentially at infinity and can be viewed as spatial modulations of linear standing waves with frequencies $\omega = \omega_c$. Oscillatory parts are $O(|\omega - \omega_c|^{1/2})$ and superpose to $O(|\omega - \omega_c|)$ static distortions of the lattice. As a consequence, we can describe the geometry of the different breathers obtained in Theorem 1. In case (i), light masses are out of phase and have larger displacements than heavy masses (see Figs. 4 and 5). In case (ii), light masses have larger displacements than heavy masses and nearest neighbours are out of phase (see Figs. 2 and 3). In case (iii), heavy masses are out of phase and have larger displacements than light masses (see Figs. 6 and 7). These results are consistent with the numerical works previously mentioned.

Note that Theorem 1 provides a condition ($B \neq 0$) valid for general potentials for the existence of small amplitude breathers while previous numerical studies were only valid for particular potentials. The existence conditions $B > 0$ and $B < 0$ have a simple interpretation. The potential $V$ is hard (resp. soft) around 0 if and only if $V^{(4)}(0) - 5/3(V^{(3)}(0))^2 > 0$ (resp. $< 0$). Thus if $B > 0$ then $V$ is a hard potential and there exist breathers with frequencies slightly above the acoustic and optic bands. On the other hand, if $V$ is a soft potential then $B < 0$ and there exist breathers with frequencies slightly below the optic band (this is the case for Lennard–Jones, Morse or Born–Mayer–Coulomb potentials used in [26,14,15]).
The proof of Theorem 1 is based on the fact that one can rewrite (1) as an ill-posed recurrence relation on a loop space and analyze small amplitude solutions using an adapted centre manifold reduction theorem. This method was initially introduced for studying discrete breathers in monoatomic FPU lattices [23] and has been extended to more general discrete systems in [24].

To be more precise, \( V' \) is locally invertible (\( V''(0) > 0 \)) and thus one can reformulate (1) using the rescaled force variable \( f_n(t) = V'(x_n - x_{n-1})(t/\omega) \). This yields the equations

\[
\omega^2 m_1 m_2 \frac{d^2}{dt^2} (V')^{-1}(f_n) = m_{n-1} f_{n+1} - (m_1 + m_2)f_n + m_n f_{n-1}, \quad n \in \mathbb{Z}
\]

(recall \( m_{2n+1} = m_1, m_{2n} = m_2 \)), where \( \omega \) is viewed as a bifurcation parameter. Setting \( Y_n = (f_{2n}, f_{2n-1}) \), problem (5) can be rewritten as a mapping on a loop space

\[
Y_{n+1} = F_\omega(Y_n) \quad \text{in} \quad \mathbb{X},
\]

where Eq. (6) holds in a Sobolev space \( \mathbb{X} \) of \( 2\pi \)-periodic functions of \( t \) and \( Y_n \) belongs to the domain \( \mathbb{D} \subset \mathbb{X} \) of \( F_\omega \) for all \( n \in \mathbb{Z} \). The map \( F_\omega \) is reversible due to the invariance \( f_n \rightarrow f_{-n-1} \) in (5). It has a fixed point \( Y = 0 \) corresponding to the lattice at rest and the linearized operator \( L_\omega = D F_\omega(0) \) is unbounded in \( \mathbb{X} \) (hence the recurrence relation (6) is ill-posed). Discrete breathers correspond to solutions of (6) homoclinic to 0, i.e. satisfying \( \lim_{n \rightarrow \pm \infty} \| Y_n \|_\mathbb{D} = 0 \).

When \( \omega \) takes one of the critical values \( \omega_c \) given in Theorem 1, \( L_\omega \) has a double eigenvalue \( +1 \) or \( -1 \) and the remaining part of the spectrum lies away from the unit circle. This spectral separation allows us to apply a centre manifold theorem ([24]), which ensures that all small amplitude solutions of (6) with \( \omega \approx \omega_c \) lie on a 2-D centre manifold invariant under \( F_\omega \). This property reduces the local study of (6) to that of a 2-D reversible mapping, which admits homoclinic orbits to 0 for good parameter values. These orbits correspond to DB solutions of (6).

Note that the centre manifold theorem used in the present paper is a discrete analogue of centre manifold theorems for ill-posed differential equations in Banach spaces ([25,28,35]). These techniques have been used recently for studying travelling waves in different types of infinite 1-D lattices (Iooss and Kirchgässner [22], Iooss [21]). These problems take the form of scalar advance-delay reversible differential equations (written in the moving frame), where the continuous space coordinate plays the role of time. Note that other techniques have been used for studying different kinds of travelling waves in FPU lattices (see [16,17,20] and references therein).

In the present case, when \( V''(0) > 0 \) (and with additional convexity assumptions), variational techniques should also work in principle and allow us to find finite amplitude discrete breathers (see [7] in the case of monoatomic FPU chains). In comparison, centre manifold theory is only local but provides more precise results concerning the shape of solutions, and works for general potentials.

Another advantage of centre manifold theory is that it does not fix a priori the spatial behaviour of solutions, since all small amplitude solutions with near-critical frequencies lie on a centre manifold. As a result, our analysis is not limited to DB and provides other types of time-periodic solutions. For good parameter values, we prove the existence of “dark breathers” (see [1] and references therein), which connect at infinity two spatially periodic solutions and have a smaller amplitude at the centre (these solutions are not spatially localized). Although this point is not treated in the present paper, a more extended analysis of the reduced mapping would also provide spatially periodic and quasi-periodic solutions.

An interesting generalization of this work would be to analyze case (iii) of Theorem 1 when \( m_2/m_1 \) belongs to the “forbidden” bands \( \{k(k+2), (k+1)^2\} (k \geq 1) \). In this case, the spectrum of \( L_{\omega_c} \) on the unit circle consists in a double eigenvalue \(-1\) and a pair of simple imaginary eigenvalues (the centre manifold is 4-D). It would be interesting to know if there still exist DB in this parameter regime. A possibility would be that DB do not exist in this case and are replaced by “nanopterons” having small oscillatory tails.

The outline of the paper is as follows. In Section 2, we formulate (1) as a mapping in an adapted function space, and we determine the spectrum of the linearized operator in Section 3 (depending on the parameters \( \omega, m_1/m_2 \)). Next we perform a centre manifold reduction when the frequency of solutions is close to one of the critical frequencies.
given in Theorem 1. This is done in Section 4, where we compute the (2-D) centre manifolds and the reversible reduced mappings in normal form (some computations are detailed in Appendix B). The reduction is an application of the general result [24], except for the treatment of reversibility. In our case, the reversibility symmetry is an unbounded operator in $D$ and thus we have to redo a part of the proof (Section 4.2 and Appendix A). We study small amplitude bifurcating orbits in Section 5 (reversibility is important at this stage). We describe the corresponding lattice vibrations in Section 6 and compare our results with previous numerical works.

2. Formulation of the mathematical problem

In this section, we formulate the FPU system for time-periodic solutions as a mapping in an adapted function space.

We first rescale Eq. (1) for decreasing the number of parameters. Setting $x_n(t) = \tilde{x}_n((V''(0)/m_2)^{1/2} t)$, $V = V''(0)\tilde{V}$ in (1) and then dropping the tildes for convenience, we get

$$\frac{d^2}{dt^2} x_{2n} = V'(x_{2n+1} - x_{2n}) - V'(x_2 - x_{2n-1}), \quad m \frac{d^2}{dt^2} x_{2n+1} = V'(x_{2n+2} - x_{2n+1}) - V'(x_{2n+1} - x_{2n}). \quad (7)$$

where $m = m_1/m_2 \in (0, 1)$, $V'(0) = 0$ and $V''(0) = 1$. We look for solutions of (7) with frequency $\omega$. To cut off the invariance of (7) under translations, we use the rescaled force variable $y_n = V'(x_n - x_{n-1}) t/\omega$. Note that $y_n$ is $2\pi$-periodic in time. By integrating (7) one observes that the time average of $y_n$ is independent of $n$. Since we are interested in spatially localized solutions, we fix $\int_0^{2\pi} y_n(t) \, dt = 0$ in the sequel.

Using variables $y_n$, problem (7) leads to the new system:

$$\omega^2 \frac{d^2}{dt^2} (W(y_{2n+1})) = y_{2n+2} - (m + 1)y_{2n+1} + m y_{2n},$$

$$m\omega^2 \frac{d^2}{dt^2} (W(y_{2n})) = m y_{2n+1} - (m + 1)y_{2n} + y_{2n-1}, \quad (8)$$

where $W(y) = (V')^{-1}(y)$ is the local inverse of $V'$ and satisfies $W(0) = 0$, $W'(0) = 1$.

Breather solutions of (7) with frequency $\omega$ correspond to $2\pi$-periodic solutions of (8) satisfying

$$\lim_{|n| \to \infty} \|y_n\|_{L^\infty} = 0.$$

We now formulate (8) as a first order recurrence relation in a loop space. For this purpose, we define $u_n = y_{2n}$, $v_n = y_{2n-1}$ and $Y_n = (u_n, v_n)$. Problem (8) can be rewritten

$$Y_{n+1} = F_{m,\omega}(Y_n) \quad (9)$$

with

$$F_{m,\omega}(u, v) = \begin{pmatrix} D \left( \frac{1}{m} (D(u) - v) \right) - mu \\ \frac{1}{m} (D(u) - v) \end{pmatrix} \quad (10)$$

and $D(u) = m\omega^2 (d^2/dt^2) W(u) + (m + 1) u$. Now $(m, \omega)$ play the role of bifurcation parameters.

We now define appropriate function spaces on which $F_{m,\omega}$ is acting. We restrict our attention to even functions of $t$ because this divides by two the dimension of the centre manifold. We introduce the spaces $H^a_p$ ($n \geq 0$) defined by

$$H^a_p = \left\{ y \in H^a(\mathbb{R}/2\pi\mathbb{Z}) / y \text{ even}, \int_0^{2\pi} y \, dt = 0 \right\}.$$
where $H^n$ denotes the classical Sobolev space (with $H^0(\mathbb{R}/2\pi\mathbb{Z}) = L^2(\mathbb{R}/2\pi\mathbb{Z})$). We look for $(u_n, v_n) \in \mathbb{D} = H^4_k \times H^2_l$. The recurrence relation (9) holds in $\mathbb{X} = H^4_k \times H^2_l$. The operator $F_{m, \omega} : \mathbb{D} \to \mathbb{X}$ is analytic in a neighbourhood $\mathcal{U}$ of $Y = 0 \in \mathbb{D}$. Note that the fixed point $Y = 0$ of $F_{m, \omega}$ corresponds to the lattice at rest.

We now examine the symmetry properties of Eq. (9). On the one hand, Eq. (9) is invariant under the symmetry $TY = Y(\cdot + \pi)$. Moreover, if $y_n$ is a solution of (8) then $\tilde{y}_n = y_{n-1}$ also satisfies (8). As a consequence, if $(u_n, v_n)$ is a solution of (9) then $(v_{n-1}, u_{n-1})$ is also a solution, i.e. Eq. (9) is reversible with respect to the symmetry $R(u, v) = (v, u)$. Note that this statement is only formal at the present stage since $R$ does not map $\mathbb{D}$ into itself (we shall make this point rigorous in Section 4.2). Reversibility is characterized by the property that for all $Y \in \mathcal{U}$ such that $RY \in \mathcal{U}$ and $RF_{m, \omega}(RY) \in \mathcal{U}$, one has $(F_{m, \omega} \circ R)^2 Y = Y$.

3. Spectral properties of the linearized operator

The linearized operator $L_{m, \omega} = DF_{m, \omega}(0)$ reads

$$L_{m, \omega}(u, v) = \begin{pmatrix}
A \left( \frac{1}{m}(Au - v) \right) - mu \\
\frac{1}{m}(Au - v)
\end{pmatrix},$$

where

$$Au = m\omega^2 \frac{d^2 u}{dt^2} + (m + 1)u. \tag{11}$$

The operator $L_{m, \omega} : \mathbb{D} \subset \mathbb{X} \to \mathbb{X}$ is unbounded in $\mathbb{X}$ (of domain $\mathbb{D}$) and closed. Its spectrum is invariant under $z \to z^{-1}$ (due to reversibility) and $z \to \bar{z}$. The following lemma states the spectral properties of $L_{m, \omega}$ in more detail.

**Lemma 1.** For all $m \in (0, 1)$ and $\omega > 0$, the spectrum of $L_{m, \omega}$ is unbounded, discrete and can be written $\sigma(L_{m, \omega}) = \{0\} \cup \Sigma_{m, \omega}$, where $0$ belongs to the essential spectrum and $\Sigma_{m, \omega}$ consists in non-zero eigenvalues. The set $\Sigma_{m, \omega}$ is contained in the union of the real axis and the unit circle, and invariant under $z \to \bar{z}$, $z \to z^{-1}$. The eigenvalues form sequences $(z_k)_{k \geq 1}$ and $(z_k^{-1})_{k \geq 1}$ (with $|z_k| \geq 1$ and $\text{Im} z_k \geq 0$) determined by the equation

$$\frac{1}{2}(z_k + z_k^{-1}) = 1 + k^2 \omega^2 (\frac{1}{2}mk^2 \omega^2 - m - 1), \quad k \geq 1. \tag{12}$$

**Proof.** For $z \in \mathbb{C}$ and $(f, g) \in \mathbb{X}$ we consider the spectral problem

$$(L_{m, \omega} - zI)(u, v) = (f, g), \quad (u, v) \in \mathbb{D}, \tag{13}$$

which reads

$$A \left( \frac{1}{m}(Au - v) \right) - (m + z)u = f, \quad \frac{1}{m}(Au - v) - zv = g.$$

The system is simplified by setting $w = (1/m)(Au - v)$ (the operator $(u, v) \mapsto (u, w)$ is bounded and invertible in $\mathbb{D}$). This yields

$$Aw - (m + z)u = f, \quad (1 + mz)w - zAu = g. \tag{14}$$

We first examine the case when $z = 0$. One obtains

$$mu = Ag - f, \quad w = g,$$

hence, $u \notin H^4_k$ and thus $z = 0$ belongs to the essential spectrum.
We now consider the case when $z \neq 0$. Problem (14) can be written

$$Aw - (m + z)u = f,$$

$$-Au + (m + z^{-1})w = z^{-1}g.$$  \hspace{1cm} (15) \hspace{1cm} (16)

For $z \neq -m^{-1}$, Eq. (16) can be solved with respect to $w$, which yields

$$A^2u - (m + z)(m + z^{-1})u = (m + z^{-1})f - z^{-1}Ag,$$

$$w = \frac{1}{1 + mz}(zAu + g).$$  \hspace{1cm} (17) \hspace{1cm} (18)

We now solve Eq. (17). The operator $A^2$ is unbounded in $H^0_γ$ (of domain $H^2_γ$), closed with a compact resolvent. It follows that (17) has a unique solution $(u, w) \in \mathbb{D}$ if and only if $(m + z)(m + z^{-1})$ is not an eigenvalue of $A^2$. Moreover, if $(m + z)(m + z^{-1})$ is an eigenvalue of $A^2$ then $z$ is an eigenvalue of $L_{m, γ}$. This occurs when $z$ satisfies the dispersion relation

$$(m + z)(m + z^{-1}) = (m + 1 - mk^2\omega^2)^2, \quad k \in \mathbb{N}^*$$  \hspace{1cm} (19)

which can be written

$$\frac{1}{z}(z + z^{-1}) = 1 + k^2\omega^2(\frac{1}{k^2} - m - 1).$$  \hspace{1cm} (20)

There remains to investigate the case when $z = -m^{-1}$. In this case, the system (15)–(16) reads

$$Aw = f + (m - m^{-1})u,$$  \hspace{1cm} (21)

$$Au = mg.$$  \hspace{1cm} (22)

Note that $A : H^0_γ \rightarrow H^{n-2}_γ$ is invertible if and only if $(m + 1)/m\omega^2 \neq k^2$ for all $k \in \mathbb{N}^*$ (see (11) for the definition of $A$). When this is the case, (21)–(22) has a unique solution $(u, w) \in \mathbb{D}$ and $z = -m^{-1}$ belongs to the resolvent set of $L_{m, γ}$. Lastly, we consider the case when $(m + 1)/m\omega^2 = k^2 (k \in \mathbb{N}^*)$. In this case, one has $Au = (m + 1)/k^2(d^2u/dr^2) + k^2u$. Eqs. (21) and (22) yield the solvability conditions

$$\int_0^{2\pi} u \cos(kt) \, dt = \frac{m}{1 - m} \int_0^{2\pi} f \cos(kt) \, dt,$$  \hspace{1cm} (23)

$$\int_0^{2\pi} g \cos(kt) \, dt = 0.$$  \hspace{1cm} (24)

When condition (24) is satisfied, Eqs. (22) and (23) determine $u \in H^2_γ$ uniquely. Then Eq. (21) has an infinite number of solutions $w \in H^2_γ$ defined up to an additive term $C \cos(kt)$. This shows that $z = -m^{-1}$ is an eigenvalue of $L_{m, γ}$ when $(m + 1)/m\omega^2 = k^2 (k \in \mathbb{N}^*)$. For these parameter values, we note that $z = -m^{-1}$ satisfies the dispersion relation (19) (or equivalently (20)).

It follows from the above analysis that $L_{m, γ}$ has essential spectrum at $z = 0$, while its eigenvalues are determined by the dispersion relation (20). Eq. (20) shows that $z + z^{-1} \in \mathbb{R}$ for all eigenvalue $z$, hence the spectrum of $L_{m, γ}$ is contained in the union of the real axis and the unit circle. Moreover, Eq. (20) shows that the spectrum is invariant under $z \rightarrow \bar{z}, z \rightarrow z^{-1}$ and is unbounded (one of the solutions of (20) tends to infinity as $k \rightarrow +\infty$). This completes the proof.
We now study the variations of the spectrum of $L_{m,\omega}$ as we vary the parameter $(\omega^2, m) \in S$, $S = (0, +\infty) \times (0, 1)$. Eq. (12) can be written

$$z_k^2 - 2 \left(1 + k^2 \omega^2 \left(\frac{m}{2} k^2 \omega^2 - m - 1\right)\right) z_k + 1 = 0.$$  \tag{25}

The discriminant of Eq. (25) is negative if and only if

$$-2 \leq k^2 \omega^2 \left(\frac{m}{2} k^2 \omega^2 - m - 1\right) \leq 0,$$

or equivalently

$$\frac{2}{m} \leq k^2 \omega^2 \leq 2 \left(1 + \frac{1}{m}\right) \quad \text{or} \quad 0 < k^2 \omega^2 \leq 2.$$

In this case, the eigenvalues $z_k, z_k^{-1}$ belong to the unit circle. Note that the intervals $[0, 2]$ and $[2/m, 2(1 + (1/m))]$ are often referred as acoustic and optic band respectively.

The eigenvalues $z_k, z_k^{-1}$ belong to the positive real axis when $k^2 \omega^2 \geq 2(1 + (1/m))$ and collide at $z = +1$ when $k^2 \omega^2 = 2(1 + (1/m))$. Note that all the eigenvalues belong to the positive real axis (and lie outside the unit circle) when $\omega^2 > 2(1 + (1/m))$.

The eigenvalues $z_k, z_k^{-1}$ belong to the negative real axis when $2 \leq k^2 \omega^2 \leq 2/m$ and collide at $z = -1$ when $k^2 \omega^2 = 2/m$ and $k^2 \omega^2 = 2$. One can see that $z_k$ does not vary monotonically for $\omega^2 \in [2/k^2, 2/k^2 m]$, and reaches its minimum $z_k = -1/m$ at the mid value $\omega^2 = (m + 1)/k^2 m$. This frequency corresponds to the case when $A$ is non-invertible.

The above analysis leads us to consider the following curves in the parameter space $S$

$$\Gamma^+_k: \omega^2 = \frac{2}{k^2} \left(1 + \frac{1}{m}\right), \quad \Gamma^-_k: \omega^2 = \frac{2}{k^2 m}, \quad \Gamma^u_k: \omega^2 = \frac{2}{k^2}.$$

The infinite collection of curves $\Gamma^+_k, \Gamma^-_k, \Gamma^u_k (k \geq 1)$ divides $S$ in different regions corresponding to different numbers of eigenvalues on the unit circle (these curves are depicted in Fig. 1 for $k = 1, 2$).

Note that in our parameter space the anticontinuous limit developed in [27] consists in following a curve $\omega^2 = \Omega^2 m$, where $\Omega > \sqrt{2}$ is fixed and $m \to 0$. This is due to the fact that the system (1) is rescaled differently in this reference (the FPU chain consists in masses 1 and $M = m_2/m_1 > 1$).

The set of possible bifurcation scenarios is broad. With the aim of finding discrete breathers as homoclinic orbits to $Y = 0$, we shall restrict ourselves to the situation when $z_1, z_1^{-1}$ are the only eigenvalues close to the unit circle and the fixed point $Y = 0$ is hyperbolic. The corresponding regions in the parameter space $S$ are depicted in Fig. 1 (shaded regions). One of these regions is located in the neighbourhood of $\Gamma^+_1$, at the right side (i.e. above the top of the optic band). An other region is in the neighbourhood of $\Gamma^-_1$, at the left side (i.e. below the bottom of the optic band). The situation is more complicated when $(\omega^2, m)$ is close to $\Gamma^u_1$ and at the right side (i.e. above the top of the acoustic band), since the curves $\Gamma^+_k$ and $\Gamma^-_k (k \geq 2)$ intersect at the point $(\omega^2, m) = (2, M_k^\pm)$ with

$$M_k^+ = \frac{1}{k^2 - 1}, \quad M_k^- = \frac{1}{k^2}.$$  \tag{26}

We shall only consider the case when $m \in (M_k^+, M_k^-) (k \geq 1)$ is fixed and $\omega^2 \approx 2$, since $z_1, z_1^{-1}$ are the only eigenvalues close to the unit circle. In the case when $m \in (M_k^-, M_k^+) (k \geq 2)$ is fixed and $\omega^2 \approx 2$, there are two additional eigenvalues $z_k, z_k^{-1}$ on the unit circle and the bifurcation problem is more complicated. Note that these
Fig. 1. Spectrum of $L_{m,ω}$ near the unit circle in some parameter regions. In the shaded regions, $z_1, z_1^{-1}$ are the only eigenvalues close to the unit circle and the fixed point $F = 0$ is hyperbolic.

eigenvalues indicate that $k\omega$ belongs to the optic band, and consequently resonance phenomena may occur in these parameter regimes. In the sequel, we shall consider the following line segments in the parameter space $S$

$$I_k^\pm : \omega^2 = 2, \quad m \in (M^{-}_{k+1}, M^+_{k}).$$

We note that Eq. (8) has the invariance $(y_n, ω) \rightarrow (y_n(kt), ω/k)$ (for any integer $k \geq 2$), and consequently the existence of a given solution $y_n$ for $(ω^2, m) \approx Γ_{1}^{\pm, a}$ implies the existence of an other solution $\tilde{y}_n = y_n(kt)$ for $(ω^2, m) \approx Γ_{k}^{\pm, a}$. This new solution is artificial since $y_n$ and $\tilde{y}_n$ both correspond to the same solution of (7). However, this remark would be useful for studying bifurcations near $Γ_{k}^{\pm, a}$ involving several modes.

Although we shall not investigate these cases in the present paper, we note the presence of simultaneous or multiple collisions of eigenvalues for particular values of $(ω^2, m)$ (codimension two bifurcations). This phenomenon may yield the existence of new interesting types of solutions when the mass ratio is close to such critical values. For $(ω^2, m) = (2, M^+_k)$, the central part of the spectrum is composed of two double eigenvalues $z_1 = -1, z_k = +1$, and for $(ω^2, m) = (2, M^-_k)$ it is composed of one quadruple eigenvalue $-1$ ($z_1^{±1} = z_k^{±1} = ±1$). Moreover, the curves $Γ^{-}_k$ and $Γ^{+}_{k+1} (k \geq 3)$ cross at $(ω^2, m) = (2/(2k + 1), m_k)$, with $m_k = (2k + 1)/k^2$ (then $z_k = -1, z_{k+1} = +1$ are double eigenvalues, and a finite number of other eigenvalues lie on the unit circle). One can even observe a triple crossing of $Γ^{-}_p, Γ^{+}_{(p^2-1)/2}$ and $Γ^{p}_{(p^2+1)/2}$ when $p \geq 3$ is odd (then $z = -1, z = +1$ are, respectively, quadruple and double eigenvalues, and a finite number of other eigenvalues lie on the unit circle).

Note also that a study of the codimension two bifurcation $(ω^2, m) \approx (2, 1)$ should be of interest. Indeed, we shall see in Section 6 that small amplitude breathers exist for $(ω^2, m) \approx (2, 1)$ if one chooses good parameter values in the domain between $Γ^{+}_1$ and $Γ^{-}_1$. This domain vanishes as $m \rightarrow 1$, and for $m = 1$ the limiting frequency $ω = \sqrt{2}$ lies inside the acoustic band.

The following lemma describes the centre space $\mathcal{Z}_c$ (invariant subspace under $L_{m,ω}$ associated with the eigenvalues on the unit circle) when parameters lie on the curves $Γ^{+}_1, Γ^{-}_1, Γ^k_1$. 

\[132\]
Lemma 2. If \((\omega^2, m) \in \Gamma_1^+, \varepsilon = +1\) is a double non-semi-simple eigenvalue of \(L_{m, \omega}\). If \((\omega^2, m) \in \Gamma_1^-\) or \((\omega^2, m) \in \Gamma_1^k\) \((k \geq 1)\), \(\varepsilon = -1\) is a double non-semi-simple eigenvalue of \(L_{m, \omega}\). Moreover, the centre space \(\mathcal{X}_c\) is the same for all these parameter values and is spanned by the vectors \(V_u = (\cos t, 0)\) and \(V_v = (0, \cos t)\). The spectral projection \(\Pi_c\) on \(\mathcal{X}_c\) reads \(\Pi_c(u, v) = (\pi_c u, \pi_c v)\), where \(\pi_c u = (1/\pi) \int_0^{2\pi} u(t) \cos t \, dt\).

Proof. The operators \(L_{m, \omega}\) and \(\Pi_c\) commute since \(\pi_c\) commutes with \(d^2/dt^2\). We now define \(\mathcal{X}_c = \Pi_c \mathcal{X} = \text{Span} \{V_u, V_v\}\), \(\mathcal{X}_h = L_{m, \omega}\mathcal{X}_c\), \(\Pi_h = I - \Pi_c\), \(\mathcal{X}_h = \Pi_h \mathcal{X}\), \(\mathbb{D}_h = \Pi_h \mathbb{D}\), \(L_c = L_{m, \omega}|_{\mathcal{X}_c}\), and \(L_h = L_{m, \omega}|_{\mathcal{X}_h}\). Going back to Lemma 1 and replacing \(\mathcal{X}, \mathbb{D}\) by \(\mathcal{X}_h, \mathbb{D}_h\), one obtains \(\sigma(L_h) = \sigma(L_{m, \omega}) \setminus \{1\}\) for \((\omega^2, m) \in \Gamma_1^+\) and \(\sigma(L_h) = \sigma(L_{m, \omega}) \setminus \{-1\}\) for \((\omega^2, m) \in \Gamma_1^-\) or \((\omega^2, m) \in \Gamma_1^k\). Moreover, \(\sigma(L_h)\) is purely hyperbolic. Now we have in the basis \([V_u, V_v]\)

\[
L_c = \begin{pmatrix} \gamma & -\beta \\ \beta & -m^{-1} \end{pmatrix},
\]

where \(\beta = 1 + m^{-1} - \omega^2\) and \(\gamma = m (\beta^2 - 1)\). Consequently, \(\varepsilon = -1\) (respectively, \(\varepsilon = +1\)) is a double non-semi-simple eigenvalue of \(L_c\) for \((\omega^2, m) \in \Gamma_1^-\) or \((\omega^2, m) \in \Gamma_1^k\) (respectively, \((\omega^2, m) \in \Gamma_1^+\)).

4. Centre manifold reduction

In this section, we locally reduce the infinite-dimensional mapping (9) to a finite-dimensional one on a centre manifold, for parameter values close to the bifurcation curves \(\Gamma_1^k\) \((k \geq 1)\), \(\Gamma_1^+, \Gamma_1^-\). Section 4.1 states the reduction theorem, and we show in Section 4.2 that the reduced mapping inherits the symmetries of the infinite-dimensional system. At this step one has to deal with a technical difficulty, due to the fact that the reversibility symmetry is an unbounded operator in \(\mathbb{D}\). In Section 4.3, we compute the principal terms in the Taylor expansions of the centre manifold and the reduced mapping, which is written in a simpler normal form in Section 4.4.

4.1. Reduction theorem

We now consider the bifurcations at a double eigenvalue \(\pm 1\) occuring on the curves \(\Gamma_1^k\) \((k \geq 1)\), \(\Gamma_1^+, \Gamma_1^-\). We set \(\omega^2 = \omega_c^2 + \mu\), where \(\mu \approx 0\) is a small bifurcation parameter and \(\omega_c^2, m\) are as follows

\[
\text{for } (\omega_c^2, m) \in \Gamma_1^+: m \in (0, 1), \quad \omega_c^2 = 2 \left(1 + \frac{1}{m}\right);
\]

\[
\text{for } (\omega_c^2, m) \in \Gamma_1^-: m \in (0, 1), \quad \omega_c^2 = 2 \frac{1}{m};
\]

\[
\text{for } (\omega_c^2, m) \in \Gamma_1^k: m \in (M_{k+1}^+, M_k^-), \quad \omega_c^2 = 2.
\]

The recurrence (9) can be written in the form:

\[
Y_{n+1} = LY_n + N(Y_n, \mu)
\]

with \(L = DF_{m, \omega}(0, \mu) = F_{m, \omega}(Y) - LY = \mathcal{O}(\|Y\|^2 + |\mu|\|Y\|\mathbb{D})\). We now use the notation \(\alpha = 1/m\). Setting \(Y = (u, v)\) and \(Qu = (\omega_c^2 d^2/dt^2 + (1 + \alpha))u\), one has:

\[
L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha} (Q^2 - 1)u - Qv \\ Qu - \alpha v \end{pmatrix}
\]
and \( N \) is given by:

\[
N(Y, \mu) = \left( \frac{1}{a} (QN(u, \mu) + N(Qu - av + N(u, \mu))) \right)
\]

with

\[
N(u, \mu) = \frac{d^2}{dt^2} (\mu W(u) + \omega_c^2 W(u) - u)).
\]

Note that \( W = (V')^{-1} \) can be expanded as

\[
W(y) = y + W_2 y^2 + W_3 y^3 + O(y^4),
\]

with

\[
W_2 = -\frac{1}{2} V(3)(0), \quad W_3 = \frac{1}{2} (V(3)(0))^2 - \left( \frac{1}{5} V(4)(0) \right).
\]

Thus we have

\[
N(u, \mu) = \frac{d^2}{dt^2} (\mu u + \omega_c^2 W_2 u^2 + W_2 \mu u^2 + \omega_c^2 W_3 u^3) + \text{h.o.t.}
\]

The part of the spectrum of \( L \) lying on the unit circle consists in a double eigenvalue \(+ 1\) or \(- 1\), which is isolated from the rest of the spectrum. Consequently, the centre manifold theorem applies (see Theorem 1 in [24]). Therefore, one can reduce locally the study of (29) to that of a 2-D map on the generalized eigenspace \( \mathbb{X}_c \). In the sequel, we denote by \( \Pi_c \) the spectral projection on \( \mathbb{X}_c \) (see Lemma 2) and define \( \Pi_h = \text{Id} - \Pi_c, \mathbb{X}_h = \Pi_h \mathbb{X}, L_h = L|\mathbb{X}_h, L_c = L|\mathbb{X}_c, Y^c = \Pi_c Y, y^h = \Pi_h Y \). A direct application of the centre manifold theorem gives the following result:

**Theorem 2.** Let us fix \( m \) and \( \omega_c \) as in (28) and \( k \geq 4 \). There exist neighbourhoods \( \Omega, \Lambda \) of 0 in \( \mathbb{D}, \mathbb{R} \), respectively, and a map \( \Psi \in C^k(\mathbb{X}_c \times \Lambda; \mathbb{D}_h) \) with \( \Psi(0, \mu) = 0 \) and \( D\Psi(0, \mu) = 0 \) such that for any \( \mu \in \Lambda \), the manifold

\[
\mathcal{M}_\mu = \{ Y \in \mathbb{D}/Y = Y^c + \Psi(Y^c, \mu), Y^c \in \mathbb{X}_c \}
\]

has the following properties.

(i) \( \mathcal{M}_\mu \) is locally invariant under \( L + N(\cdot, \mu) \), i.e. if \( Y \in \mathcal{M}_\mu \cap \Omega \) then \( LY + N(Y, \mu) \in \mathcal{M}_\mu \).

(ii) If \( Y_n \) is a solution of (29) such that \( Y_n \in \Omega \) for all \( n \in \mathbb{Z} \), then \( Y_n \in \mathcal{M}_\mu \) for all \( n \in \mathbb{Z} \) and \( Y^c_n \) satisfies the recurrence relation

\[
\forall n \in \mathbb{Z}, Y^c_{n+1} = f(Y^c_n, \mu)
\]

with \( f(Y, \mu) = L_c Y + \Pi_c N(Y + \Psi(Y, \mu), \mu) \).

(iii) Conversely, if \( Y^c_n \) is a solution of (33) such that \( Y^c_n \in \Omega \) for all \( n \in \mathbb{Z} \), then \( Y_n = Y^c_n + \Psi(Y^c_n, \mu) \) satisfies (29).

Note that the validity domain of the reduction (i.e. the size of the neighbourhood \( \Omega \times \Lambda \)) depends on the spectral gap between the unit circle and the hyperbolic part of the spectrum (see [24]). More precisely, this neighbourhood shrinks as other eigenvalues approach the unit circle, which is the case when \( \omega_c^2 = 2 \) and \( m \to M^+_k(k \geq 2) \). It also shrinks as \( m \to 0 \) since this limit is singular in (9).

The situation is different as \( m \to 1 \) on \( \Gamma^+_1, \Gamma^-_1 \) and \( \Gamma^+_1 \) (the hyperbolic part of the spectrum remains at a nonzero distance). In these limits, the validity domain of the reduction is uniform with respect to \( m \). However, the frequency domains in which different classes of bifurcating solutions of the reduced mapping (33) exist may
vanish as $m \to 1$. Indeed, the reduced mapping structure changes drastically at $(\omega^2, m) = (2, 1)$ since the double eigenvalue $-1$ becomes semi-simple (see (27)).

4.2. Symmetries

We have seen in Section 2 that the mapping (29) is equivariant with respect to the symmetry $TY = Y(\cdot + \pi)$ and reversible with respect to $R(u, v) = (v, u)$. We now have to check that the centre manifold reduction preserves these symmetries. This means that the centre manifold is invariant under $T$ and $R$, and that the reduced mapping is equivariant under $T$ and reversible under $R$. As we shall see, these properties simplify the computation of the centre manifold and play a fundamental role in the reduced mapping dynamics.

We now state the main result of this section.

**Theorem 3.** The maps $f$ and $\Psi$ in Theorem 2 satisfy

\[ \forall Y \in X_c \cap \Omega, \quad R\Psi(Y, \mu) = \Psi(RY, \mu), \quad (f(\cdot, \mu) \circ R)^2 = Id, \tag{34} \]

\[ \forall Y \in X_c \cap \Omega, \quad T\Psi(Y, \mu) = \Psi(TY, \mu), \quad f(TY, \mu) = Tf(Y, \mu). \tag{35} \]

The main difficulty is related to the reversibility symmetry $R$. Indeed, the proof of Theorem 3 requires to modify (29) by a cut-off preserving reversibility. The existence of a suitable cut-off is not as automatic for maps as for vector fields, due to the fact that a reversible vector field $F$ is conjugate to $-F$ whereas a reversible map is conjugate to its inverse. This difficulty has been treated in [24] for a class of problems in which $R$ is bounded in $\mathbb{D}$.

However, these results are not directly applicable here since $R$ is unbounded in $\mathbb{D}$. More precisely, $R : \mathbb{D} \subset \mathbb{D} \to \mathbb{D}$ is closed and the domain $\mathbb{D}_R$ of $R$ is $\mathbb{D}_R = H^4 \times H^4$. Consequently we have to adapt the proof of reversibility preservation given in [24] (Section 4 and 5.1). The idea consists in using a cut-off function on the finite-dimensional centre space $X_c$ (instead of $\mathbb{D}$) since $R$ is bounded in $X_c$. We shall only point out which steps are modified and refer to [24] for the complete proof.

The modification of (29) by a cut-off consists in replacing $N(\cdot, \mu)$ by a map $N_c(\cdot, \mu)$ equal to $N(\cdot, \mu)$ in a small neighbourhood of 0, globally bounded and lipschitzian on $X_c$ (with small Lipschitz constant). This comes from the necessity of using the contraction mapping theorem in a space of sequences with possibly unbounded central parts (see section 3.1 in [24]). One first proves a global centre manifold reduction result for the truncated problem

\[ Y_{n+1} = LY_n + N_c(Y_n, \mu), \tag{36} \]

which gives a local centre manifold reduction result for the original problem (29). In order to prove that the centre manifold reduction preserves reversibility, one has to construct $N_c$ such that the mapping (36) is also reversible. This means that one has to modify the nonlinear terms in (8) in such a way that the invariance $y_n \to y_{-n-1}$ is preserved. In what follows, we introduce an adapted cut-off of (8).

Setting $\omega^2 = \omega^2_c + \mu$ and $\alpha = m^{-1}$ in (8), one obtains

\[ \omega^2_c \frac{d^2}{dt^2}(y_{2n+1}) + N(y_{2n+1}, \mu) = \alpha y_{2n+2} - (1 + \alpha)y_{2n+1} + y_{2n}, \]

\[ \omega^2_c \frac{d^2}{dt^2}(y_{2n}) + N(y_{2n}, \mu) = y_{2n+1} - (1 + \alpha)y_{2n} + \alpha y_{2n-1}, \tag{37} \]

where $N(\cdot, \mu) = d^2/dt^2 g(y, \mu)$ and $g(y, \mu) = \mu W(y) + \omega^2_c (W(y) - y)$. We now consider the truncated problem

\[ \omega^2_c \frac{d^2}{dt^2}(y_{2n+1}) + N_c(y_{2n+1}, \mu) = \alpha y_{2n+2} - (1 + \alpha)y_{2n+1} + y_{2n}, \]

\[ \omega^2_c \frac{d^2}{dt^2}(y_{2n}) + N_c(y_{2n}, \mu) = y_{2n+1} - (1 + \alpha)y_{2n} + \alpha y_{2n-1}. \tag{38} \]
where \( N_e(y, \mu) = \frac{d^2}{dt^2} g_\varepsilon(y, \mu), \ g_\varepsilon(y, \mu) = g(y, \mu) \chi(\varepsilon^{-1}[(y, \cos t)]) \), \( (y, \cos t) = \int_0^{2\pi} y(t) \cos rt \) and \( \chi \in C^\infty([0, +\infty), [0, 1]) \) is a cut-off function satisfying \( \chi(x) = 1 \) for \( x \in [0, 1] \) and \( \chi(x) = 0 \) for \( x \geq 2 \). One can check that the invariance \( y_n \to y_{n-1} \) is preserved in (38).

Setting \( Y_n = (u_n, v_n) = (y_{2n}, y_{2n-1}) \), problem (38) takes the form (36) with

\[
L(u \ v) = \begin{pmatrix}
-\frac{1}{\alpha}(Q^2 - 1)u - Qu \\
Q_1^2 \frac{d^2}{dt^2} + (1 + \alpha)
\end{pmatrix}, \quad Qu = \begin{pmatrix} \alpha^2 \frac{d^2}{dt^2} + (1 + \alpha) \end{pmatrix} u,
\]

\[
N_e(Y, \mu) = \begin{pmatrix}
\frac{1}{\alpha}(QN_e(u, \mu) + N_e(Qu - \alpha v + N_e(u, \mu))) \\
N_e(u, \mu)
\end{pmatrix}.
\]

The cut-off in (38) is only performed on the (finite-dimensional) centre space. Indeed, since \( R \) is unbounded in \( \mathbb{D} \) it is not possible to use a cut-off function on the whole of \( \mathbb{D} \) (as it is done in [24]). This procedure will slightly modify the subsequent choice of spaces.

We now detail some important properties of \( \mathcal{N}_e \). Let us consider the strip

\[
B^h = \left\{ Y \in \mathbb{D}/\|Y^h\|_\mathbb{D} \leq \epsilon \right\}.
\]

One can find \( \epsilon_0 > 0 \) such that \( \mathcal{N}_e \in C_0 \left( B^h_{\epsilon_0} \times (-\epsilon_0, \epsilon_0), \mathbb{X} \right) \). Moreover, one obtains using Lemma A.11 of Appendix A

\[
\|\mathcal{N}_e \|_{C_0^1 \left( B^h_{\epsilon_0} \times (-\epsilon, \epsilon), \mathbb{X} \right)} = O(\epsilon^2), \quad D_Y \|\mathcal{N}_e \|_{C_0^1 \left( B^h_{\epsilon_0} \times (-\epsilon, \epsilon), L(\mathbb{D}, \mathbb{X}) \right)} = O(\epsilon)
\]

as \( \epsilon \to 0^+ \). In addition, one has \( \mathcal{N}_e(., \mu) = \mathcal{N}(., \mu) \) in a ball \( B_{\epsilon} \) having the form

\[
B_{\epsilon} = \left\{ Y \in \mathbb{D}/\|Y\|_\mathbb{D} < C_\epsilon \right\}.
\]

and thus problems (29) and (36) are locally identical.

We now examine more closely the reversibility symmetry in (36). One can check that if \( u \in \mathbb{D}_e \), \( Ru \in B^h_{\epsilon_0} \), \( (L + \mathcal{N}_e(., \mu)) \circ Ru \in \mathbb{D}_e \) and \( R(L + \mathcal{N}_e(., \mu)) \circ Ru \in B^h_{\epsilon_0} \), then

\[
((L + \mathcal{N}_e(., \mu)) \circ R)u = u.
\]

This property is due to the invariance \( y_n \to y_{n-1} \) in (38) and characterizes the fact that (36) is reversible with respect to \( R \).

In the case when \( R \) is bounded in \( \mathbb{D} \), property (40) immediately implies that for any solution \( Y_n \) of (36), \( Z_n = RY_{n-1} \) is also a solution. The situation is more complicated in our case since \( R \) is unbounded in \( \mathbb{D} \) and \( Y_{n-1} \) does not a priori belong to \( \mathbb{D}_e \).

However, one can show that the solutions of (36) have more regularity than the regularity of \( \mathbb{D} \) and it turns out that \( Y_{n-1} \in \mathbb{D}_e \). More precisely, the following result is proved in Appendix A (see Lemma A.15).

**Lemma 3.** Fix \( p \geq 2 \). There exist \( \epsilon_0, \gamma > 0 \) (depending on \( p \)) such that for all \( \epsilon < \epsilon_0, \mu \in [-\epsilon, \epsilon] \), any solution of (36) such that \( Y_n \in B^h_\epsilon \) for all \( n \in \mathbb{Z} \) satisfies \( Y_n \in H^{p+2} \times H^p \) and \( \|Y_n\|_{H^{p+2}_e \times H^p_e} \leq \gamma \epsilon \) for all \( n \in \mathbb{Z} \).

For proving Lemma 3 one rewrites (36) as an evolutionary system

\[
\frac{d^2}{dt^2} (\alpha^2 y_{2n+1} + g_\varepsilon(y_{2n+1}, \mu)) = \alpha y_{2n+2} - (1 + \alpha) y_{2n+1} + y_{2n},
\]

\[
\frac{d^2}{dt^2} (\alpha^2 y_{2n} + g_\varepsilon(y_{2n}, \mu)) = y_{2n+1} - (1 + \alpha) y_{2n} + \alpha y_{2n-1},
\]

where the right side belongs to \( H^p_e \times H^p_e \) (with \( i = 2 \)), and one proceeds by induction on \( i \).
Now property (40) and Lemma 3 yield the following result.

Lemma 4. If $Y_n$ is a solution of (36), then $Z_n = RY_{-n}$ is also a solution.

Proof. According to Lemma 3 we have $Y_n \in H^h_\theta \times H^\delta_\theta \subseteq \mathbb{D}_R$, with the estimate $\|Y^h_n\|_{H^h_\theta \times H^\delta_\theta} \leq \gamma^\varepsilon$. Consequently, we have $Z_n, RZ_n \in \mathbb{D}_R$ and $Z_n, RZ_n \in B^h_{\mu_0}$ for $\varepsilon$ small enough. Setting $n \to -n$ in (36) yields

$$(L + \mathcal{N}_\varepsilon(\cdot, \mu))(RZ_n) = RZ_{n-1} \in \mathbb{D}_R,$$

with $R(L + \mathcal{N}_\varepsilon(\cdot, \mu))(RZ_n) \in B^h_{\mu_0}$. Consequently, property (40) implies

$$Z_n = ((L + \mathcal{N}_\varepsilon(\cdot, \mu))R)^2Z_n = (L + \mathcal{N}_\varepsilon(\cdot, \mu))Z_{n-1},$$

which completes the proof. \qed

We now give a global centre manifold reduction result (Theorem 4 below) which preserves the reversible character of the truncated problem (36). The proof is completely parallel to [24] (Section 4), except one has to make some obvious changes of spaces (since the cut-off is performed on $\mathcal{X}_c$ instead of $\mathbb{D}$). In Appendix A, we recall the principal steps of the proof and detail the modifications specific to our case.

We first introduce a suitable space of sequences for $Y = (Y_n)_{n \in \mathbb{Z}}$. Given a Banach space $E$ and $\nu \in (0, 1]$, we define the Banach space

$$B_\nu(E) = \left\{ Y/Y_n \in E \|Y\|_{B_\nu(E)} < +\infty \right\},$$

where $\|Y\|_{B_\nu(E)} = \sup_{n \in \mathbb{Z}} \|y_n\|_E$ (note that $B_1(E) = \ell_\infty(E)$). Now we look for $Y$ in the set

$$B^\nu_\varepsilon(\mathbb{D}) = \{ Y \in B_\nu(\mathbb{D})/Y^h \in B_1(\mathbb{D}_h), \|Y^h\|_{B_1(\mathbb{D}_h)} \leq \varepsilon \}$$

(sequences $Y \in B^\nu_\varepsilon(\mathbb{D})$ can have an unbounded central part). The set $B^\nu_\varepsilon(\mathbb{D})$ is a closed (convex) subset of $B_\nu(\mathbb{D})$ and consequently a complete metric space for the distance $d(Y, Z) = \|Y - Z\|_{B_\nu(\mathbb{D})}$.

One can prove the following global reduction result using (39), the spectral properties of $L$ and Lemma 4 (see Appendix A).

Theorem 4. There exists $r \in (0, 1)$ such that for $r < \zeta < \nu^k < \nu < 1$, for $\varepsilon$ small enough, for all $\mu \in [-\varepsilon, \varepsilon]$ and $x \in \mathcal{X}_c$, the problem

$$Y_{n+1} = LY_n + \mathcal{N}_\varepsilon(Y_n, \mu), \quad Y \in B^\nu_\varepsilon(\mathbb{D}), \quad Y^0 = x,$$

has a unique solution $Y_n = \phi^\nu_\varepsilon(x, \mu)$ with

$$\phi^\nu \in C^0(\mathcal{X}_c \times [-\varepsilon, \varepsilon], B^\nu_\varepsilon(\mathbb{D})) \cap C^k(\mathcal{X}_c \times [-\varepsilon, \varepsilon], B^\nu_\varepsilon(\mathbb{D})).$$

Moreover one has $Y^h_n = \psi_\varepsilon(Y^c_n, \mu)$, where $\psi_\varepsilon = \Pi h \phi^\nu_0 \in C^k(\mathcal{X}_c \times [-\varepsilon, \varepsilon], \mathbb{D}_h)$ and $\|\psi_\varepsilon\|_{C^k(\mathcal{X}_c \times [-\varepsilon, \varepsilon], \mathbb{D}_h)} = O(\varepsilon^4)$. The central part of $Y_n$ satisfies the reduced recurrence relation

$$Y^c_{n+1} = f_\varepsilon(Y^c_n, \mu), \quad \forall n \in \mathbb{Z},$$

(43)
where

\[ f_\epsilon(x, \mu) = L_\epsilon x + \Pi_\epsilon N_\epsilon(x + \psi_\epsilon(x, \mu), \mu). \]

We have in addition \( R\psi_\epsilon(., \mu) = \psi_\epsilon(., \mu) \circ R \) and \((f_\epsilon(., \mu) \circ R)^2 = I \).

We now turn back to Theorems 2 and 3, where \( \Omega \) denotes the ball of radius \( \varepsilon/2 \) in \( \mathbb{D} \), \( \Lambda \) the interval \((-\varepsilon, \varepsilon)\) and \( \psi = \psi_\epsilon \). Note that \( f(x, \mu) = f_\epsilon(x, \mu) \) for all \( x \in \mathbb{X}_c \cap \Omega \) since \( \|x + \psi(x, \mu)\|_\mathbb{D} < \varepsilon \) (one has \( \|\psi_\epsilon\|_{C_0(\mathbb{X}_c \times (-\varepsilon, \varepsilon), \mathbb{D}_h)} = \mathcal{O}(\varepsilon^2) \)). It follows immediately from Theorem 4 that the maps \( f \) and \( \psi \) in Theorem 2 satisfy

\[ \forall Y \in \mathbb{X}_c \cap \Omega, \quad R\psi(Y, \mu) = \psi(RY, \mu), \quad (f(\cdot, \mu) \circ R)^2 = \text{Id}. \]

Consequently, we have shown that reversibility is preserved throughout the reduction.

There remains to check that the reduction preserves the equivariance under \( TY = Y(\cdot + \pi) \). Since \( T \) commutes with \( L + N_\epsilon(., \mu) \), it follows that the maps \( \psi_\epsilon(., \mu) \) and \( f_\epsilon(., \mu) \) in Theorem 4 commute with \( T \) (see [24]), which completes the proof of Theorem 3.

4.3. Centre manifold computation

In this section, we compute the Taylor expansions of the function \( \psi \) and the reduced map \( f \). For calculating \( \psi \), we project (29) on \( \mathbb{X}_h \) and \( \mathbb{X}_c \):

\[
Y_n^{h+1} = L_h Y_n^h + \Pi_h N_h(Y_n^c + Y_n^h, \mu),
\]

\[
Y_n^{c+1} = L_c Y_n^c + \Pi_c N_c(Y_n^c + Y_n^h, \mu).
\]

Choosing \( Y_n \in \mathcal{M}_\mu \cap \Omega \) and using property (i) of Theorem 1, we find:

\[
\psi(Y_n^{c+1}, \mu) = L_h \psi(Y_n^c, \mu) + \Pi_h N_h(Y_n^c + \psi(Y_n^c, \mu), \mu), \tag{45}
\]

\[
Y_n^{c+1} = L_c Y_n^c + \Pi_c N_c(Y_n^c + \psi(Y_n^c, \mu), \mu) = f(Y_n^c, \mu). \tag{46}
\]

Replacing (46) into (45) yields:

\[
\psi(L_c Y_n^c + \Pi_c N_c(Y_n^c + \psi(Y_n^c, \mu), \mu), \mu) = L_h \psi(Y_n^c, \mu) + \Pi_h N_h(Y_n^c + \psi(Y_n^c, \mu), \mu). \tag{47}
\]

The Taylor expansion of \( \psi \) at \((Y^c, \mu) = 0\) can be computed by identifying the terms of equal orders in \((Y_n^c, \mu)\) obtained by expanding (47). We identify \( \mathbb{X}_c \) with \( \mathbb{R}^2 \) by setting:

\[
Y^c = \begin{pmatrix} a \\ b \end{pmatrix} \cos(t).
\]

The symmetry properties of \( \psi \) (Theorem 3) imply that

\[
T\psi(a, b, \mu) = \psi(-a, -b, \mu), \quad R\psi(a, b, \mu) = \psi(b, a, \mu). \tag{48}
\]

Consequently, the Taylor expansion of \( \psi \) has the form:

\[
\psi(a, b, \mu) = \psi_{011}a\mu + \psi_{101}b\mu + \psi_{020}a^2 + \psi_{110}ab + \psi_{200}b^2 + \psi_{101}a\mu + \psi_{011}b\mu + \psi_{200}a^2 + \psi_{110}ab + \psi_{020}b^2 + \text{h.o.t.} \tag{49}
\]
where \( \Psi_{pqr} \in H_g^4 \). Due to the invariance under \( T \), we have also:
\[
\Psi_{011}, \Psi_{101} \in \langle \cos((2k + 1)t)/k \in \mathbb{Z} \rangle = \mathcal{V}_{\text{odd}}, \quad \Psi_{020}, \Psi_{110}, \Psi_{200} \in \langle \cos(2kt)/k \in \mathbb{Z} \rangle = \mathcal{V}_{\text{even}}.
\] (50)

Setting
\[
Y_n^c = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos(t),
\]
in (47), expanding (47) in powers of \((a_n, b_n, \mu)\) and identifying quadratic terms leads to a linear system for the corresponding coefficients \( \Psi_{pqr} \) (calculations are given in the appendix). Solving this system yields:
\[
\Psi_{011} = \Psi_{101} = 0,
\]
and \( \Psi_{020}, \Psi_{200}, \Psi_{110} \) have the general form:
\[
\Psi_{020} = p(m)W_2 \cos(2t), \quad \Psi_{200} = q(m)W_2 \cos(2t), \quad \Psi_{110} = r(m)W_2 \cos(2t).
\] (51)

with \( p, q, r \) depending on the considered value of \( \omega_c^2 \) in (28). For \( \omega_c^2 = 2(1 + \alpha) \) we find:
\[
p(m) = -\frac{1}{16} \frac{8m - 1}{m}, \quad q(m) = \frac{1}{16m}, \quad r(m) = \frac{1}{8m}.
\] (52)

For \( \omega_c^2 = 2\alpha \) we get:
\[
p(m) = -\frac{1}{16} \frac{7m^2 - 34m + 3}{m(m - 3)}, \quad q(m) = \frac{1}{16} \frac{m^2 - 6m - 3}{m(m - 3)}, \quad r(m) = \frac{1}{8} \frac{m^2 + 2m - 3}{m(m - 3)}.
\] (53)

Considering the case \( \omega_c^2 = 2 \) yields:
\[
p(m) = -\frac{1}{16} \frac{24m^3 + 5m^2 - 6m + 1}{m^2(3m - 1)}, \quad q(m) = \frac{1}{16} \frac{3m^2 + 6m - 1}{m^2(3m - 1)}, \quad r(m) = \frac{1}{8} \frac{5m^2 - 6m + 1}{m^2(3m - 1)}.
\] (54)

When \( \omega_c^2 = 2 \), note that \( \Psi \) is not defined for \( m = 1/3 = M_3^+ \) (see the comments following Theorem 2).

The Taylor expansion of \( f \) is obtained by injecting (51) in equation (46). One obtains (see Appendix B)
\[
\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} F_1(a_n, b_n, \mu) \\ F_2(a_n, b_n, \mu) \end{pmatrix} = f(a_n, b_n, \mu)
\] (55)

with
\[
\alpha F_1(a, b, \mu) = (\beta^2 - 1)a - \alpha b - 2\beta a \mu + \alpha b \mu - \omega_c^2 \alpha (\beta_4 a^3 + \beta_5 a^2 b + \beta_6 ab^2 + \beta_7 b^3)
\]
\[+ o((a, b)\|\mu + \| (a, b)\|^3),
\]
\[
F_2(a, b, \mu) = \beta a - \alpha b - \alpha \mu - \omega_c^2 (\beta_1 a^3 + \beta_2 ab^2 + \beta_3 a^2 b) + o((a, b)\|\mu + \| (a, b)\|^3),
\]
\[
\beta_1 = p(m)W_2^2 + \frac{3}{4} W_3, \quad \beta_2 = q(m)W_2^2, \quad \beta_3 = r(m)W_2^2,
\] (56)
\[ \alpha \beta_4 = \beta_1 + \beta^3 \beta_1 + \beta \gamma^2 \beta_2 + \beta^2 \gamma \beta_3, \quad \alpha \beta_5 = -3 \beta^2 \alpha \beta_1 - (\alpha \gamma^2 + 2 \gamma \beta^2) \beta_2 + (\beta - \beta^3 - 2 \alpha \beta \gamma) \beta_3, \]
\[ \alpha \beta_6 = 3 \beta \alpha \beta_1 + (\beta + \beta^3 + 2 \alpha \beta \gamma) \beta_2 + (2 \beta^2 \alpha + \alpha^2 \gamma) \beta_3, \]
\[ \beta_7 = -\alpha^2 \beta_1 - \beta^2 \beta_2 - \alpha \beta \beta_3, \quad (57) \]
and \( \beta = 1 + \alpha - \omega^2, \gamma = (\beta^2 - 1) \alpha. \)

The reduced map \( f \) has the structure:
\[ f(a, b, \mu) = L_c \begin{pmatrix} a \\ b \end{pmatrix} + \mu f_{11} \begin{pmatrix} a \\ b \end{pmatrix} + f_{30} \begin{pmatrix} a \\ b \end{pmatrix} + o(\| (a, b) \| \mu + \| (a, b) \|^3) \quad (58) \]
where
\[ L_c = \begin{pmatrix} \gamma & -\beta \\ \beta & -\alpha \end{pmatrix}, \quad (59) \]
\[ f_{11} = \begin{pmatrix} -2 \beta^2 & 1 \\ -1 & 0 \end{pmatrix}, \quad (60) \]
\[ f_{30} = \begin{pmatrix} P_3 \\ Q_3 \end{pmatrix}, \quad (61) \]
and \( P_3, Q_3 \) are homogeneous cubic polynomials in \((a, b)\). The reduced map \( f \) inherits the symmetry properties of (29), i.e.
\[ f(-a, -b, \mu) = -f(a, b, \mu), \quad (62) \]
\[ (f(\cdot, \mu) \circ R)^2 = Id, \text{ or equivalently } f^{-1}(\cdot, \mu) = R \circ f(\cdot, \mu) \circ R. \quad (63) \]

4.4. Normal form computation

In this section, we write the reduced mapping (55) in normal form, i.e. we perform a change of variables which only keeps its essential terms. This greatly simplifies the recurrence relation, which takes the form
\[ A_{n+1} \pm 2A_n + A_{n-1} = c \mu A_n + d A_n^3 + \text{h.o.t.}, \]
the sign \( \pm \) depending whether the bifurcation occurs at a double eigenvalue \(-1\) or \(1\). Moreover, one can choose the change of variables such that the reversibility symmetry is transformed into an involution (nonlinear symmetry) which remains linear up to higher order terms. These results are detailed in the following lemma.

**Lemma 5.** There exists a \( C^4 \) local diffeomorphism \( h_\mu \) defined on a neighbourhood of \((a, b) = 0\) which transforms (55) into the following mapping:
\[ \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \quad (64) \]
where \((A_n, B_n) = h_\mu(a_n, b_n)\) and
\[ G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + g_\mu(A_n, B_n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (65) \]
with \(a + \) sign in (65) for \((\omega^2, m) \in \Gamma_1^+\) and \(a - \) sign for \((\omega^2, m) \in \Gamma_1^-\) or \(\Gamma_1^p\) \(p \geq 1\),

\[
g_\mu(A_n, B_n) = -\frac{\beta}{\alpha} \left(2\mu A_n - \frac{B_0\omega^2}{8} A_n^3\right) + o\left(\|A_n, B_n\|_3 + \mu\|(A_n, B_n)\|\right), \tag{66}
\]

with

\[
B = \frac{1}{2} V^{(4)}(0) - (V^{(3)}(0))^2. \tag{67}
\]

The coefficient \(\beta\) in (66) depends on the value of \((\omega^2, m)\) considered in (28) and one has:

\[
\begin{align*}
\text{for} \quad (\omega^2, m) & \in \Gamma_1^+ : \quad \beta = -1 - \alpha, \\
\text{for} \quad (\omega^2, m) & \in \Gamma_1^- : \quad \beta = 1 - \alpha, \\
\text{for} \quad (\omega^2, m) & \in \Gamma_1^p : \quad \beta = \alpha - 1.
\end{align*} \tag{68}
\]

The maps \(h_\mu\) and \(G_\mu\) commute with \(-\text{Id}\). Moreover, the map \(G_\mu\) is reversible with respect to the involution:

\[
\mathcal{R}_\mu = h_\mu \circ R \circ h_\mu^{-1}, \tag{69}
\]

which depends on the considered value of \((\omega^2, m)\). For \((\omega^2, m) \in \Gamma_1^+\), we note \(\mathcal{R}_\mu = \mathcal{R}_\mu^+\) and one has:

\[
\mathcal{R}_\mu^+ \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o\left(\|A, B\|_3 + \mu\|(A, B)\|\right). \tag{70}
\]

For \((\omega^2, m) \in \Gamma_1^-\), we note \(\mathcal{R}_\mu = \mathcal{R}_\mu^-\) with:

\[
\mathcal{R}_\mu^- \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o\left(\|A, B\|_3 + \mu\|(A, B)\|\right). \tag{71}
\]

For \((\omega^2, m) \in \Gamma_1^p\), we note \(\mathcal{R}_\mu = \mathcal{R}_\mu^a\) with:

\[
\mathcal{R}_\mu^a \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o\left(\|A, B\|_3 + \mu\|(A, B)\|\right). \tag{72}
\]

The principal part of \(h_\mu\) is given by:

\[
\begin{align*}
\text{for} \quad (\omega^2, m) & \in \Gamma_1^+ : \quad h_\mu = \begin{pmatrix} -(\alpha + 2) & -\alpha \\ -2(1 + \alpha) & -2(1 + \alpha) \end{pmatrix} + o\|(a, b)\|, \\
\text{for} \quad (\omega^2, m) & \in \Gamma_1^- : \quad h_\mu = \begin{pmatrix} -\alpha & 2 - \alpha \\ 2(\alpha - 1) & 2(1 - \alpha) \end{pmatrix} + o\|(a, b)\|, \\
\text{for} \quad (\omega^2, m) & \in \Gamma_1^p : \quad h_\mu = \begin{pmatrix} \alpha & -\alpha \\ -2(\alpha - 1) & 2(1 - \alpha) \end{pmatrix} + o\|(a, b)\|. \tag{73}
\end{align*}
\]

**Proof.** In the sequel, the \(\pm\) sign has to be understood as \(+\) sign for \((\omega^2, m) \in \Gamma_1^+\) and \(-\) sign for \((\omega^2, m) \in \Gamma_1^-\) or \(\Gamma_1^p\).
We look for a change of variables \((A_n, B_n) = h_{\mu}(a_n, b_n)\) having the form:

\[
\begin{align*}
A_n &= P(a_n, b_n, \mu), \\
B_n &= A_n - (\pm A_{n-1}) = P(a_n, b_n, \mu) - (\pm P \circ f^{-1}(a_n, b_n, \mu)),
\end{align*}
\]

which transforms (55) into (64). We choose \(P\) such that the term \(g_\mu\) in (65) has the form:

\[
g_\mu(A, B) = c \mu A + d A^3 + o(\mu \|(A, B)|| + \|(A, B)||^3).
\]

The mapping (64) can be written as a second order scalar recurrence of the form:

\[
A_{n+1} - 2(\pm A_n) + A_{n-1} = c \mu A_n + d A_n^3 + o(\|(A_n, A_{n-1})\||^3 + \mu \|(A_n, A_{n-1})\|). 
\tag{75}
\]

We choose \(P\) as a cubic polynomial in \((a, b, \mu)\) which preserves the symmetry \(-Id\) of (55), hence \(P(-a, -b, \mu) = -P(a, b, \mu)\). Note that the mapping (64) is reversible with respect to the involution \(R_{\mu} = h_{\mu} \circ R \circ h_{\mu}^{-1}\). We write \(P = M_1 + \mu M_{11} + Q\) with \(M_1, M_{11}\) linear and

\[
Q(a, b) = \delta_1 a^3 + \delta_2 a^2 b + \delta_3 ab^2 + \delta_4 b^3. 
\]

Eq. (75) yields:

\[
P \circ f(a_n, b_n, \mu) - 2(\pm P(a_n, b_n, \mu)) + P \circ f^{-1}(a_n, b_n, \mu) = c \mu P(a_n, b_n, \mu) + dP(a_n, b_n, \mu)^3 + \text{h.o.t.} 
\tag{76}
\]

Using property (63), Eq. (76) can be written:

\[
P \circ f(a_n, b_n, \mu) - 2(\pm P(a_n, b_n, \mu)) + P \circ f \circ R(a_n, b_n, \mu) \\
= c \mu P(a_n, b_n, \mu) + dP(a_n, b_n, \mu)^3 + \text{h.o.t.} 
\tag{77}
\]

Moreover, we choose a change of variable which transforms the reversibility symmetry \(R\) into an involution \(R_{\mu}\) which has the simplest possible form. In the case when \((\omega_1^2, m) \in \Gamma_1^+\), Eq. (70) is equivalent to

\[
P \circ R \circ f(a, b, \mu) = -P(a, b, \mu) + o(\|(a, b)||^3 + \mu \|(a, b)||) 
\tag{78}
\]

(use Eqs. (69), (74), (63)). For \((\omega_1^2, m) \in \Gamma_1^-\), Eq. (71) yields similarly

\[
P \circ R \circ f(a, b, \mu) = P(a, b, \mu) + o(\|(a, b)||^3 + \mu \|(a, b)||). 
\tag{79}
\]

For \((\omega_1^2, m) \in \Gamma_1^a\), Eq. (72) leads to

\[
P \circ R \circ f(a, b, \mu) = -P(a, b, \mu) + o(\|(a, b)||^3 + \mu \|(a, b)||). 
\tag{80}
\]

We now start by computing \(P\) and the normal form (75) for \((\omega_1^2, m) \in \Gamma_1^+\). Let us determine \(M_1, M_{11}, Q, c, d\) by identification in the Taylor expansions of Eqs. (77) and (78).

The identification of the linear terms in (77) and (78) leads to:

\[
M_1 \circ L_c - 2M_1 + M_1 \circ L_c^{-1} = 0, 
\tag{81}
\]

\[
M_1 \circ RL_c = -M_1. 
\tag{82}
\]

We recall that

\[
L_c = \begin{pmatrix}
\gamma - \beta \\
\beta - \alpha
\end{pmatrix} = \begin{pmatrix}
\alpha + 2 & 1 + \alpha \\
-1 - \alpha & -\alpha
\end{pmatrix}. 
\tag{83}
\]
Since \( L_c + L_c^{-1} = 2Id \), it follows that (81) is satisfied for all \( M_1 \). Let us note \( M_1(x, y) = a_1 x + b_1 y \). Eq. (82) gives the condition:

\[
 a_1 \alpha - b_1 \gamma = 0.
\]

Consequently, \( M_1 \) is defined up to a multiplicative constant and we take \( M_1(x, y) = -(\gamma x + \alpha y) \). Then the principal part of \( h_\mu \) is given by:

\[
 h_\mu = \begin{pmatrix}
 -\alpha + 2 & -\alpha \\
 -2(1 + \alpha) & -2(1 + \alpha)
\end{pmatrix} + o(\|(a, b)\|) = M + o(\|(a, b)\|).
\]

Note that \( M \) is invertible and thus \( h_\mu \) defines a local diffeomorphism.

The identification of \( O(\mu) \) linear terms in (77) and (78) leads to:

\[
 M_{11} \circ (L_c - 2Id + L_c^{-1}) + M_1 \circ (f_{11} + Rf_{11} - cId) = 0, \quad (84)
\]

\[
 M_{11} \circ (RL_c + Id) = -M_1 \circ f_{11}, \quad (85)
\]

where \( f_{11} \) is defined in (60). We note that \( f_{11} + Rf_{11} = -2(\beta/\alpha)Id \). Consequently, Eq. (84) yields \( c = -2(\beta/\alpha) \).

Let us note \( M_{11}(x, y) = a_{11} x + b_{11} y \). Then Eq. (85) is equivalent to:

\[
 -a_{11} \alpha + b_{11} \gamma = -\alpha,
\]

hence one can fix \( M_{11}(x, y) = x \) (others choices are possible).

The identification of the cubic terms in Eqs. (77) and (78) leads to:

\[
 Q \circ L_c - 2Q + Q \circ L_c^{-1} = dM_1^3 - M_1 \circ (f_{30} + R \circ f_{30} \circ R), \quad (86)
\]

\[
 Q \circ RL_c + Q = -M_1 \circ f_{30}, \quad (87)
\]

where \( f_{30} \) is defined in (61). The identification of the powers \( a^i b^j \) in (86) and (87) leads to a couple of linear systems:

\[
 A_1 \begin{pmatrix}
 \delta_1 \\
 \delta_2 \\
 \delta_3 \\
 \delta_4
\end{pmatrix} = V_1, \quad A_2 \begin{pmatrix}
 \delta_1 \\
 \delta_2 \\
 \delta_3 \\
 \delta_4
\end{pmatrix} = V_2 \quad (88)
\]

with

\[
 A_1 = \begin{pmatrix}
 \gamma^3 - \alpha^3 - 2 & \gamma^2 \beta - \beta \alpha^2 & \gamma \beta^2 - \alpha \beta^2 & 0 \\
 3\beta \alpha^2 - 3\gamma^2 \beta & \delta & 0 & 3\beta^2 \gamma - 3\beta^2 \alpha \\
 3\beta^2 \gamma - 3\beta^2 \alpha & 0 & \delta & 3\beta \alpha^2 - 3\gamma^2 \beta \\
 0 & \gamma \beta^2 - \alpha \beta^2 & \gamma^2 \beta - \beta \alpha^2 & \gamma^3 - \alpha^3 - 2
\end{pmatrix},
\]

\[
 \delta = \alpha^2 \gamma + 2 \alpha \beta^2 - \alpha \gamma^2 - 2 \gamma \beta^2 - 2,
\]
\[ V_1 = \begin{pmatrix} -d\gamma^3 - \omega_2^2(\gamma\beta_4 + \alpha(\beta_2 + \beta_1)) \\ -3d\alpha\gamma^2 - \omega_2^2(\gamma(\beta_5 + \beta_2) + \alpha(\beta_3 + \beta_6)) \\ -3d\alpha^2\gamma - \omega_2^2(\alpha(\beta_5 + \beta_2) + \gamma(\beta_3 + \beta_6)) \\ -d\alpha^3 - \omega_2^2(\alpha\beta_4 + \gamma(\beta_1 + \beta_1)) \end{pmatrix}, \]

\[ A_2 = \begin{pmatrix} \beta^3 + 1 & \gamma\beta^2 & \gamma^2\beta & \gamma^3 \\ -3\beta^2\alpha & 1 - \beta^3 - 2\alpha\beta\gamma & -\alpha\gamma^2 - 2\gamma\beta^2 & -3\beta\gamma^2 \\ 3\beta\alpha^2 & \gamma\alpha^2 + 2\alpha\beta\gamma & 1 + \beta^3 + 2\alpha\beta\gamma & 3\gamma\beta^2 \\ -\alpha^3 & -\alpha^2\beta & -\beta^2\alpha & 1 - \beta^3 \end{pmatrix}, \]

and

\[ V_2 = -\omega_2^2 \begin{pmatrix} \alpha\beta_4 + \gamma\beta_1 \\ \alpha\beta_5 + \gamma\beta_3 \\ \alpha\beta_6 + \gamma\beta_2 \\ \alpha\beta_7 \end{pmatrix}. \]

where the coefficients \( \beta_i \) are defined in (56) and (57). The couple of systems seems to be overdetermined but we shall see in the sequel that this is not the case for a suitable choice of \( d \).

We now solve (88) and start with the linear system:

\[ A_1 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_1 \quad (89) \]

which corresponds to the identification of cubic terms in (86). Since \( \gamma = \alpha + 2 \) and \( \beta = -1 - \alpha \), \( A_1 \) has a simpler form

\[ A_1 = \beta^2 \begin{pmatrix} 6 & -4 & 2 & 0 \\ 12 & -6 & 0 & 6 \\ 6 & 0 & -6 & 12 \\ 0 & 2 & -4 & 6 \end{pmatrix}. \]

We have \( \text{rank}(A_1) = 2 \). This yields the conditions of compatibility:

\[ \det \begin{pmatrix} 0 & 2 & -d\alpha^3 - \omega_2^2(\alpha\beta_4 + \gamma(\beta_7 + \beta_1)) \\ 6 & 0 & -3d\alpha^2\gamma - \omega_2^2(\alpha(\beta_5 + \beta_2) + \gamma(\beta_3 + \beta_6)) \\ 12 & -6 & -3d\alpha\gamma^2 - \omega_2^2(\gamma(\beta_5 + \beta_2) + \alpha(\beta_3 + \beta_6)) \end{pmatrix} = 0 \quad (90) \]
and

\[
\begin{vmatrix}
0 & 2 & -3d \alpha^2 - \omega_c^2 (\alpha \beta_4 + \gamma (\beta_7 + \beta_1)) \\
6 & 0 & -3d \alpha^2 \gamma - \omega_c^2 (\alpha \beta_4 + \gamma (\beta_3 + \beta_6)) \\
6 & -4 & -d \gamma^3 - \omega_c^2 (\gamma \beta_4 + \alpha (\beta_7 + \beta_1)) \\
\end{vmatrix} = 0.
\]  

(91)

Thanks to relations (57) (which are due to the reversibility of \( f \)), conditions (90) and (91) are both satisfied for:

\[
3d = -\omega_c^2 (3 \gamma (\beta_1 + \beta_7) + 3 \alpha \beta_4 + (\alpha - 2 \gamma) (\beta_3 + \beta_6) + (\gamma - 2 \alpha) (\beta_2 + \beta_3)) / (\alpha^3 + \alpha \gamma^2 - 2 \alpha^2 \gamma).
\]

(92)

Injecting Eq. (57) in (92) yields:

\[
12 \alpha d = -6 \omega_c^2 \beta (\beta_1 + \beta_2 - \beta_3).
\]

Using (56) and (52) gives

\[
d = \frac{\beta \alpha^2}{8 \alpha} B,
\]

where \( B \) is given in (67). Consequently, we have computed the coefficients \( c, d \) in the normal form (75).

There remains to check the existence of a solution \((\delta_1, \delta_2, \delta_3, \delta_4)\) of (88). The system (89) is equivalent to:

\[
\beta^2 \begin{pmatrix} 0 & 6 & -6 & 12 \\ 0 & 2 & -4 & 6 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} V_{13} \\ V_{14} \end{pmatrix}
\]

where \( V_{13} \) and \( V_{14} \) are, respectively, the third and the fourth component of \( V_1 \). We now study the second linear system in (88) corresponding to the identification of cubic terms in (87). We have \( \text{rank}(A_2) = 2 \). The compatibility conditions read:

\[
\begin{vmatrix}
\beta^3 + 1 & \gamma \beta^2 & -\omega_c^2 (\alpha \beta_4 + \gamma \beta_1) \\
-\alpha^3 & -\alpha^2 \beta & -\omega_c^2 (\alpha \beta_7) \\
-3 \beta^2 \alpha^2 (1 - \beta^3 - 2 \alpha \beta_4) & -\omega_c^2 (\alpha \beta_3 + \gamma \beta_3) \\
\end{vmatrix} = 0.
\]

(93)

and

\[
\begin{vmatrix}
\beta^3 + 1 & \gamma \beta^2 & -\omega_c^2 (\alpha \beta_4 + \gamma \beta_1) \\
-\alpha^3 & -\alpha^2 \beta & -\omega_c^2 (\alpha \beta_7) \\
3 \beta \alpha^2 (\gamma \alpha^2 + 2 \alpha \beta_2) & -\omega_c^2 (\alpha \beta_6 + \gamma \beta_2) \\
\end{vmatrix} = 0.
\]

(94)

Thanks to relations (57), conditions (93) and (94) are both satisfied and the second system is equivalent to:

\[
\begin{pmatrix}
\beta^3 + 1 & \gamma \beta^2 & \gamma^2 \beta & \gamma^3 \\
-\alpha^3 & -\alpha^2 \beta & -\beta^2 \alpha (1 - \beta^3) \\
\end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} V_{21} \\ V_{24} \end{pmatrix}.
\]

Here, we denote by \( V_{21} \) and \( V_{24} \), respectively, the first and the fourth component of \( V_2 \).
As a conclusion, (88) is equivalent to:

\[ d = \frac{\beta \omega^2}{8\alpha} B \]

and

\[ A_3(\delta_1, \delta_2, \delta_3, \delta_4) = (V_{13}, V_{14}, V_{21}, V_{24})^T, \]

where

\[ A_3 = \begin{pmatrix} 6\beta^2 & 0 & -6\beta^2 & 12\beta^2 \\ 0 & 2\beta^2 & -4\beta^2 & 6\beta^2 \\ \beta^3 + 1 & \gamma \beta^2 & \gamma^2 \beta & \gamma^3 \\ -\alpha^3 & -\alpha^2 \beta & -\beta^3 \alpha & 1 - \beta^3 \end{pmatrix}. \]

We have \( \text{rank}(A_3) = 3 \). The compatibility condition reads

\[ \det \begin{pmatrix} 6\beta^2 & 0 & -6\beta^2 & V_{13} \\ 0 & 2\beta^2 & -4\beta^2 & V_{14} \\ \beta^3 + 1 & \gamma \beta^2 & \gamma^2 \beta & V_{21} \\ -\alpha^3 & -\alpha^2 \beta & -\beta^3 \alpha & V_{24} \end{pmatrix} = 0. \]

Thanks to Eq. (57), this condition is again satisfied and thus we can find a (non unique) solution to (88). As a conclusion, we have proved Lemma 5 in the case when \( (\omega^2, m) \in \Gamma_1^+ \).

We now consider the case when \( (\omega^2, m) \in \Gamma_1^- \). Since the computations are similar to previous ones, we keep the same notations and only give the outline of the proof. We recall that \( \omega^2 = 2\alpha, \beta = 1 - \alpha \) and \( \gamma = \alpha - 2 \).

The identification of the linear terms in (77) and (79) leads to \( M_1(x, y) = \gamma x + \alpha y \). Then the principal part of \( h_\mu \) is given by

\[ h_\mu = \begin{pmatrix} \alpha - 2 & \alpha \\ -2(1 - \alpha) & -2(1 - \alpha) \end{pmatrix} + o(\|(a, b)\|) = M + o(\|(a, b)\|). \]

We note that \( M \) is invertible and \( h_\mu \) defines a local diffeomorphism. The identification of \( O(\mu) \) linear terms in (77) and (79) leads to \( c = -2\beta/\alpha \) and \( M_{11}(x, y) = x \) (this choice is non unique). The identification of cubic terms in (77) and (79) leads to:

\[ A_1 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_1, \quad A_2 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_2 \]

This gives (96).
with

\[
A_1 = \begin{pmatrix}
\gamma^3 - \alpha^3 + 2 & \gamma^2 \beta - \beta \alpha^2 & \gamma \beta^2 - \alpha \beta^2 & 0 \\
3 \beta \alpha^2 - 3 \gamma^2 \beta & \delta & 0 & 3 \beta \gamma^2 - 3 \beta \alpha^2 \\
3 \beta^2 \gamma - 3 \beta^2 \alpha & 0 & \delta & 3 \beta \alpha^2 - 3 \gamma \beta^2 \\
0 & \gamma \beta^2 - \alpha \beta^2 & \gamma^2 \beta - \beta \alpha^2 & \gamma^3 - \alpha^3 + 2
\end{pmatrix},
\]

\[
\delta = \alpha^2 \gamma + 2 \alpha \beta^2 - \alpha \gamma^2 - 2 \gamma \beta^2 + 2.
\]

\[
V_1 = \begin{pmatrix}
d \gamma^3 + \omega_c^2 (\gamma \beta_4 + \alpha (\beta_1 + \beta_1)) \\
3 d \alpha \gamma^2 + \omega_c^2 (\gamma (\beta_5 + \beta_2) + \alpha (\beta_3 + \beta_6)) \\
3 d \alpha^2 \gamma + \omega_c^2 (\alpha (\beta_5 + \beta_2) + \gamma (\beta_3 + \beta_6)) \\
d \alpha^3 + \omega_c^2 (\alpha (\beta_1 + \gamma (\beta_1 + \beta_1))
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
\beta^3 - 1 & \gamma \beta^2 & \gamma^2 \beta & \gamma^3 \\
-3 \beta^2 \alpha - 1 - \beta^3 - 2 \alpha \beta \gamma & -\alpha \gamma^2 - 2 \gamma \beta^2 & -3 \beta \gamma^2 \\
3 \beta \alpha^2 & \gamma \alpha^2 + 2 \alpha \beta^2 & -1 + \beta^3 + 2 \alpha \gamma \beta & 3 \gamma \beta^2 \\
-\alpha^3 & -\alpha \gamma^2 & -\beta^2 \alpha & -1 - \beta^3
\end{pmatrix},
\]

and

\[
V_2 = \omega_c^2 \begin{pmatrix}
\alpha \beta_4 + \gamma \beta_1 \\
\alpha \beta_5 + \gamma \beta_3 \\
\alpha \beta_6 + \gamma \beta_2 \\
\alpha \beta_1
\end{pmatrix},
\]

where the coefficients \( \beta_i \) are defined in (56) and (57).

Let us study the first system in (96). Since \( \beta = 1 - \alpha \) and \( \gamma = \alpha - 2 \), \( A_1 \) has a simpler form:

\[
A_1 = \beta^2 \begin{pmatrix}
-6 & 4 & -2 & 0 \\
-12 & 6 & 0 & -6 \\
-6 & 0 & 6 & -12 \\
0 & -2 & 4 & -6
\end{pmatrix},
\]

The compatibility conditions have the same form (90) and (91) and we find:

\[
d = -\frac{\beta \omega_c^2}{2 \alpha} (\beta_1 + \beta_2 - \beta_3).
\]

Injecting (56) and (53) into (97), we obtain \( d = B \beta \omega_c^2 / 2 \alpha \). Then it is lengthy but straightforward to check the existence of a solution \((\delta_1, \delta_2, \delta_3, \delta_4)\) of (96) (these calculations are similar to previous ones). This completes the proof of Lemma 5 for \((\omega_c^2, m) \in \Gamma_1^-\).

We now give the main steps of the proof for \((\omega_c^2, m) \in \Gamma_1^P\). We recall that \( \omega_c^2 = 2 \), \( \beta = \alpha - 1 \) and \( \gamma = \alpha + 2 \).
The identification of the linear terms in (77) and (80) leads to \( M_1(x, y) = \alpha x - \gamma y \). Then the principal part of \( h_\mu \) is given by

\[
h_\mu = \begin{pmatrix} \alpha & 2 - \alpha \\ 2(\alpha - 1) & 2(1 - \alpha) \end{pmatrix} + o(\|(a, b)\|) = M + o(\|(a, b)\|).
\]

Note that \( M \) is invertible and thus \( h_\mu \) defines a local diffeomorphism. The identification of \( O(\mu) \) linear terms in (77) and (80) leads to \( c = -2\beta/\alpha \) and \( M_{11}(x, y) = x \) (this choice is non unique). The identification of cubic terms in (77) and (80) leads to:

\[
A_1 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_1, \quad A_2 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_2
\]

(98)

with

\[
A_1 = \begin{pmatrix} \gamma^2 - \alpha^3 + 2 \gamma^2 \beta - \beta \alpha^2 \gamma \beta^2 - \alpha \beta^2 & 0 \\ 3\beta \alpha^2 - 3\gamma^2 \beta & \delta & 0 & 3\beta^2 \gamma - 3\beta^2 \alpha \\ 3\beta^2 \gamma - 3\beta^2 \alpha & 0 & \delta & 3\beta \alpha^2 - 3\gamma^2 \beta \\ 0 & \gamma \beta^2 - \alpha \beta^2 & \gamma^2 \beta - \beta \alpha^2 & \gamma^3 - \alpha^3 + 2 \end{pmatrix},
\]

\[
\delta = \alpha^2 \gamma + 2\alpha \beta^2 - \alpha \gamma^2 - 2\gamma \beta^2 + 2
\]

\[
V_1 = \begin{pmatrix} d\alpha^3 + \omega_x^2(\alpha \beta_4 - \gamma(\beta_4 + \beta_1)) \\ -3d\alpha^2 \gamma + \omega_x^2(\alpha(\beta_5 + \beta_2) - \gamma(\beta_5 + \beta_6)) \\ 3d\alpha \gamma^2 + \omega_x^2(-\gamma(\beta_5 + \beta_2) + \alpha(\beta_5 + \beta_6)) \\ -d\gamma^3 + \omega_x^2(-\gamma \beta_4 + \alpha(\beta_7 + \beta_1)) \end{pmatrix}
\]

\[
A_2 = \begin{pmatrix} \beta^3 + 1 & \gamma \beta^2 & \gamma^2 \beta & \gamma^3 \\ -3\beta^2 \alpha & 1 - \beta^3 - 2\alpha \beta \gamma & -\alpha \gamma^2 - 2\gamma \beta^2 - 3\beta \gamma^2 \\ 3\beta \alpha^2 & \gamma \alpha^2 + 2\alpha \beta^2 & 1 + \beta^3 + 2\alpha \beta \gamma & 3\gamma \beta^2 \\ -\alpha^3 & -\alpha^2 \beta & -\beta^2 \alpha & 1 - \beta^3 \end{pmatrix}
\]

and

\[
V_2 = \omega_x^2 \begin{pmatrix} -\gamma \beta_4 + \alpha \beta_1 \\ -\gamma \beta_5 + \alpha \beta_3 \\ -\gamma \beta_6 + \alpha \beta_2 \\ -\gamma \beta_7 \end{pmatrix},
\]

where the coefficients \( \beta_i \) are defined in (56) and (57).
We now study the first system in (98). Since \( \beta = \alpha - 1 \) and \( \gamma = \alpha - 2 \), \( A_1 \) has a simpler form:

\[
A_1 = \beta^2 \begin{pmatrix} -6 & -4 & -2 & 0 \\ 12 & 6 & 0 & -6 \\ -6 & 0 & 6 & 12 \\ 0 & -2 & -4 & -6 \end{pmatrix}.
\]

The compatibility conditions yield:

\[
d = -\frac{\beta \omega^2}{2\alpha} (\beta_1 + \beta_2 + \beta_3)
\]

(note that this expression is slightly different from (97)).

Using (56) and (54), we obtain \( d = B \beta \omega^2/2\alpha \) and one checks as above that there exists a solution \( (\delta_1, \delta_2, \delta_3, \delta_4) \) of (98). This complete the proof of Lemma 5.

\[\square\]

5. Study of the reduced mapping

In this section, we study small amplitude bifurcating solutions of the reduced mapping (55). We focus our attention on homoclinic and heteroclinic solutions, which will be related in Section 6 to breather and “dark breather” solutions of the FPU system. As we shall see, these results heavily rely on the reversibility of the reduced system. Note that one could complete our analysis by studying the existence of quasi-periodic orbits on invariant tori ([31]) and periodic orbits.

We shall not examine the question of transverse intersections of stable and unstable manifolds in homoclinic orbits. The case of a transverse intersection is generic and yields a rich variety of solutions, because there exists an invariant Cantor set on which some iterate of the map is topologically conjugate to a full shift on \( N \) symbols (([16])). Proving transverse intersections is particularly difficult in our context because the splitting size is beyond all orders in \( \mu \) (hence Melnikov theory cannot be applied directly) and the map is a priori not analytic (see [24]). The lack of analyticity enables us to use recent techniques for proving exponentially small splitting of separatrices (see [19,9,10] and references therein).

For studying the reduced mapping we shall consider the normal form (64). A mapping having the same form as (64) has been studied in [24] (see Section 6.2.3). The difference with respect to [24] is that higher order terms are present in the reversibility symmetry (see (70)–(72)) but this detail does not change the results. Consequently we shall refer to [24] for the study of the normal form (existence of homoclinic and heteroclinic orbits are obtained using reversibility and approximation by a flow).

Let us start with the case when \((\omega_c^2, m) \in \Gamma_1^+\).

**Lemma 6.** Assume \((\omega_c^2, m) \in \Gamma_1^+\) and \( B = (1/2)V^{(4)}(0) - (V^{(3)}(0))^2 \neq 0 \). For \( \mu = \omega^2 - \omega_c^2 = \omega^2 - 2(1 + (1/m)) \approx 0 \), the recurrence relation (55) has the following solutions.

(i) For \( \mu > 0 \) and \( B > 0 \), (55) has at least two homoclinic solutions \((a_1^+, b_1^+), (a_2^+, b_2^+)\) such that

\[
\lim_{n \to \pm \infty} (a_n^+, b_n^+) = 0.
\]

These solutions have the symmetries

\[
-R(a_{-n}^+, b_{-n}^+) = (a_n^+, b_n^+), \quad -R(a_{-n+1}^+, b_{-n+1}^+) = (a_n^+, b_n^+)
\]

and satisfy for some \( C > 0 \):

\[
0 < b_n^+ \leq C \mu^{1/2} |z_1|^{-|n|}, \quad -C \mu^{1/2} |z_1|^{-|n|} \leq a_n^+ < 0 \quad \text{and} \quad |a_n^+ + b_n^+| \leq C \mu |z_1|^{-|n|} (i = 1, 2), \text{where} \ |z_1| = 1 + O(\mu^{1/2}) > 1.
\]
(ii) If $\mu$ and $B$ have the same sign, (55) has two symmetric fixed points $\pm(a^*, -a^*)$ with $a^* = O(|\mu|^{1/2})$.

(iii) For $\mu < 0$ and $B < 0$, (55) has at least two heteroclinic solutions $(a_n^{3+}, b_n^{3+})$, $(a_n^{4+}, b_n^{4+})$ (with the other solutions $-(a_n^{3+}, b_n^{3+})$, $-(a_n^{4+}, b_n^{4+})$) such that $\lim_{n \to \pm\infty} (a_n^{\pm}, b_n^{\pm}) = \pm(a^*, -a^*)$. These solutions have the symmetries

\[ R(a_{n+1}^{3+}, b_{n+1}^{3+}) = (a_{n}^{3+}, b_{n}^{3+}), \quad R(a_{n+1}^{4+}, b_{n+1}^{4+}) = (a_{n}^{4+}, b_{n}^{4+}). \]

Moreover, $(a_n^{3+}, b_n^{3+})$, $(a_n^{4+}, b_n^{4+})$ are $O(|\mu|^{1/2})$ as $n \to \pm\infty$ and $O(|\mu|)$ for bounded values of $n$.

**Proof.** Using Lemma 5, we write (55) in the normal form

\[
\begin{pmatrix}
  A_{n+1} \\
  B_{n+1}
\end{pmatrix} = G_\mu \begin{pmatrix}
  A_n \\
  B_n
\end{pmatrix},
\]

where

\[
G_\mu \begin{pmatrix}
  A_n \\
  B_n
\end{pmatrix} = \begin{pmatrix}
  1 & 1 \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  A_n \\
  B_n
\end{pmatrix} + g_\mu(A_n, B_n) \begin{pmatrix}
  1 \\
  1
\end{pmatrix},
\]

\[
g_\mu(A_n, B_n) = 2(m + 1)\mu A_n - \frac{1}{4}m(1 + \alpha)^2 B A_n^4 + o(||(A_n, B_n)||^3 + \mu ||(A_n, B_n)||).\]

The map $G_\mu$ commutes with $-I$. Moreover, (99) is reversible with respect to involutions $\mathcal{R}_\mu^+$ and $-\mathcal{R}_\mu^+$ given by

\[ \mathcal{R}_\mu^+ \begin{pmatrix}
  A \\
  B
\end{pmatrix} = \begin{pmatrix}
  -1 & 1 \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  A \\
  B
\end{pmatrix} + o(||(A, B)||^3 + \mu ||(A, B)||) \]

and $\mathcal{R}_\mu^-$ commutes with $-I$. Since the involutions $\pm\mathcal{R}_\mu^+$ are reversors of (99), it is a classical result that (99) is also reversible with respect to $\pm\mathcal{R}_\mu^+ G_\mu^p$ ($p \in \mathbb{Z}$). For example, $\pm\mathcal{S}_\mu^+ = \pm\mathcal{R}_\mu^+ G_\mu$ are also reversors. An orbit $(A_n, B_n)$ is said to be reversible with respect to $\mathcal{R}_\mu^+$ if it has the symmetry $(A_{-n}, B_{-n}) = \mathcal{R}_\mu^-(A_n, B_n)$. One can check that any reversible orbit with respect to $\pm\mathcal{R}_\mu^+ G_\mu^p$ is a shift of a reversible orbit under $\pm\mathcal{R}_\mu^+$ or $\pm\mathcal{S}_\mu^+$. Consequently, we only consider the reversors $\pm\mathcal{R}_\mu^+$ and $\pm\mathcal{S}_\mu^+$ in the sequel.

We now discuss the existence of small amplitude homoclinic and heteroclinic solutions of (99) for $\mu \approx 0$. These results heavily rely on the reversibility of (99). Their proof can be found e.g. in [24], Section 6.2.3. The difference with respect to [24] is that higher order terms are present in (102), but this detail does not change the results.

The fixed point $(A, B) = 0$ of (99) is hyperbolic for $\mu > 0$ and elliptic for $\mu < 0$. In the case when $\mu$ and $B$ have the same sign, (99) has two other symmetric fixed points $(\pm A^*, 0)$ satisfying $\mathcal{R}_\mu^+(A^*, 0) = (\pm A^*, 0)$. These fixed points are elliptic for $\mu > 0$, $B > 0$, hyperbolic for $\mu < 0$, $B < 0$ and one has $A^* = (8\mu/(1 + \alpha)B)^{1/2} + O(|\mu|)$.

In the case when $\mu > 0$ and $B > 0$, (99) has reversible solutions $(A_n^{1+}, B_n^{1+})$, $(A_n^{2+}, B_n^{2+})$ homoclinic to $(A, B) = 0$ and satisfying

\[
-\mathcal{R}_\mu^+(A_{n-1}^{1+}, B_{n-1}^{1+}) = (A_n^{1+}, B_n^{1+}), \quad -\mathcal{S}_\mu^+(A_{n-1}^{2+}, B_{n-1}^{2+}) = (A_n^{2+}, B_n^{2+})
\]

(the same holds for the symmetric solutions $-(A_n^{1+}, B_n^{1+})$, $-(A_n^{2+}, B_n^{2+})$). Moreover, one has $0 < A_n^{1+} \leq C_1 |\mu|^{1/2} |\xi_1|^{-|n|}$ and $|B_n^{1+}| \leq C_2 |\mu| |\xi_1|^{-|n|}$ ($i = 1, 2$, where $|\xi_1| = 1 + O(\mu^{1/2}) > 1$).

In the case when $B < 0$ and $\mu > 0$ ($\mu \approx 0$), the local stable and unstable manifolds of $(A, B) = 0$ do not intersect in a small neighbourhood of 0. Consequently, there exist no small amplitude homoclinic orbits to $(A, B) = 0$ in this parameter range.
In the case when $\mu < 0$ and $B < 0$, (99) has reversible heteroclinic solutions $(A_n^{3+}, B_n^{3+}), (A_n^{4+}, B_n^{4+})$ (with also $-(A_n^{3-}, B_n^{3+}), -(A_n^{4+}, B_n^{4+})$) connecting the hyperbolic fixed points $(\pm A^*, 0)$. They satisfy

$$\lim_{n \to \pm \infty} (A_n^{\pm}, B_n^{\pm}) = (\pm A^*, 0),$$

with

$$R_\mu^+(A_n^{3+}, B_n^{3+}) = (A_n^{3+}, B_n^{3+}), \quad S_\mu^+(A_n^{4+}, B_n^{4+}) = (A_n^{4+}, B_n^{4+}).$$

Moreover, $B_n^{3^\pm}, B_n^{4^\pm}$ are $O(|\mu|)$, and $A_n^{3^\pm}, A_n^{4^\pm}$ are $O(|\mu|^{1/2})$ as $n \to \pm \infty$, $O(|\mu|)$ for bounded values of $n$.

The above analysis of the normal form (64) allows us to describe small amplitude homoclinic and heteroclinic solutions of the original mapping (55), which completes the proof.

We now consider the case when $(\omega_c^2, m) \in \Gamma_1^-$.

**Lemma 7.** Assume $(\omega_c^2, m) \in \Gamma_1^-$ and $B = (1/2)V^{(4)}(0) - (V^{(3)}(0))^2 \neq 0$. For $\mu = \omega^2 - \omega_c^2 = \omega^2 - (2/m) \approx 0$, the recurrence relation (55) has the following solutions.

(i) For $\mu < 0$ and $B < 0$, (55) has at least two homoclinic solutions $(a_n^{1-}, b_n^{1-}), (a_n^{2-}, b_n^{2-})$ such that

$$\lim_{n \to \pm \infty} (a_n^{1-}, b_n^{1-}) = 0.$$ 

These solutions have the symmetries

$$-R(a_n^{1-}, b_n^{1-}) = (a_n^{1-}, b_n^{1-}), \quad R(a_n^{2-}, b_n^{2-}) = (a_n^{2-}, b_n^{2-})$$

and satisfy for some $C > 0$: $0 < b_n^{1-} \leq C|\mu|^{1/2}|z_1|^{-|n|}$, $-C|\mu|^{1/2}|z_1|^{-|n|} \leq a_n^{1-} < 0$ and $|a_n^{1-} + b_n^{1-}| \leq C|\mu||z_1|^{-|n|} (i = 1, 2)$, where $|z_1| = 1 + O(|\mu|^{1/2}) > 1$.

(ii) If $\mu$ and $B$ have the same sign, (55) has a period-2 orbit $(a_n^{0}, b_n^{0}) = (-1)^n(a^*, -a^*)$ with $a^* = O(|\mu|^{1/2})$.

(iii) For $\mu > 0$ and $B > 0$, (55) has at least two heteroclinic solutions $(a_n^{3-}, b_n^{3-}), (a_n^{4-}, b_n^{4-})$ (with the other solutions $- (a_n^{3-}, b_n^{3-}), -(a_n^{4-}, b_n^{4-})$) such that $\lim_{n \to \pm \infty} |(a_n^{3-}, b_n^{3-})| = 0$. These solutions have the symmetries

$$R(a_n^{3-}, b_n^{3-}) = (a_n^{3-}, b_n^{3-}), \quad -R(a_n^{4-}, b_n^{4-}) = (a_n^{4-}, b_n^{4-}).$$

Moreover, $(a_n^{1-}, b_n^{1-}), (a_n^{2-}, b_n^{2-})$ are $O(|\mu|^{1/2})$ as $n \to \pm \infty$ and $O(\mu)$ for bounded values of $n$.

**Proof.** Using Lemma 5, we write (55) in the normal form

$$
\begin{pmatrix}
A_{n+1} \\
B_{n+1}
\end{pmatrix}
= G_\mu
\begin{pmatrix}
A_n \\
B_n
\end{pmatrix},
$$

(103)

where

$$G_\mu
\begin{pmatrix}
A_n \\
B_n
\end{pmatrix}
= \begin{pmatrix}
-1 & -1 \\
0 & -1
\end{pmatrix}
+ g_\mu(A_n, B_n)
\begin{pmatrix}
1 \\
1
\end{pmatrix},$$

(104)

$$g_\mu(A_n, B_n) = 2(1 - m)\mu A_n + \frac{1 - \alpha}{4} BA_n^3 + o(||(A_n, B_n)||^3 + \mu ||(A_n, B_n)||).$$

(105)
In order to recover the case of a double eigenvalue $+1$ considered above, one makes the change of variable $(\tilde{A}_n, \tilde{B}_n) = (-1)^n (A_n, B_n)$ (this yields an autonomous mapping since $G_\mu$ commutes with $-I$). We obtain
\[
\begin{pmatrix}
\tilde{A}_{n+1} \\
\tilde{B}_{n+1}
\end{pmatrix} = \tilde{G}_\mu \begin{pmatrix}
\tilde{A}_n \\
\tilde{B}_n
\end{pmatrix},
\]
where $\tilde{G}_\mu = -G_\mu$ has the same structure as (100). The maps $G_\mu, \tilde{G}_\mu$ commute with $-I$ and are reversible with respect to involutions $\pm R_\mu^-$ having the form
\[
R_\mu^-(A, B) = \begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix} (A, B) + o(\|(A, B)\|^3 + \mu\|(A, B)\|).
\]
We shall also consider the complementary reversors $\pm S_\mu^- = \pm R_\mu^- G_\mu$.

The description of small amplitude homoclinic and heteroclinic solutions of (106) has been done previously and yields the following results for the original mapping (103).

The fixed point $(A, B) = 0$ of (103) is hyperbolic for $\mu < 0$ and elliptic for $\mu > 0$. In the case when $\mu$ and $B$ have the same sign, (103) has a period-2 orbit $(A_0^0, B_0^0) = (-1)^n (A_0^0, 0)$ with $R_\mu^-(A_0^0, 0) = (-A_0^0, 0)$ and $A_0^+ = ((8m/B)\mu)^{1/2} + O(\mu))$. This orbit is elliptic for $\mu < 0$, $B < 0$ and hyperbolic for $\mu > 0$, $B > 0$. In the case when $\mu < 0$ and $B < 0$, (103) has reversible solutions $(A_1^-, B_1^-), (A_2^-, B_2^-)$ homoclinic to $(A, B) = 0$ and satisfying
\[
-R_\mu^-(A_1^-, B_1^-) = (A_1^-, B_1^-), \quad S_\mu^-(A_2^-, B_2^-) = (A_2^-, B_2^-)
\]
(the same holds for the symmetric solutions $-(A_1^-, B_1^-), -(A_2^-, B_2^-))$. Moreover, one has $0 < (-1)^n a_i A_i^- \leq C\mu^{1/2} |z|^{-[n]}$ and $|B_i^-| \leq C\mu |z|^{-[n]}$ ($i = 1, 2$, where $|z| = 1 + O(\mu^{1/2}) > 1$).

In the case when $B > 0$ and $\mu < 0$ ($\mu \approx 0$), the stable and unstable manifolds of $(A, B) = 0$ do not intersect in a small neighbourhood of 0 and thus there exist no small amplitude homoclinic orbits to $(A, B) = 0$.

In the case when $\mu > 0$ and $B > 0$, (103) has reversible homoclinic solutions $(A_3^-, B_3^-), (A_4^-, B_4^-)$ (with also $-(A_3^-, B_3^-), -(A_4^-, B_4^-)$) connecting the period-2 orbit $(A_0^0, B_0^0)$ with the shifted orbit $(A_{n+1}^0, B_{n+1}^0)$. Note that these solutions can be considered as heteroclinic solutions connecting $(A_0^0, B_0^0)$ with $-(A_0^0, B_0^0)$ (one has $(A_{n+1}^0, B_{n+1}^0) = -(A_n^0, B_n^0)$). They satisfy $\lim_{n \to \pm \infty} (-1)^n (A_n^-, B_n^-) = (\pm A^+, 0)$, with
\[
R_\mu^-(A_3^-, B_3^-) = (A_3^-, B_3^-), \quad -S_\mu^-(A_4^-, B_4^-) = (A_4^-, B_4^-).
\]
Moreover, $B_3^-, B_4^-$ are $O(\mu)$, and $A_3^-, A_4^-$ are $O(\mu^{1/2})$ as $n \to \pm \infty$, $O(\mu)$ for bounded values of $n$.

The above analysis of the normal form (64) provides small amplitude homoclinic and heteroclinic solutions of the original mapping (55), which completes the proof.

Lastly, we consider the case when $(\omega^2, m) \in \Gamma_1^0 (p \geq 1)$.

**Lemma 8.** Assume $(\omega^2, m) \in \Gamma_1^0 (p \geq 1)$ and $B = (1/2)V^2(0) - (V^3(0))^2 \neq 0$. For $\mu = \omega^2 - \omega_0^2 = \omega^2 - 2 \approx 0$, the recurrence relation (55) has the following solutions.

(i) For $\mu > 0$ and $B > 0$, (55) has at least two homoclinic solutions $(a_{n}^2, b_{n}^2)$, $(a_{n}^2, b_{n}^2)$ such that $\lim_{n \to \pm \infty} (a_{n}^2, b_{n}^2) = 0$. These solutions have the symmetries
\[
R(a_{n}^{2a}, b_{n}^{2a}) = (a_{n}^{2a}, b_{n}^{2a}), \quad -R(a_{n}^{2a}, b_{n}^{2a}) = (a_{n}^{2a}, b_{n}^{2a})
\]
and satisfy for some $C > 0: 0 < a_n^{1i} \leq C \mu^{1/2} |z_1|^{-|n|}, 0 < b_n^{1i} \leq C \mu^{1/2} |z_1|^{-|n|},$ and $|a_n^{1i} - b_n^{1i}| \leq C \mu |z_1|^{-|n|} (i = 1, 2),$ where $|z_1| = 1 + O(\mu^{1/2}) > 1$.

(ii) If $\mu$ and $B$ have the same sign, (55) has a period-2 orbit $(a_n^0, b_n^0) = (-1)^n (a^*, a^*)$ with $a^* = O(|\mu|^{1/2})$.

(iii) For $\mu < 0$ and $B < 0$, (55) has at least two heteroclinic solutions $(a_n^{3i}, b_n^{3i}), (a_n^{4i}, b_n^{4i})$ (with the other solutions $-(a_n^{3i}, b_n^{3i}), -(a_n^{4i}, b_n^{4i})$) such that $\lim_{n \to \pm \infty} |(a_n^{1i}, b_n^{1i}) \mp (a_n^0, b_n^0)| = 0$. These solutions have the symmetries

$$-R(a_n^{3i}, b_n^{3i}) = (a_n^{3i}, b_n^{3i}), \quad R(a_n^{4i}, b_n^{4i}) = (a_n^{4i}, b_n^{4i}).$$

Moreover, $(a_n^{3i}, b_n^{3i}), (a_n^{4i}, b_n^{4i})$ are $O(|\mu|^{1/2})$ as $n \to \pm \infty$ and $O(|\mu|)$ for bounded values of $n$.

**Proof.** Using Lemma 5, we write (55) in the normal form

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = G_{\mu} \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \quad \text{(108)}$$

where

$$G_{\mu} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + g_{\mu}(A_n, B_n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{(109)}$$

$$g_{\mu}(A_n, B_n) = 2(m - 1)\mu A_n + \frac{1}{3} (1 - m) BA_n^3 + o(||(A_n, B_n)||^3 + \mu ||(A_n, B_n)||). \quad \text{(110)}$$

The map $G_{\mu}$ commutes with $-I$ and is reversible with respect to involutions $\pm R_{\mu}^a$ having the form

$$R_{\mu}^a \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o(||(A, B)||^3 + \mu ||(A, B)||). \quad \text{(111)}$$

We shall also consider the complementary reversors $\pm S_{\mu}^a = \pm R_{\mu}^a G_{\mu}$.

The structure of (109) is the same as in (104) but coefficients of $g_{\mu}$ have opposite signs. Moreover, reversers (71), (72) have opposite principal parts. These variations induce several differences with the bifurcation results obtained for (104).

The fixed point $(A, B) = 0$ of (103) is hyperbolic for $\mu > 0$ and elliptic for $\mu < 0$. In the case when $\mu$ and $B$ have the same sign, (108) has a period-2 orbit $(A_n^0, B_n^0) = (-1)^n (A^*, 0)$ with $R_{\mu}^a(A^*, 0) = (A^*, 0)$ and $A^* = (8\mu/B)^{1/2} + O(|\mu|).$ This orbit is elliptic for $\mu > 0$, $B > 0$ and hyperbolic for $\mu < 0, B < 0$.

In the case when $\mu > 0$ and $B > 0$, (103) has reversible solutions $(A_n^{1a}, B_n^{1a}), (A_n^{2a}, B_n^{2a})$ homoclinic to $(A, B) = 0$ and satisfying

$$R_{\mu}^a(A_n^{1a}, B_n^{1a}) = (A_n^{1a}, B_n^{1a}), \quad -S_{\mu}^a(A_n^{2a}, B_n^{2a}) = (A_n^{2a}, B_n^{2a})$$

(the same holds for the symmetric solutions $-A_n^{1a}, B_n^{1a}), -(A_n^{2a}, B_n^{2a})$). Moreover, one has $0 < (-1)^n A_n^{1a} \leq C \mu^{1/2} |z_1|^{-|n|}$ and $|B_n^{1a}| \leq C \mu |z_1|^{-|n|} (i = 1, 2),$ where $|z_1| = 1 + O(\mu^{1/2}) > 1$.

In the case when $B \leq 0$ and $\mu > 0 (\mu \approx 0)$, the stable and unstable manifolds of $(A, B) = 0$ do not intersect in a small neighbourhood of 0 and thus there exist no small amplitude homoclinic orbits to $(A, B) = 0$.

In the case when $\mu < 0$ and $B < 0$, (108) has reversible homoclinic solutions $(A_n^{3a}, B_n^{3a}), (A_n^{4a}, B_n^{4a})$ (with also $-(A_n^{3a}, B_n^{3a}), -(A_n^{4a}, B_n^{4a}))$ connecting the period-2 orbit $(A_n^0, B_n^0)$ with $(A_n^{3a}, B_n^{3a})$. These solutions can again be considered as heteroclinic solutions connecting $(A_n^0, B_n^0)$ with $-(A_n^0, B_n^0)$, since $(A_n^{3a}, B_n^{3a}) = -(A_n^{0}, B_n^{0})$. They
satisfy \( \lim_{n \to \pm \infty} (-1)^n (A_n^{4a}, B_n^{4a}) = (\pm A^*, 0) \), with
\[
-R_\mu^a(A_{-n}^{3a}, B_{-n}^{3a}) = (A_{n}^{3a}, B_{n}^{3a}), \quad S_\mu^a(A_{-n}^{4a}, B_{-n}^{4a}) = (A_{n}^{4a}, B_{n}^{4a}).
\]

Moreover, \( B_{n}^{3a}, B_{n}^{4a} \) are \( O(1) \), and \( A_{n}^{3a}, A_{n}^{4a} \) are \( O(1) \) as \( n \to \pm \infty \), \( O(1) \) for bounded values of \( n \).

This analysis of the normal form (64) provides small amplitude homoclinic and heteroclinic solutions of the original mapping (55), which completes the proof. \( \square \)

If \( (\omega^2, m) \in \Gamma^+_1 \) or \( (\omega^2, m) \in \Gamma^0_1, \) \( B < 0 \) and \( \mu > 0 \) \( (\mu \approx 0) \), note that the local stable and unstable manifolds of \((a, b) = 0\) do not intersect and thus (55) has no small amplitude homoclinic solution. In the same way, (55) has no small amplitude homoclinic solution to \((a, b) = 0\) for \((\omega^2, m) \in \Gamma^-_1, B > 0 \) and \( \mu < 0 \) \( (\mu \approx 0) \).

6. Breathers and “dark” breathers

In this section, we deduce the existence of breathers and “dark” breathers from the reduced mapping properties and study their spatial geometry. Their symmetries are of interest since they seem strongly related to stability (see [26] and references therein). We have divided our analysis into two parts.

Section 6.1 describes the solutions of (9) corresponding to the homoclinic and heteroclinic orbits found in Section 5 (breathers and “dark” breathers). It gives also a rather simple explanation of the fact that the coefficient \( B \) determining their existence is independent of \( m \).

Section 6.2 describes the shape of DB using the original displacement variable \( x_n \) of system (7). These results are compared to different numerical studies ([18,26,14]).

6.1. Existence results

According to Theorem 2, each solution \((a_n, b_n)\) in Lemmas 6, 7 and 8 corresponds to a solution \( Y_n = (u_n, v_n) \) of (9) for \( \omega^2 = \mu + \omega^2 \). This solution is given by
\[
Y_n = (a_n, b_n) \cos t + \Psi(a_n, b_n, \mu),
\]
where \( \Psi \in C^{1}(\mathbb{R}; \mathbb{D}) \) has the symmetries
\[
T \Psi(a, b, \mu) = \Psi(-a, -b, \mu), \quad R \Psi(a, b, \mu) = \Psi(b, a, \mu).
\]
We describe some of these solutions in Theorems 5–7 below (these results follow directly from Lemmas 6–8 and Eqs. (112), (113)). Solutions \( Y_n^{1+}, Y_n^{2+}, Y_n^{4a}, Y_n^{4a} \) below correspond to discrete breathers (they are time-periodic and spatially localized). They have a large extent \((\ln |z_1|)^{-1} \approx 1) \) but decay exponentially at infinity. Solutions \( Y_n^{3+}, Y_n^{4+}, Y_n^{4a}, Y_n^{4a} \) are sometimes referred as dark breathers, by analogy with dark solitons (see e.g. [1]). Indeed, oscillations have an amplitude \( O(|\omega - \omega_c|) \) as \( n \to \pm \infty \) and a smaller amplitude \( O(|\omega - \omega_c|) \) in the centre.

Let us start with the case when \( (\omega^2, m) \in \Gamma^+_1 \).

Theorem 5. Suppose \( B = (1/2)V^{(4)}(0) - (V^{(3)}(0))^2 \neq 0 \) and \( m \in (0, 1) \). For \( \omega \approx \omega_c = (2 + 2/m)^{1/2} \), problem (9) has the following solutions with \( Y_n \in \mathbb{D} \) for all \( n \in \mathbb{Z} \).

(i) For \( \omega > \omega_c \) and \( B > 0 \), (9) has at least two homoclinic solutions \( Y_n^{1+}, Y_n^{2+} \) (and also \( T Y_n^{1+}, T Y_n^{2+} \)) such that \( \lim_{n \to \pm \infty} \| Y_n^{1+} \|_{\mathbb{D}} = 0 \). These solutions satisfy
\[
TRY_n^{1+} = Y_n^{1+}, \quad TRY_n^{2+} = Y_n^{2+}.
\]
or equivalently
\[ T v^{1+}_n = u^{1+}_n, \quad T v^{2+}_{-n+1} = u^{2+}_n. \]

They have the form
\[ y^{l+}_n = (-b^{l+}_n, b^{l+}_n) \cos t + O(|\omega - \omega_c|), \]
where \( 0 < b^{l+}_n \leq C(\omega - \omega_c)^{1/2} |z_1|^{-|n|}, |z_1| = 1 + O((\omega - \omega_c)^{1/2}) > 1. \)

(ii) If \( \omega = \omega_c \) and \( B \) have the same sign, (9) has two symmetric fixed points \( Y^{0+}, TY^{0+} \in D \). They have the form \( Y^{0+} = (0), Ty^{0+} \) with \( y^{0+}(t) = \alpha^* \cos t + O(|\omega - \omega_c|) (y^{0+} \in H^0_2) \) and \( \alpha^* = O(|\omega - \omega_c|^{1/2}) \).

(iii) For \( \omega < \omega_c \) and \( B < 0, (9) \) has at least two heteroclinic solutions \( Y^{3+}_n, Y^{4+}_n \) (and also \( TY^{3+}_n, TY^{4+}_n \)) such that \( \lim_{n \to -\infty} \|Y^{l+}_n - Ty^{l+}_n\|_D = 0, \lim_{n \to +\infty} \|Y^{l+}_n - Ty^{l+}_n\|_D = 0 \). These solutions satisfy
\[ \begin{align*}
RY^{3+}_{-n} &= Y^{3+}_n, \\
RY^{4+}_{-n+1} &= Y^{4+}_n,
\end{align*} \]
or equivalently
\[ \begin{align*}
u^{3+}_{-n} &= u^{3+}_n, \\
\nu^{4+}_{-n+1} &= u^{4+}_n.
\end{align*} \]

Moreover, \( \|Y^{3+}_n\|_D, \|Y^{4+}_n\|_D \) are \( O(|\omega - \omega_c|^{1/2}) \) as \( n \to \pm \infty \) and \( O(|\omega - \omega_c|) \) for bounded values of \( n \).

We now consider the case when \( (\omega_c^2, m) \in \varGamma_1^+ \).

**Theorem 6.** Suppose \( B = (1/2)V^{(4)}(0) - (V^{(3)}(0))^2 \neq 0 \) and \( m \in (0, 1) \). For \( \omega \approx \omega_c = (2/m)^{1/2} \), problem (9) has the following solutions with \( Y_n \in D \) for all \( n \in \mathbb{Z} \).

(i) For \( \omega < \omega_c \) and \( B < 0, (9) \) has at least two homoclinic solutions \( Y^{1-}_n, Y^{2-}_n \) (and also \( TY^{1-}_n, TY^{2-}_n \)) such that \( \lim_{n \to \pm \infty} \|Y^{l-}_n\|_D = 0 \). These solutions satisfy
\[ \begin{align*}
TRY^{1-}_{-n} &= Y^{1-}_n, \\
RY^{2-}_{-n+1} &= Y^{2-}_n,
\end{align*} \]
or equivalently
\[ \begin{align*}
v^{1-}_{-n} &= u^{1-}_n, \\
v^{2-}_{-n+1} &= u^{2-}_n.
\end{align*} \]

They have the form
\[ y^{l-}_n = (-b^{l-}_n, b^{l-}_n) \cos t + O(|\omega - \omega_c|), \]
where \( 0 < b^{l-}_n \leq C(\omega - \omega_c)^{1/2} |z_1|^{-|n|}, |z_1| = 1 + O((\omega - \omega_c)^{1/2}) > 1. \)

(ii) If \( \omega = \omega_c \) and \( B \) have the same sign, (9) has a solution \( y^{0-} \) being 2-periodic in \( n \). It has the form \( y^{0-} = (T^n y^0, T^{n+1} y^0) \) with \( y^0(t) = \alpha^* \cos t + O(|\omega - \omega_c|) (y^0 \in H^0_2) \) and \( \alpha^* = O(|\omega - \omega_c|^{1/2}) \).

(iii) For \( \omega > \omega_c \) and \( B > 0, (9) \) has at least two heteroclinic solutions \( Y^{3-}_n, Y^{4-}_n \) (and also \( TY^{3-}_n, TY^{4-}_n \)) such that \( \lim_{n \to -\infty} \|Y^{l-}_n - Ty^{l-}_n\|_D = 0, \lim_{n \to +\infty} \|Y^{l-}_n - Ty^{l-}_n\|_D = 0 \). These solutions satisfy
\[ \begin{align*}
RY^{3-}_{-n} &= Y^{3-}_n, \\
TRY^{4-}_{-n+1} &= Y^{4-}_n,
\end{align*} \]
or equivalently
\[ v_{-n}^3 - u_{n}^3 = T v_{-n+1}^4 - u_{n}^4. \]

Moreover, \( \|Y_{n}^{3}\|_{\mathbb{D}}, \|Y_{n}^{4}\|_{\mathbb{D}} \) are \( O(|\omega - \omega_{c}|^{1/2}) \) as \( n \to \pm \infty \) and \( O(|\omega - \omega_{c}|) \) for bounded values of \( n \).

Lastly, we consider the case when \((\omega_{c}^2, m) \in I_{1}^{k}, (k \geq 1)\).

**Theorem 7.** Suppose \( B = (1/2)V^{(4)}(0) - (V^{(3)}(0))^2 \neq 0 \) and \( m \in (1/k(k + 2), 1/k^2) (k \geq 1) \). For \( \omega \approx \omega_{c} = \sqrt{2}, \) problem (9) has the following solutions with \( Y_{n} \in \mathbb{D} \) for all \( n \in \mathbb{Z} \).

(i) For \( \omega > \omega_{c} \) and \( B > 0 \), (9) has at least two homoclinic solutions \( Y_{n}^{1a}, Y_{n}^{2a} \) (and also \( T Y_{n}^{1a}, T Y_{n}^{2a} \)) such that \( \lim_{n \to \pm \infty} \|Y_{n}^{1a}\|_{\mathbb{D}} = 0 \). These solutions satisfy
\[ R Y_{-n}^{1a} = Y_{n}^{1a}, \quad T R Y_{-n+1}^{2a} = Y_{n}^{2a}, \]
or equivalently
\[ v_{-n}^{1a} = u_{n}^{1a}, \quad T v_{-n+1}^{2a} = u_{n}^{2a}. \]

They have the form
\[ Y_{n}^{1a} = (b_{n}^{1a}, b_{n}^{2a}) \cos t + O(|\omega - \omega_{c}|). \]
where \( 0 < b_{n}^{1a} \leq C|\omega - \omega_{c}|^{1/2}|z_{1}|^{-|n|}, |z_{1}| = 1 + O(|\omega - \omega_{c}|^{1/2}) > 1 \).

(ii) If \( \omega - \omega_{c} \) and \( B \) have the same sign, (9) has a solution \( Y_{n}^{0a} \) being 2-periodic in \( n \). It has the form \( Y_{n}^{0a} = T^n(y^0, y^0) \) with \( y^0(t) = a^r \cos t + O(|\omega - \omega_{c}|) \) \( y^0 \in H_{0}^2 \) and \( a^r = O(|\omega - \omega_{c}|^{1/2}) \).

(iii) For \( \omega < \omega_{c} \) and \( B < 0 \), (9) has at least two heteroclinic solutions \( Y_{n}^{3a}, Y_{n}^{4a} \) (and also \( T Y_{n}^{3a}, T Y_{n}^{4a} \)) such that \( \lim_{n \to -\infty} \|Y_{n}^{3a} - Y_{n}^{0a}\|_{\mathbb{D}} = 0, \lim_{n \to +\infty} \|Y_{n}^{4a} - Y_{n}^{0a}\|_{\mathbb{D}} = 0 \). These solutions satisfy
\[ T R Y_{-n}^{3a} = Y_{n}^{3a}, \quad R Y_{-n+1}^{4a} = Y_{n}^{4a}, \]
or equivalently
\[ T v_{-n}^{3a} = u_{n}^{3a}, \quad v_{-n+1}^{4a} = u_{n}^{4a}. \]
Moreover, \( \|Y_{n}^{3a}\|_{\mathbb{D}}, \|Y_{n}^{4a}\|_{\mathbb{D}} \) are \( O(|\omega - \omega_{c}|^{1/2}) \) as \( n \to \pm \infty \) and \( O(|\omega - \omega_{c}|) \) for bounded values of \( n \).

In addition, note that for \( B < 0 \) and \( (\omega_{c}^2, m) \in I_{1}^{k} \) or \( I_{2}^{k} \), there exists no small amplitude discrete breather solution \( Y_{n} \in \mathbb{D} \) with \( \omega > \omega_{c} \) and \( \omega \approx \omega_{c} \) (since (55) has no small amplitude solution homoclinic to 0). In the same way, for \( B > 0 \) and \( (\omega_{c}^2, m) \in I_{1}^{k} \), there exists no small amplitude breather \( Y_{n} \in \mathbb{D} \) with \( \omega < \omega_{c} \) and \( \omega \approx \omega_{c} \).

The above results provide a condition \( (B \neq 0) \) valid for general potentials for the existence of small amplitude breathers while previous numerical studies were only valid for particular potentials. The existence conditions \( B > 0 \) and \( B < 0 \) have a simple interpretation. The potential \( V \) is hard (resp. soft) around 0 if and only if \( V^{(4)}(0) - (5/3)(V^{(3)}(0))^2 > 0 \) (resp. < 0). Thus if \( B > 0 \) then \( V \) is a hard potential and there exist breathers with frequencies slightly above the acoustic band \( (\omega \approx \sqrt{2}, \omega > \sqrt{2}) \) and slightly above the optic band \( (\omega \approx (2 + 2/m)^{1/2}, \omega > (2 + 2/m)^{1/2}) \).
On the other hand, if \( V \) is a soft potential then \( B < 0 \) and there exist breathers with frequencies slightly below the optic band \( (\omega \approx (2/m)^{1/2}, \omega < (2/m)^{1/2}) \). This is the case for Lennard–Jones, Morse or Born–Mayer–Coulomb potentials used in [26,14,15].

We now discuss the a priori surprising fact that coefficient \( B \) in the DB existence conditions is independent of \( m \). Let us first investigate the case when \( (\omega_0^2, m) \in \Gamma^+_1 \). We have a bifurcation at a double eigenvalue \(-1\) and the analysis of the reduced mapping shows that the existence of small amplitude homoclinic solutions \( Y_n = (u_n, v_n) \) is equivalent to the existence of a pair of bifurcating fixed points. Consequently, let us look for non trivial solutions \((u_n, v_n) = (u, v)\) of (9), where \( u, v \) are time-periodic functions. The system (8) reads

\[
m \omega^2 \frac{d^2}{dt^2}(W(v)) = (m + 1)(u - v), \quad m \omega^2 \frac{d^2}{dt^2}(W(u)) = (m + 1)(v - u).
\]

We note \( \omega^2 = \omega_0^2(1 + (1/m)\bar{\mu}) = (1 + (1/m))(2 + \bar{\mu}) \), where \( \bar{\mu} \approx 0 \). This yields

\[
(2 + \bar{\mu}) \frac{d^2}{dt^2}(W(v)) = (u - v), \quad (2 + \bar{\mu}) \frac{d^2}{dt^2}(W(u)) = (v - u).
\]

The condition \( B > 0 \) leading to a pair of time-periodic solutions bifurcating from \((u, v) = (0, 0)\) as \( \bar{\mu} \approx 0^+ \) could be obtained from (115) by a standard Lyapounov–Schmidt procedure. We shall not detail this point here but it is clear that the condition \( B > 0 \) is independent of the mass ratio (since (115) does not depend on \( m \)).

We now consider the case when \((\omega_0^2, m) \in \Gamma^+_1 \). We have a bifurcation at a double eigenvalue \(-1\) and the study of the reduced mapping proves that the existence of small amplitude discrete breathers is equivalent to the existence of a small amplitude period 2 orbit \( Y_n^0 = (u_n, v_n) = (T^n y^0, T^{n+1} y^0) \) (see property ii) of Theorem 6). Consequently, let us look for solutions of (9) with \( u_{2n} = u, v_{2n} = v, u_{2n-1} = v \) and \( v_{2n-1} = u, \) where \( u, v \) are time-periodic functions. Let us note that in this case \( \omega^2 = \omega_0^2 + (1/m)\bar{\mu} = (1/m)(2 + \bar{\mu}) \approx 0 \). Then system (8) again takes the form (115) independent of \( m \). Consequently, the condition \( B < 0 \) for the existence of small amplitude DB is independent of the mass ratio \( m \).

We conclude with the case \((\omega_0^2, m) \in \Gamma^+_1 \), which also corresponds to a bifurcation at a double eigenvalue \(-1\). The existence of small amplitude homoclinic solutions is equivalent to the existence of a small amplitude solution \( Y_n^0 = (u_n, v_n) = T^n y^0, T^{n+1} y^0 \) being 2 periodic in \( n \) (see property (ii) of Theorem 7). Consequently, let us look for solutions of (9) with \( u_{2n} = u, v_{2n} = v, u_{2n-1} = v \) and \( v_{2n-1} = u \), where \( u, v \) are time-periodic functions. Setting \( \omega^2 = \omega_0^2 + \bar{\mu} = 2 + \bar{\mu} \approx 0 \), system (8) again takes the form (115) independent of \( m \).

Note that we have not investigated all the possible breather solutions for a given bifurcation. As we precised in Section 5 for the reduced map, the intersection of stable and unstable manifolds is generically transverse. In this case, the reduced map admits an infinity of homoclinic orbits to 0, each one corresponding to a different breather solution of the FPU system.

The next section examines the shape of the discrete breathers described in Theorems 5–7, using the original displacement variable \( x_n \). We shall focus our attention on breather solutions, but the case of dark breathers could be treated in the same way.

### 6.2. Discrete breathers geometry

The displacement variable \( x_n \) can be recovered using the following formula (integrate (7) and use the evenness of \( x_n \))

\[
M_n x_n(t) = M_n x_n(0) + \frac{1}{\omega^2} \int_0^t \int_0^{\tau} (y_{n+1}(s) - y_n(s)) \, ds \, d\tau,
\]

(116)
where \( M_{2n} = 1 \), \( M_{2n+1} = m \) and \( x_n(0) \) is obtained up to an arbitrary constant \( x_0(0) \) by the case \( t = 0 \) of
\[
x_n(t) = x_0(t) + \sum_{k=1}^{n} W(y_k(\omega t)), \quad n \geq 1, \quad x_n(t) = x_0(t) - \sum_{k=n}^{\infty} W(y_{k+1}(\omega t)), \quad n \leq -1.
\]
(117)

Given a solution \( Y_n \) of (9), formula (116)–(117) determine \( x_n \) up to an additive constant \( c \), due to the invariance \( x_n \to x_n + c \) in (7).

In the sequel, \( x^{1\pm}_n, x^{2\pm}_n \) denote displacement variables associated with the homoclinic solutions \( Y^{1\pm}_n, Y^{2\pm}_n \), and similarly \( x^{1a}_n, x^{2a}_n \) correspond to the homoclinic solutions \( Y^{1a}_n, Y^{2a}_n \). Displacements are described in the following lemma.

**Lemma 9.** Solutions \( Y^{i\pm}_n, Y^{ia}_n \) \((i = 1, 2)\) of (9) provided by Theorems 5, 6, 7 (property i) correspond via formula (116)–(117) to DB solutions of (7) \( x^{i\pm}_n, x^{ia}_n \) having the form
\[
x^{i\pm}_n(t) = c + d_n + X^{i\pm}_n(t)
\]
where \( X^{i\pm}_n \) is time-periodic (with frequency \( \omega \)), with 0 time-average and \( \| X^{i\pm}_n \|_{L\infty} \) decays exponentially as \( n \to \pm\infty \).

The stationary term \( d_n \) satisfies \( d_n = O(|\mu|) \) for any fixed \( \mu = \omega^2 - \omega^2 \) and \( \lim_{n\to\pm\infty} d_n = O(|\mu|^{1/2}) \). It has a kink shape if \( V(3)(0) \neq 0 \) (decreasing if \( V(3)(0) > 0 \) and increasing if \( V(3)(0) < 0 \). One has \( d_n = 0 \) in the special case when \( V \) is even. The constant \( c \in \mathbb{R} \) is arbitrary. The oscillatory parts \( X^{i\pm}_n \) have the form
\[
x^{i\pm}_{2n}(t) = -\frac{2b^{i\pm}_n}{\omega^2} \cos(\omega t) + O(|\mu|), \quad X^{i\pm}_{2n-1}(t) = \frac{2b^{i\pm}_n}{m\omega^2} \cos(\omega t) + O(|\mu|),
\]
(118)
\[
x^{i\pm}_{2n-1}(t) = O(|\mu|), \quad X^{i\pm}_{2n-1}(t) = \frac{2b^{i\pm}_n}{m\omega^2} \cos(\omega t) + O(|\mu|),
\]
(119)
\[
x^{ia}_n(t) = \frac{2b^{ia}_n}{\omega^2} \cos(\omega t) + O(|\mu|), \quad X^{ia}_{2n-1}(t) = O(|\mu|),
\]
(120)
where \( b^{i\pm}_n = O(|\mu|^{1/2}) \) are the homoclinic solutions of the reduced mapping described in Lemmas 6, 7, 8 (property i).

**Proof.** We recall that \( Y_n = (u_n, v_n) = (y_{2n}, y_{2n-1}) \) with in addition \( y_n = V(x_n - x_{n-1})(t/\omega) \). Eq. (7) yields
\[
x^{\mu}_{2n}(t) = v_{n+1}(\omega t) - u_n(\omega t), \quad mX^{\mu}_{2n-1}(t) = u_n(\omega t) - v_n(\omega t).
\]
(121)
Let us consider the case when \((\omega^2, m) \in \mathcal{F}_1^+ \) (we drop the \( i \) index in the notations). One obtains using (121) and Theorem 5 (property i)
\[
x^{\mu}_{2n}(t) = (b_n + b_{n+1}) \cos(\omega t) + O(|\mu|), \quad mX^{\mu}_{2n-1}(t) = -2b_n \cos(\omega t) + O(|\mu|).
\]
(122)
The orbits \((a_n, b_n)\) in Lemma 6 (property i) satisfy
\[
(a_{n+1}, b_{n+1}) = L_c(a_n, b_n) + O(|\mu|^{3/2}),
\]
(123)
where
\[
L_c = \begin{pmatrix} \alpha + 2 & 1 + \alpha \\ -1 - \alpha & -\alpha \end{pmatrix}
\]
\((\alpha = 1/m)\) and \(a_n = -b_n + O(|\mu|)\). This implies that
\[
b_{n+1} = b_n + O(|\mu|).
\]
(124)
One obtains Eq. (118) by inserting (124) in (122) and integrating twice.

We now turn to the case when \((\omega^2, m) \in I_i^\ast\). Using (121) and Theorem 6 (property i) leads again to Eqs. (122) and (123), with

\[
L_c = \begin{pmatrix}
\alpha - 2 \\ 1 - \alpha
\end{pmatrix}
\]

and \(a_n = -b_n + O(|\mu|)\) (see Lemma 7, property i). This implies that

\[
b_{n+1} = -b_n + O(|\mu|)
\]

and Eq. (119) follows by inserting (125) in (122) and integrating twice.

We now consider the case when \((\omega^2, m) \in I_i^0\). One obtains using (121) and Theorem 7 (property i)

\[
X''_{2n}(t) = (-b_n + b_{n+1}) \cos(\omega t) + O(|\mu|), \quad mX''_{2n-1}(t) = O(|\mu|).
\]

In Eq. (123), we have

\[
L_c = \begin{pmatrix}
\alpha - 2 & 1 - \alpha \\
1 - \alpha & -\alpha
\end{pmatrix}
\]

and \(a_n = b_n + O(|\mu|)\) (see Lemma 8, property i). This implies that

\[
b_{n+1} = -b_n + O(|\mu|)
\]

and Eq. (120) follows by inserting (127) in (126) and integrating twice.

We now consider the steady part of \(x_n\). According to Eq. (117) one has \(\bar{x}_n = c + d_n\) (the bar denotes time-average) where

\[
d_n = \sum_{k=1}^{n} W(y_{k}(\omega t)), \quad n \geq 1; \quad d_n = -\sum_{k=n+1}^{0} W(y_{k}(\omega t)), \quad n \leq -1,
\]

\(d_0 = 0\) and \(c = \bar{x}_0\). We recall that \(W(y_n) = y_n - (1/2)V^{(3)}(0)y_n^2 + O(|\mu|)\) and \(\bar{y}_n = 0\). The estimates on \(d_n\) follow from the decay properties of \(y_n\) (Theorems 5, 6, 7, property i). In addition, if \(V^{(3)}(0) \neq 0\) then \(d_{n+1} - d_n\) has the sign of \(-V^{(3)}(0)\) for \(\mu\) small enough.

If \(V\) is even then \(W\) is odd and Eq. (9) is invariant under \(-I\). It follows that the centre manifold is invariant under \(-I\), and consequently \(TY_n = -Y_n\) (since \(T = -I\) on the centre space). This implies that \(W(y_n(\omega t)) = 0\) and thus \(d_n = 0\). \(\square\)

A suitable choice of the additive constant \(c\) yields the symmetric DB solutions of (7) listed in Lemma 10. In what follows we note \(T x = x(\cdot + \pi/\omega)\).

**Lemma 10.** One can choose a constant \(c = O(|\mu|)\) in Lemma 9 such that

\[
x_{-n}^{1+} = -\bar{T}x_{-n}^{1+}(t), \quad x_{-n}^{2+} = -Tx_{-n}^{2+}(t),
\]
Fig. 2. Case $B > 0$, $\omega \approx \omega_c = [2\kappa(1/m_1 + 1/m_2)]^{1/2}$ and $\omega > \omega_c$. Sketch of the breather solution $x_n^{1+}$, as a function of $n$ and at fixed $t$. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_n^{1+}(t) = x_{n-2}^{1+}(t + \pi/\omega)$. Light masses displacements are larger and nearest neighbours are out of phase.

\[ x_{-n}^{-1}(t) = -\tilde{T}x_{n-2}^{-1}(t), \quad x_{-n}^{-2}(t) = -x_n^{-2}(t), \]
\[ x_{-n}^{1a}(t) = -x_{n-2}^{1a}(t), \quad x_{-n}^{2a}(t) = -\tilde{T}x_n^{2a}(t). \]

**Proof.** We prove the lemma for $x_n^{1+}$, the other cases being similar. To shorten notations we drop the $1+ \text{ index}$ in the computations. We shall prove that $x_{-n} = -\tilde{T}x_{n-2}$ for all $n \in \mathbb{Z}$, for a suitable choice of the translation constant $c$. Note that it suffices to show this equality for $n \geq 0$ (the case $n \leq -1$ follows by symmetry).

According to Theorem 5(i), the homoclinic solution $Y_n = (u_n, v_n)$ has the symmetry

\[ T v_{-n} = u_n. \]

Recalling $(u_n, v_n) = (y_{2n}, y_{2n-1})$, this means that

\[ T y_{-n+1} = y_{n-2}. \quad (129) \]

Fig. 3. Case $B > 0$, $\omega \approx \omega_c = [2\kappa(1/m_1 + 1/m_2)]^{1/2}$ and $\omega > \omega_c$. Sketch of the breather solution $x_n^{2+}$, as a function of $n$ and at fixed $t$. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_n^{2+}(t) = x_{n+2}^{2+}(t + \pi/\omega)$. Light masses displacements are larger and nearest neighbours are out of phase.
Fig. 4. Case $B < 0$, $\omega \approx \omega_c = (2\kappa/m_1)^{1/2}$ and $\omega < \omega_c$. Sketch of the breather solution $x_n^1$, as a function of $n$ and at fixed $t$. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_n^1(t) = x_n^{-1}(t + \pi/\omega)$. Light masses are out of phase and have larger displacements than heavy masses.

Using (117), we have for $n \geq 1$

$$x_{-n}(t) = -\sum_{i=-n}^{-1} W(y_{i+1}(\omega t)) + x_0(t) = -\sum_{i=1}^{n} W(y_{-i+1}(\omega t)) + x_0(t)$$

$$= -\tilde{T} \sum_{i=1}^{n} W(y_{i-2}(\omega t)) + x_0(t) = -\tilde{T} x_{n-2}(t) + x_0(t) + \tilde{T} x_{-2}(t).$$

Now one observes that $x_0 + \tilde{T} x_{-2}$ is independent of $t$ (differentiate twice, use (7) and (129)). Consequently, one can use the invariance $x_n \rightarrow x_n + c$ in (7) and choose the constant $c$ such that $x_0 + \tilde{T} x_{-2} = 0$. In this case, one has $c = O(|\mu|)$ (use Lemma 9 and Eq. (124)) and one obtains

$$x_{-n} = -\tilde{T} x_{n-2}$$

for all $n \geq 0$. □

Fig. 5. Case $B < 0$, $\omega \approx \omega_c = (2\kappa/m_1)^{1/2}$ and $\omega < \omega_c$. Sketch of the breather solution $x_n^2$, as a function of $n$ and at fixed $t$. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_n^2(t) = x_n^{-2}(t)$. Light masses are out of phase and have larger displacements than heavy masses.
Fig. 6. Case $B > 0$, $m_2/m_1 \in (k^2, k(k + 2))$ (for some integer $k \geq 1$), $\omega \approx \omega_k = (2k/m_2)^{1/2}$ and $\omega > \omega_k$. Sketch of the breather solution $s_n^{1o}$, as a function of $n$ and at fixed $t$. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-s_n^{1o}(t) = s_{n-2}^{1o}(t)$. Heavy masses are out of phase and have larger displacements than light masses.

Fig. 7. Case $B > 0$, $m_2/m_1 \in (k^2, k(k + 2))$ (for some integer $k \geq 1$), $\omega \approx \omega_k = (2k/m_2)^{1/2}$ and $\omega > \omega_k$. Sketch of the breather solution $s_n^{2o}$, as a function of $n$ and at fixed $t$. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-s_n^{2o}(t) = s_{n-2}^{2o}(t + \pi/\omega)$. Heavy masses are out of phase and have larger displacements than light masses.

Lemmas 9 and 10 allow us to sketch the shape of the above discrete breather solutions. We plot their $\cos(\omega t)$ Fourier component in Figs. 2–7 (the error with respect to $x_n$ is $O(|\mu|)$). Since Eq. (1) has been rescaled in the form (7), we fix $\kappa = 1$, $m_1 = m$, $m_2 = 1$ in the captions of Figs. 2–7.

For the case $B < 0$, $\omega \approx \omega_c = (2/m)^{1/2}$ and $\omega < \omega_c$, light masses are out of phase and have larger displacements than heavy masses (see Figs. 4 and 5). For the case $B > 0$, $\omega \approx \omega_c = (2 + 2/m)^{1/2}$ and $\omega > \omega_c$, light masses have larger displacements than heavy masses and nearest neighbours are out of phase (see Figs. 2 and 3). For the case $B > 0$, $1/m \in (k^2, k(k + 2))$ (for some integer $k \geq 1$), $\omega \approx \omega_k = \sqrt{2}$ and $\omega > \omega_c$, heavy masses are out of phase and have larger displacements than light masses (see Figs. 6 and 7). These results are in agreement with previous numerical works (see e.g. [8,26,14]).

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Appendix A. Global reduction preserving reversibility

This appendix is aimed at proving Theorem 4 of Section 4.2 (to which we refer for part of the notations). Our analysis is organized as follows.

We first derive basic estimates on $N_{\epsilon}$ and use them (in conjunction with the spectral properties of $L$) to obtain a global centre manifold reduction result for the truncated problem. This part is merely an adaptation of [24] (the difference is that the cut-off is performed on $\mathcal{X}_\epsilon$ instead of $\mathbb{D}$).

Next we show that our global reduction procedure preserves reversibility. The reversibility symmetry $R$ is unbounded in $\mathbb{D}$ and thus we need a regularity result on each solution $Y_n$ (Lemma 3) for proving that $RY_{n-1}$ is also a solution. The remainder of the proof is identical to [24], Section 5.1.

The truncated problem reads

$$Y_{n+1} = LY_n + N_{\epsilon}(Y_n, \mu),$$  \hspace{1cm} (A.1)

where

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}(Q^2 - 1)u - Qv \\ Qu - \alpha v \end{pmatrix},$$

$$Qu = \left( \omega_x^2 \frac{d^2}{dt^2} + (1 + \alpha) \right) u,$$

$$N_{\epsilon}(y, \mu) = \begin{pmatrix} \frac{1}{\alpha}(QN_{\epsilon}(u, \mu) + N_{\epsilon}(Qu - \alpha v + N_{\epsilon}(u, \mu), \mu)) \\ N_{\epsilon}(u, \mu) \end{pmatrix},$$

$N_{\epsilon}(y, \mu) = (d^2/\alpha)g_{\epsilon}(y, \mu)$ and $g_{\epsilon}(y, \mu) = g(y, \mu)\chi(\epsilon^{-1} |(y, \cos t)|)$. We recall that $g(y, \mu) = \mu W(y) + \omega_x^2(W(y) - y), (y, \cos t) = \int_0^{2\pi} y(t) \cos t \, dt$ and $\chi \in C^\infty([0, +\infty), [0, 1])$ is a cut-off function satisfying $\chi(x) = 1$ for $x \in [0, 1]$ and $\chi(x) = 0$ for $x \geq 2$.

The first step consists in estimating $g_{\epsilon}(y, \mu)$. In the sequel, we use the notations $\pi_{\epsilon}u = (1/\pi) \int_0^{2\pi} u(t) \cos t \, dt \cos t$ and $\pi_{b} = I - \pi_{\epsilon}$. Given $n \geq 1$, we denote by $H^{n}_{\epsilon}$ the subspace of $H^{n}(\mathbb{R}/2\pi\mathbb{Z})$ consisting in even functions of $t$ (equipped with the usual norm $\|\cdot\|_{H^n}$). We shall also consider

$$H^{n}_{\epsilon} = \left\{ y \in H^{n}_{\epsilon}/ \int_0^{2\pi} y \, dt = 0 \right\}.$$

The following estimates follow from the fact that $g(0, \mu) = 0$, $(\partial g/\partial y)(0, 0) = 0$.

**Lemma A.11.** There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $g_{\epsilon} : H^{n}_{\epsilon} \to H^{n}_{\epsilon}$ is $C^{\infty}$. Moreover, there exists $C > 0$ (depending on $n$) such that for all $y \in H^{n}_{\epsilon}$ and $\mu \in [-\varepsilon, \varepsilon]$ one has

$$\|g_{\epsilon}(y, \mu)\|_{H^{n}} \leq C\varepsilon^2,$$

$$\|D_y g_{\epsilon}(y, \mu)\|_{L(H^{n}_{\epsilon}, H^{n}_{\epsilon})} \leq C\varepsilon.$$
We are now ready to estimate \( N_\varepsilon \) in the strip \( B^h_\varepsilon = \{ Y \in \mathbb{D} / \| Yh^h \|_\mathbb{D} \leq \varepsilon \} \). An application of Lemma A.11 yields as \( \varepsilon \to 0^+ \)

\[
\| N_\varepsilon \|_{C^0_0(B^h_\varepsilon \times (\varepsilon, 0), \mathbb{X})} = O(\varepsilon^2), \quad \| DY N_\varepsilon \|_{C^0_0(B^h_\varepsilon \times (\varepsilon, 0), \mathcal{L}(\mathbb{D}, \mathbb{X}))} = O(\varepsilon).
\]

(A.2)

Our next step is to prove a global centre manifold reduction theorem for (A.1), where we look for \( Y_n \) in the following closed subspace of \( B_\varepsilon(\mathbb{D}) \)

\[
B^c_\varepsilon(\mathbb{D}) = \{ Y \in B_\varepsilon(\mathbb{D}) / Yh^h \in B_1(\mathbb{D}_h), \| Yh^h \|_{B_1(\mathbb{D}_h)} \leq \varepsilon \}.
\]

The method consists in formulating (A.1) as a fixed point equation in \( B^c_\varepsilon(\mathbb{D}) \) and then apply the contraction mapping theorem.

For this purpose, the following results on the projected affine equations are essential. They have been proved in [24] (Section A) and originate from the spectral properties of \( L \).

**Lemma A.12.** The affine recurrence relation in \( \mathbb{X}_c \)

\[
Y^c_0 = a, \quad Y^c_{n+1} = L_c Y^c_n + f^c_n \quad \forall n \in \mathbb{Z}
\]

has a unique solution

\[
Y^c = L_c^a a + K_c f^c.
\]

(A.4)

For all \( v \in (0, 1) \) one has \( Y^c \in B_v(\mathbb{X}_c) \) and \( K_c \in \mathcal{L}(B_v(\mathbb{X}_c)) \).

**Lemma A.13.** There exists \( r \in (0, 1) \) such that for all \( v \in (r, 1) \), for any \( f^h \in B_v(\mathbb{X}_h) \), the problem

\[
Y^h \in B_v(\mathbb{D}_h), \quad Y^h_{n+1} = L_h Y^h_n + f^h_n \quad \forall n \in \mathbb{Z}
\]

has a unique solution \( Y^h = K_h f^h \) with \( K_h \in \mathcal{L}(B_v(\mathbb{X}_h), B_v(\mathbb{D}_h)) \).

In Lemma A.13, the constant \( r \) must be chosen so that \( \sigma(L_h) \) lies strictly inside the ball of radius \( r \) or outside the ball of radius \( r^{-1} \).

Splitting (A.1) on \( \mathbb{X}_c, \mathbb{X}_h \) and choosing \( v \in (r, 1) \) as in Lemmas A.12, A.13, one obtains the equivalent problem

\[
Y \in B^c_\varepsilon(\mathbb{D}), \quad Y = L_c Y^c_0 + (K_c \Pi_c + K_h \Pi_h) N_\varepsilon(Y, \mu).
\]

(A.6)

We now solve (A.6) when \( \varepsilon \) is small enough and \((Y^c_0, \mu) \in \mathbb{X}_c \times [-\varepsilon, \varepsilon]\) is given. In (A.6), we view \( N_\varepsilon(\cdot, \mu) \) as a map from \( B^c_\varepsilon(\mathbb{D}) \) into \( B_\varepsilon(\mathbb{X}) \). Using (A.2) gives

\[
\sup_{Y \in B^c_\varepsilon(\mathbb{D}), |\mu| \leq \varepsilon} \| N_\varepsilon(Y, \mu) \|_{B_\varepsilon(\mathbb{X})} = O(\varepsilon^2),
\]

(A.7)

and

\[
\sup_{Y \neq Z \in B^c_\varepsilon(\mathbb{D}), |\mu| \leq \varepsilon} \frac{\| N_\varepsilon(Y, \mu) - N_\varepsilon(Z, \mu) \|_{B_\varepsilon(\mathbb{X})}}{\| Y - Z \|_{B_\varepsilon(\mathbb{D})}} = O(\varepsilon).
\]

(A.8)

By combining (A.7)–(A.8) with Lemmas A.12 and A.13, one finds \( \varepsilon_0(v) > 0 \) such that for all \( \varepsilon < \varepsilon_0 \) and \((x, \mu) \in \mathbb{X}_c \times [-\varepsilon, \varepsilon]\), the map

\[
Y \mapsto F_\varepsilon(Y, x, \mu) = L^n_c x + (K_c \Pi_c + K_h \Pi_h) N_\varepsilon(Y, \mu)
\]

maps \( B^c_\varepsilon(\mathbb{D}) \) into itself and is a contraction. Consequently, for a given \((Y^c_0, \mu) \in \mathbb{X}_c \times [-\varepsilon, \varepsilon]\) it follows from the contraction mapping theorem that (A.6) has a unique solution \( Y = \phi^\varepsilon(Y^c_0, \mu) \) with \( \phi^\varepsilon \in C^{0,1}(\mathbb{X}_c \times [-\varepsilon, \varepsilon], B^c_\varepsilon(\mathbb{D})) \).
Since for any fixed \( p \in \mathbb{Z} \) the shifted sequence \( Y_{n+p} \) is also a solution, one has by uniqueness \( Y_{n+p} = \phi_{n+p}^c(Y_0^c, \mu) = \phi_{n+p}^c(Y_p^c, \mu) \). Setting \( n = 0 \) gives in particular

\[
Y_p = \phi_p^c(Y_0^c, \mu) = \phi_p^c(Y_p^c, \mu) \quad \forall p \in \mathbb{Z}.
\]

Consequently, \( Y \) is solution of (A.6) if and only if

\[
Y_n = \phi_0^c(Y_n^c, \mu) \quad \forall n \in \mathbb{Z}.
\]

This implies that

\[
Y_n^h = \psi^c(Y_n^c, \mu),
\]

where \( \psi^c = \Pi_h \phi_0^c \in C^0(\mathbb{X}_c \times [-\varepsilon, \varepsilon], \mathcal{D}_h) \) (with \( \psi^c = O(\varepsilon^2) \)). The next step is to prove that \( \psi^c \in C^4(\mathbb{X}_c \times [-\varepsilon, \varepsilon], \mathcal{D}_h) \). This property does not follow directly from the implicit function theorem since \( \mathcal{N}_c(\cdot, \mu) : B_h^c(\mathbb{D}) \to B_h(\mathbb{X}) \) is not differentiable (due to the possible divergence of \( Y \in B_h^c(\mathbb{D}) \)). The \( C^4 \)-regularity of \( \psi^c \) is obtained in the same way as in [24,35,34], to which we refer for details. The proof is based on the fiber contraction theorem and the fact that \( \mathcal{N}_c(\cdot, \mu) \in C^4(B_h^c(\mathbb{D}), B_c(\mathbb{X})) \) for \( \zeta < \nu^0 < \nu < 1 \).

Differentiability in \( B_h^c(\mathbb{D}) \) has to be understood as follows. Consider a Banach space \( E \) and an operator \( N : B_h^c(\mathbb{D}) \to E \). One has \( N \in C^4(B_h^c(\mathbb{D}), E) \) if there exists a mapping \( DN \in C^0(B_h^c(\mathbb{D}), \mathcal{L}(B_h(\mathbb{D}), E)) \) such that for all \( Y, Z \in B_h^c(\mathbb{D}) \)

\[
\|N(Y) - N(Z) - DN(Z)(Y-Z)\|_E = o(\|Y-Z\|_{B_h(\mathbb{D})}), \quad \|Y-Z\|_{B_h(\mathbb{D})} \to 0.
\]

Obviously one has \( N \in C^4(B_h^c(\mathbb{D}), E) \) if \( D^jN \in C^4(B_h^c(\mathbb{D}), \mathcal{L}(B_h(\mathbb{D}), E)) \) for all \( j = 0, \ldots, k-1 \).

Thanks to property (A.10), one obtains a (global) reduced recurrence relation by projecting (A.1) on \( \mathbb{X}_c \). This yields

\[
Y_{n+1}^c = f(x, \mu) \quad \forall n \in \mathbb{Z},
\]

where

\[
f(x, \mu) = L_c x + \Pi_c \mathcal{N}_c(x + \psi^c(x, \mu), \mu).
\]

We now address the question of the reversibility of (A.11) and the invariance under \( R \) of the centre manifold. As we mentioned previously, one has to prove Lemma 3 as a first step. For this purpose we need the following intermediate result.

**Lemma A.14.** There exist \( \varepsilon_0, \gamma > 0 \) (depending on \( n \)) such that for all \( \varepsilon < \varepsilon_0, \mu \in [-\varepsilon, \varepsilon] \) and \( f \in H^p_\varepsilon \), the problem

\[
\frac{d^2}{dr^2}(\omega^2_c y + g_c(y, \mu)) = f, \quad y \in H^{n+2}_\varepsilon
\]

has a unique solution \( y \).

**Proof.** We define \( \pi_0 u = u - (1/2\pi) \int_0^{2\pi} u(t) \, dt \). Applying \( \pi_0 \) to Eq. (A.12) and integrating twice yields

\[
\omega^2_c y + \pi_0 g_c(y, \mu) = F,
\]

where \( d^2 F/dt^2 = f \) and \( F \in H^{n+2}_\varepsilon \) (this determines \( F \) uniquely). Consequently, we are led to solve the fixed point equation

\[
y = U_{\varepsilon, \mu}(y).
\]
where \( U_{\epsilon, \mu}(y) = (1/\omega^2)(F - \pi_0 g_\epsilon(y, \mu)) \). Choosing \( \gamma > 1/\omega^2 \) and \( \epsilon \) small enough, it follows from Lemma A.11 that \( U_{\epsilon, \mu} \) is a contraction on \( H^{n+2}_{\gamma\epsilon} \). Therefore, according to the contraction mapping theorem (A.12) has a unique solution \( y \) in \( H^{n+2}_{\gamma\epsilon} \).

We are now ready to prove Lemma 3, which can be formulated in the following way.

**Lemma A.15.** Fix \( p \geq 2 \). There exist \( \epsilon_0, \gamma > 0 \) (depending on \( p \)) such that for all \( \epsilon < \epsilon_0 \) and \( \mu \in [-\epsilon, \epsilon] \), any solution \( Y \in B_\epsilon^p(\mathbb{D}) \) of (A.1) satisfies \( Y \in H^{n+2}_{\gamma\epsilon} \times H^p_{\gamma\epsilon} \).

**Proof.** We prove this result by induction. Firstly we have \( Y_n \in H^n_{\epsilon} \times H^2_{\epsilon} \). Secondly, assume \( p > 2 \) and \( Y_n \in H^{k+2}_{\epsilon} \times H^k_{\epsilon} \) with \( 2 \leq k < p \). Setting \( Y_n = (u_n, v_n) \), problem (A.1) yields

\[
[Q + N_\epsilon(\cdot, \mu)]v_n = au_{n+1} + u_n,
\]

\[
[Q + N_\epsilon(\cdot, \mu)]u_n = \alpha v_n + v_{n+1}.
\]

Eq. (A.15) can be written

\[
\frac{d^2}{dt^2}(w^2v_{n+1} + g_\epsilon(v_n, \mu)) = au_{n+1} + u_n - (1 + \alpha)v_{n+1} \in H^k_{\epsilon\epsilon}.
\]

Consequently, Lemma A.14 ensures that \( v_n \in H^{k+2}_{\gamma\epsilon\epsilon} \). Turning to Eq. (A.16), we get

\[
\frac{d^2}{dt^2}(w^2u_n + g_\epsilon(u_n, \mu)) = \alpha v_n + v_{n+1} - (1 + \alpha)u_n \in H^{k+2}_{\epsilon\epsilon},
\]

and Lemma A.14 yields \( u_n \in H^{k+4}_{\gamma\epsilon\epsilon} \). Consequently we have obtained \( Y_n \in H^{k+4}_{\epsilon\epsilon} \times H^{k+2}_{\epsilon\epsilon} \), and the proof follows by induction.

From Lemma A.15 it follows that if \( Y_n \) is a solution of (A.1) then \( RY_{-n} \) is also a solution (see Lemma 4 of Section 4.2). Proceeding as in [24] (Lemma 10), one obtains in this case \( R\psi_\epsilon(\cdot, \mu) = \psi_\epsilon(\cdot, \mu) \circ R \) and \( (f_\epsilon(\cdot, \mu) \circ R)^2 = I \). This completes the proof of Theorem 4.

**Appendix B. Centre manifold computation**

When \( \mu \) is small, the centre manifold Theorem 2 ensures that the solutions of (9) staying in the neighbourhood of \( Y = 0 \) in \( \mathbb{D} \) for any \( n \in \mathbb{Z} \) have the form

\[
Y_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos(t) + \Psi(a_n, b_n, \mu),
\]

where \( \Psi \in C^k(\mathbb{R}^3; \mathbb{D}_b) \). The coordinates \( (a_n, b_n) \) satisfy a recurrence relation in \( \mathbb{R}^2 \):

\[
\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} F_1(a_n, b_n, \mu) \\ F_2(a_n, b_n, \mu) \end{pmatrix} = f(a_n, b_n, \mu).
\]

(B.1)

In this appendix, we compute the leading order terms in the Taylor expansions of \( \Psi \) and the reduced map (B.1).
Computation of $\Psi$: We first compute the Taylor expansion of $\Psi$ in $(a, b, \mu)$ at $(a, b, \mu) = 0$. Due to the invariance under $R$, it has the form:

$$\Psi(a, b, \mu) = \left(\Psi_{011}a\mu + \Psi_{010}b\mu + \Psi_{020}a^2 + \Psi_{110}ab + \Psi_{200}b^2\right) + \text{h.o.t.} \quad (B.2)$$

The invariance under $T$ implies that:

$$T\Psi_{011} = -\Psi_{011}, \quad T\Psi_{010} = -\Psi_{010}, \quad T\Psi_{020} = \Psi_{020}, \quad T\Psi_{110} = \Psi_{110}, \quad T\Psi_{200} = \Psi_{200}. \quad (B.3)$$

Hence we have:

$$\Psi_{011}, \Psi_{010} \in (\cos((2k + 1)\tau)/k \in \mathbb{Z}) = \mathcal{V}_\text{odd}, \quad \Psi_{020}, \Psi_{110}, \Psi_{200} \in (\cos(2k\tau)/k \in \mathbb{Z}) = \mathcal{V}_\text{even}. \quad (B.4)$$

Setting $Y_n = (u_n, v_n)$, one has:

$$u_n = a_n \cos(t) + \Psi_{011}a_n\mu + \Psi_{010}b_n\mu + \Psi_{020}a_n^2 + \Psi_{110}a_n b_n + \Psi_{200}b_n^2 + \text{h.o.t.},$$

$$v_n = b_n \cos(t) + \Psi_{011}a_n\mu + \Psi_{010}b_n\mu + \Psi_{020}a_n^2 + \Psi_{110}a_n b_n + \Psi_{200}b_n^2 + \text{h.o.t.} \quad (B.5)$$

Problem (9) can be written

$$m\omega^2 \frac{d^2}{dt^2} W(v_{n+1}) = u_{n+1} - (m + 1)v_{n+1} + mu_n, \quad m\omega^2 \frac{d^2}{dt^2} W(u_n) = mv_{n+1} - (m + 1)u_n + v_n. \quad (B.6)$$

or equivalently

$$\left(\mu + \omega_c^2\right) \frac{d^2}{dt^2} W(v_{n+1}) = \alpha u_{n+1} - (1 + \alpha)v_{n+1} + u_n, \quad \left(\mu + \omega_c^2\right) \frac{d^2}{dt^2} W(u_n) = v_{n+1} - (1 + \alpha)u_n + \alpha v_n, \quad (B.7)$$

with $\alpha = 1/m$. We calculate the Taylor expansion of $\Psi$ as explained in Section 4.3. We identify the quadratic terms $a_n\mu$, $b_n\mu$, $a_n^2$, $b_n^2$ and $a_n b_n$ in the Taylor expansion of (47). Due to the symmetries of $\Psi$ (see (B.2)), we just need the second component of (47) to perform the identification. In the sequel, we use the notations $\pi_c, u = (1/\pi) \int_0^{2\pi} u(t) \cos t \, dt \cos t$ and $\pi_h = I - \pi_c$.

We first derive a system which allows to compute $(\Psi_{020}, \Psi_{200}, \Psi_{110})$. The identification at order $a_n^2$ gives:

$$\omega_c^2 \frac{d^2}{dt^2} \psi_{020} + W_2 \omega_c^2 \frac{d^2}{dt^2} \pi_h(\cos^2(\tau)) = \beta\psi_{110} - \left(\frac{\beta}{\alpha}\right)^2 \psi_{200} + \beta^2 \psi_{020} - (1 + \alpha)\psi_{020} + \alpha \psi_{200}, \quad (B.8)$$

where $W_2$ is defined in (31) and $\beta = 1 + \alpha - \omega_c^2$. Moreover the identification at order $b_n^2$ yields:

$$\omega_c^2 \frac{d^2}{dt^2} \psi_{200} = \beta^2 \psi_{200} + \alpha \beta \psi_{110} + \alpha^2 \psi_{020} - (1 + \alpha)\psi_{200} + \alpha \psi_{200}. \quad (B.9)$$

Identification at order $a_n b_n$ leads to:

$$-\omega_c^2 \frac{d^2}{dt^2} \psi_{110} = 2\beta^2 - 1 \alpha \psi_{200} + 2\beta^2 \psi_{110} + 2\alpha \beta \psi_{020}. \quad (B.10)$$

By expanding $\psi_{200}, \psi_{020}, \psi_{110}$ in Fourier series in (B.8), (B.9) and (B.10), one shows that these functions are colinear to $\cos(2\tau)$. Let us note $\psi_{200} = \psi_{200}\cos(2\tau)$ and define $\psi_{110} \text{ and } \psi_{020}$ in the same way. Eqs. (B.8), (B.9)
and (B.10) yield:

$$D(\alpha, \omega_c^2) \begin{pmatrix} \psi_{200} \\ \psi_{020} \\ \psi_{110} \end{pmatrix} = b(W_2, \omega_c^2)$$

(B.11)

with

$$b(W_2, \omega_c^2) = \begin{pmatrix} -2W_2\omega_c^2 \\ 0 \\ 0 \end{pmatrix}$$

and

$$D(\alpha, \omega_c^2) = \begin{pmatrix} \frac{\beta^2 - 1}{\alpha} + \beta^2 - (\beta - 3\omega_c^2) & \frac{\beta^2 - 1}{\alpha} \\ \frac{\beta^2 - 1}{\alpha} & \alpha(\alpha + 1) & \alpha \beta \\ \frac{\beta^2 - 1}{\alpha} & \alpha \beta & \beta^2 - 2\omega_c^2 \end{pmatrix}.$$  

(B.12)

Now we solve the linear system (B.11) for the different critical values listed in (28).

Starting with $\omega_c^2 = 2(1 + \alpha)$, we find:

$$\psi_{020} = -\frac{1}{16} \frac{8m - 1}{m} W_2 \cos(2t), \quad \psi_{200} = \frac{1}{16m} W_2 \cos(2t), \quad \psi_{110} = \frac{1}{8m} W_2 \cos(2t).$$  

(B.13)

For $\omega_c^2 = 2\alpha$, one has:

$$\psi_{020} = -\frac{1}{16} \frac{7m^2 - 34m + 3}{m(m - 3)} W_2 \cos(2t), \quad \psi_{200} = \frac{1}{16} \frac{m^2 - 6m - 3}{m(m - 3)} W_2 \cos(2t),$$

$$\psi_{110} = \frac{1}{8} \frac{m^2 + 2m - 3}{m(m - 3)} W_2 \cos(2t).$$  

(B.14)

Considering the case $\omega_c^2 = 2$, we obtain:

$$\psi_{020} = -\frac{1}{16} \frac{24m^3 + 5m^2 - 6m + 1}{m^2(3m - 1)} W_2 \cos(2t), \quad \psi_{200} = \frac{1}{16} \frac{3m^2 + 6m - 1}{m^2(3m - 1)} W_2 \cos(2t),$$

$$\psi_{110} = \frac{1}{8} \frac{5m^2 - 6m + 1}{m^2(3m - 1)} W_2 \cos(2t).$$  

(B.15)

The computations of the other coefficients $\psi_{011}$ and $\psi_{101}$ are similar. Identification at order $a_n \mu$ and $b_n \mu$ leads to $\psi_{011} = 0$ and $\psi_{101} = 0$.

As a conclusion we have achieved the centre manifold computation. To simplify the notations, we shall write

$$\psi_{020} = \rho(m) W_2 \cos(2t), \quad \psi_{200} = q(m) W_2 \cos(2t), \quad \psi_{110} = r(m) W_2 \cos(2t).$$  

(B.16)

**Reduced map computation:** Now we compute the leading order terms in the Taylor expansion of the reduced map (B.1). Projecting (B.7) on $\mathbb{K}_c$ yields (we use the fact that $\pi_c(d^2/dt^2) = -\pi_c$)

$$-(\mu + \omega_c^2)(u^c_{n+1} + \pi_c G(u_{n+1})) = \alpha u^c_{n+1} - (1 + \alpha) u^c_{n+1} + u^c_n,$$

$$-(\mu + \omega_c^2)(u^c_n + \pi_c G(u_n)) = v^c_{n+1} - (1 + \alpha) u^c_n + \alpha u^c_n.$$  

(B.17)
where \((u_n', v_n') = \Pi_c(u_n, v_n) = (\pi_c u_n, \pi_c v_n) = (a_n, b_n) \cos(t)\) and

\[ G(y) = W(y) - y = W_2 y^2 + W_3 y^3 + O(y^4). \]

The recurrence relation (B.1) is obtained by inserting expressions (B.5) in Eq. (B.17). The second equation of (B.17) then gives:

\[ b_{n+1} = \beta a_n - \alpha b_n - a_n \mu - (\mu + \omega_c^2) I \pi_c G(u_n) \]

where we note \(I(a \cos(t)) = a\). To simplify the computation of \(\pi_c G(u_n)\), we note that \(\pi_c V_{\text{even}} = 0\) (see definition (B.4)). Using expressions (B.5), (B.16) and the symmetry properties (B.4), one obtains:

\[ G(u_n) = (2W_2 \Psi_{020} \cos(t) + W_3 \cos^3(t))a_n^3 + 2W_2 \Psi_{110} \cos(t) a_n^2 b_n + 2W_2 \Psi_{200} \cos(t) a_n b_n^2 + k_n(t) + \text{h.o.t.} \]

with \(k_n \in V_{\text{even}}\). Finally, we have:

\[ b_{n+1} = F_2(a_n, b_n, \mu) = \beta a_n - \alpha b_n - a_n \mu - \omega_c^2 (\beta_1 a_n^3 + \beta_2 a_n b_n^2 + \beta_3 a_n^2 b_n) + \text{h.o.t.} \quad (B.18) \]

with:

\[ \beta_1 = I \pi_c (2W_2 \Psi_{020} \cos(t) + W_3 \cos^3(t)) = p(m) W_2^3 + \frac{3}{4} W_3, \quad \beta_2 = I \pi_c (2W_2 \Psi_{110} \cos(t)) = q(m) W_2^3, \]

\[ \beta_3 = I \pi_c (2W_2 \Psi_{200} \cos(t)) = r(m) W_2^3. \quad (B.19) \]

In the sequel, we note

\[ F_2(a, b, \mu) = L_2(a, b) - a \mu + Q(a, b) + \text{h.o.t.}, \quad (B.20) \]

where \(L_2(a, b) = \beta a - \alpha b\) and \(Q(a, b) = -\omega_c^2 (\beta_1 a^3 + \beta_2 a b^2 + \beta_3 a^2 b)\).

We now expand the first equation \(a_{n+1} = F_1(a_n, b_n, \mu)\) in (B.1). Let us note

\[ F_1(a, b, \mu) = L_1(a, b) + \mu M_1(a, b) + P(a, b) + \text{h.o.t.}, \quad (B.21) \]

where \(L_1, M_1\) are linear forms and \(P\) is a homogeneous cubic polynomial. The first equation of (B.17) yields

\[ \alpha a_{n+1} = \beta b_{n+1} - a_n - \mu b_{n+1} - (\mu + \omega_c^2) I \pi_c G(v_{n+1}). \quad (B.22) \]

Inserting (B.18) in Eq. (B.22), one finds

\[ L_1(a, b) = \frac{\beta^2 - 1}{\alpha} a - \beta b, \quad M_1(a, b) = -\frac{\beta}{\alpha} a + b. \]

Cubic terms of (B.21) could be obtained by combining Eqs. (B.22), (B.18), (B.7) \((v_{n+1} \text{ is expressed as a function of } (u_n, v_n))\) and (B.5). Here we proceed in a simpler way, using the reversibility of (B.1) under \(R(a, b) = (b, a)\) (see Theorem 3). One has

\[ f \circ R \circ f \circ R = \text{Id} \quad (B.23) \]

and consequently

\[ F_2(F_2(b, a, \mu), F_1(b, a, \mu)) = b. \quad (B.24) \]
Now we identify cubic terms in (B.24) and find
\[ L_2(Q(b, a), P(b, a)) + Q(L_2(b, a), L_1(b, a)) = 0. \]

This yields
\[ \alpha P(b, a) = \beta Q(b, a) + Q(L_2(b, a), L_1(b, a)). \]

(B.25)

One obtains consequently \( P(a, b) = -\alpha^2(\beta_3a^3 + \beta_5a^2b + \beta_6ab^2 + \beta_7b^3), \) with
\[ \alpha \beta_3 = \beta \beta_1 + \beta^3 \beta_1 + \beta^3 \gamma \beta_2 + \beta^2 \gamma \beta_3, \]
\[ \alpha \beta_5 = -3\beta^2\alpha \beta_1 - (\alpha \gamma^2 + 2\gamma \beta^2)\beta_2 + (\beta - \beta^3 - 2\alpha \beta \gamma)\beta_3, \]
\[ \alpha \beta_6 = 3\beta \alpha^2 \beta_1 + (\beta + \beta^3 + 2\alpha \beta \gamma)\beta_2 + (2\beta^2 \alpha + \alpha^2 \gamma)\beta_3, \]
\[ \beta_7 = -\alpha^2 \beta_1 - \beta^2 \beta_2 - \alpha \beta_3 \] (B.26)

and \( \gamma = (\beta^2 - 1)/\alpha. \) As a conclusion, we find
\[ a_{n+1} = F_1(a_n, b_n, \mu) = \gamma a_n - \beta b_n - 2\beta^2 a_n \mu + b_n \mu - a_n^2(\beta_3 a_n^3 + \beta_5 a_n^2 b_n + \beta_6 a_n b_n^2 + \beta_7 b_n^3) + \text{h.o.t.} \] (B.27)

References


