Localized waves in nonlinear oscillator chains

Gérard Iooss
Institut Universitaire de France, INL, UMR CNRS-UNSA 6618, 1361 route des Lucioles, F-06560 Valbonne, France

Guillaume James
Laboratoire Mathématiques pour l’Industrie et la Physique (UMR 5640), INSA de Toulouse, 135 avenue de Rangueil, 31077 Toulouse Cedex 4, France

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This paper reviews results about the existence of spatially localized waves in nonlinear chains of coupled oscillators, and provides new results for the Fermi–Past–Ulam (FPU) lattice. Localized solutions include solitary waves of permanent form and traveling breathers which appear time periodic in a system of reference moving at constant velocity. For FPU lattices we analyze the case when the breather period and the inverse velocity are commensurate. We employ a center manifold reduction method introduced by Iooss and Kirchgässner in the case of traveling waves, which reduces the problem locally to a finite dimensional reversible differential equation. The principal part of the reduced system is integrable and admits solutions homoclinic to quasi-periodic orbits if a hardening condition on the interaction potential is satisfied. These orbits correspond to approximate travelling breather solutions superposed on a quasi-periodic oscillatory tail. The problem of their persistence for the full system is still open in the general case. We solve this problem for an even potential if the breather period equals twice the inverse velocity, and prove in that case the existence of exact traveling breather solutions superposed on an exponentially small periodic tail. © 2005 American Institute of Physics. [DOI: 10.1063/1.1836151]

This paper concerns spatially localized waves in nonlinear chains of coupled oscillators. The models we consider are referred to as Fermi–Past–Ulam (FPU) or Klein–Gordon (KG) lattices, depending whether nonlinearity takes the form of an anharmonic nearest-neighbors interaction potential or an on-site potential. Localized solutions include solitary waves of permanent form, and traveling breathers which appear time periodic in a system of reference moving at constant velocity. Approximate traveling breather solutions have been previously constructed in the form of modulated plane waves, whose envelopes satisfy the nonlinear Schrödinger equation. For KG chains and in the case of traveling waves (where the phase velocity of the plane wave equals the group velocity of the wave packet), the existence of nearby exact solutions has been proved by Iooss and Kirchgässner, who have obtained exact solitary wave solutions superposed on an exponentially small periodic tail. By a center manifold reduction they reduce the problem locally to a finite dimensional reversible system of ordinary differential equations, which admits homoclinic solutions to periodic orbits. It has been recently shown by James and Sire that the center manifold approach initiated by Iooss and Kirchgässner is still applicable when the breather period and the inverse group velocity are commensurate. The particular case when the breather period equals twice the inverse group velocity has been worked out explicitly for KG chains, and yields the same type of reduced system as for traveling waves if the on-site potential is symmetric. In that case, the existence of exact traveling breather solutions superposed on an exponentially small periodic tail has been proved. In this paper we apply the same method to the FPU system and treat the commensurate case in full generality (we give the main steps of the analysis and shall provide the details in a forthcoming paper). We reduce the problem locally to a finite dimensional reversible system of ordinary differential equations, whose dimension can be arbitrarily large and is of the order of the number of resonant phonons. Its principal part is integrable, and admits solutions homoclinic to quasi-periodic orbits if a hardening condition on the potential is satisfied. These orbits correspond to approximate traveling breather solutions superposed on a quasi-periodic oscillatory tail. The problem of their existence for the full system is still open in the general case, and constitutes the final step for proving the existence of exact traveling breather solutions. In the particular case of an even potential and if the breather period equals twice the inverse group velocity, we prove indeed the existence of exact traveling breather solutions superposed on an exponentially small periodic tail.

I. MODELS AND LITERATURE REVIEW

We consider one-dimensional lattices described by the system

$$\frac{d^2 u_n}{dt^2} + W'(u_n) = V'(u_{n+1} - u_n) - V'(u_{n} - u_{n-1}), \quad n \in \mathbb{Z}$$

(1)

where $u_n$ is the displacement of the $n$th particle from an equilibrium position. This system describes a chain of particles nonlinearly coupled to their first neighbors, in a local
anharmoic potential. The interaction potential \( V \) and on-site potential \( W \) are assumed analytic in a neighborhood of \( u=0 \), with \( V'(0)=W'(0)=0 \), \( V''(0), W''(0) > 0 \). System (1) is referred as Fermi–Pasta–Ulam (FPU) lattice for \( W=0 \) and Klein–Gordon (KG) lattice if \( V \) is harmonic \( V(t) = (\gamma/2)t^2 \). These models have been used for the description of a broad range of physical phenomena, such as crystal dislocation,\(^{10}\) localized excitations in ionic crystals,\(^{11}\) thermal denaturation of DNA.\(^{12}\)

In this paper, we consider solutions of (1) satisfying

\[ u_n(t) = u_{n-p}(t-p\tau), \]

(2)

for a fixed integer \( p \geq 1 \) (\( p \) being the smallest possible) and \( \tau \in \mathbb{R} \). The case when \( p=1 \) in (2) corresponds to traveling waves with velocity \( 1/\tau \). Solutions satisfying (2) for \( p \neq 1 \) consist of pulsating traveling waves, which are exactly translated by \( p \) sites after a fixed propagation time \( p\tau \) and are allowed to oscillate as they propagate on the lattice. Solutions of type (2) having the additional property of spatial localization \( u_n(t) = 0 \) as \( n \to \pm \infty \) are known as exact traveling breathers (with velocity \( 1/\tau \)) for \( p \geq 2 \) and solitary waves for \( p=1 \).

A. Exact and approximate traveling breathers

Approximate traveling breather solutions propagating on the lattice at a constant velocity have drawn a lot of attention. They have been numerically observed in various one-dimensional nonlinear lattices such as FPU lattices,\(^{13–16}\) KG chains,\(^{17,18}\) and the discrete nonlinear Schrödinger (DNLS) equation.\(^{19,20}\) Other references are available in the review paper.\(^{21}\) One way of generating approximate traveling breathers consists of “kicking” static breathers consisting of spatially localized and time periodic oscillations (see the basic papers\(^{21–25}\) for more details on these solutions). Static breathers are put into motion by perturbation in the direction of a pinning mode.\(^{17}\) The possible existence of an energy barrier that the breather has to overcome in order to become mobile has drawn a lot of attention see, e.g., Refs. 14, 17, 18, and 26, and the review paper.\(^{27}\) Approximate traveling breathers can be formally obtained via effective Hamiltonians, which approximately describe the motion of the breather center on the lattice, at a nonconstant velocity.\(^{26,28}\)

It is a delicate task to examine the existence of exact traveling breathers using numerical computations. Indeed, these solutions might not exist without being superposed on a small nonvanishing oscillatory tail which violates the property of spatial localization. Solitary waves\(^{29}\) and traveling breathers\(^{30} \) superposed on a small oscillatory tail have been numerically observed in KG lattices (Ref. 29 provides examples in which the tail can be made very small and is difficult to detect at the scale of central oscillations). Numerical results indicate similar phenomena for the propagation of kinks.\(^{30–32}\) Fine analysis of numerical convergence problems also suggests that different nonlinear lattices do not support exact solitary waves or traveling breathers in certain parameter regimes.\(^{33,34}\) Note that travelling breather solutions have been previously computed in the DNLS equation\(^{35,36}\) (in fact gauge symmetry reduces the problem to the computation of traveling waves), but the possible existence of a very small oscillatory tail has not been addressed.

Different situations have been considered for the existence of exact traveling breathers in various simpler models. On the one hand, exact traveling breathers can be explicitly computed in the integrable Ablowitz–Ladik lattice,\(^{37}\) and other examples of nonlinear lattices supporting exact traveling breathers can be obtained using an inverse method.\(^{38}\) On the other hand, traveling breather solutions of the Ablowitz–Ladik lattice are not robust under various non-Hamiltonian reversible perturbations as shown in Ref. 39.

B. The multiscale expansion approach

Formal multiscale expansions have been used by several authors to obtain traveling breather solutions of (1). The case of KG lattices has been treated by Remoissenet.\(^{3} \) A multiscale expansion provides an evolution equation for the envelopes of well-prepared initial conditions corresponding to modulated plane waves

\[ u_n(t) = \varepsilon A(\psi, \xi(n-ct)) e^{(iqn-\omega t)} + c.c. + O(\varepsilon^2). \]

(3)

Here \( \omega=\omega(q) \) is given by the dispersion relation for the linear phonons and \( \varepsilon=\varepsilon(q) \) is the group velocity of the wave packet. The amplitude \( A(s, \xi) \) satisfies the nonlinear Schrödinger (NLS) equation

\[ i\partial_s A = -\frac{1}{2}w(q)\partial_x^2 A + h[A^2 A] \]

(4)

\( h \) depending on \( q \) and \( V, W \). In the focusing case \( w(q)h <0 \), the NLS equation admits sech-shaped solutions corresponding (at least formally) to travelling breather solutions

\[ u_n(t) = \varepsilon A(\psi, \xi(n-ct)) e^{(iqn-\omega t)} + c.c. + O(\varepsilon^2) \]

(5)

[\( O(\varepsilon^2) \) correction to \( \omega \) has been left in higher order terms]. One can write alternatively \( u_n(t) = u(n-ct, t) + O(\varepsilon^2) \) where

\[ u(\xi, t) = \varepsilon A(\psi, \xi) e^{(iq\xi-\omega t)} + c.c. \]

(6)

and \( \omega = \omega - \omega_c \). The function \( u(\xi, t) \) is localized in \( \xi \) and time-periodic with frequency \( \omega_b \) (denoted as traveling breather frequency).

The multiscale approach has been used by Tsuru\(^{3} \) and Flytzanis et al.\(^{40} \) for the FPU lattice. For system (1) with \( V''(0)>0 \) the validity of the nonlinear Schrödinger equation on large but finite time intervals has been proved recently by Giannoulis and Mielke.\(^{31,42} \)

C. Generalized solitary waves in Klein–Gordon lattices

It is a challenging problem to determine if the approximate solutions (5) could constitute the principal part of exact traveling breather solutions of the Klein–Gordon system. This would imply that linear dispersion is balanced by nonlinear terms at any order in the above mentioned multiscale expansion.

This problem has been solved by Loos and Kirchgässner for the KG system in the case of travelling waves,\(^4 \) where the
phase velocity of the plane wave equals the group velocity of the wave packet, i.e., \( c = \omega / q \). In that case approximate solutions (5) read \( u_0(t) = u_n(t-n\tau) + O(\varepsilon^2) \). Traveling wave solutions of (1) with \( V(x) = (\gamma/2)x^2 \) [and \( p = 1 \) in (2)] have the form \( u_n(t) = u_0(t-n\tau) \) and are determined by the scalar advance-delay differential equation

\[
\frac{d^2u_0}{dt^2} + W'(u_0) = \gamma(u_0(t + \tau) - 2u_0 + u_0(t - \tau)).
\]

Iooss and Kirchgässner have studied small amplitude solutions of (7) in different parameter regimes and have obtained in particular “nanopteron” (or generalized solitary waves) consisting of a solitary wave superposed on an exponentially small oscillatory tail. The leading order part of these solutions (excluding their tail) coincides with approximate solutions obtained via the NLS equation.

Their analysis is based on a reduction to a center manifold in the infinite dimensional case as described in Refs. 43–45. Equation (7) is rewritten as a reversible evolution problem in a suitable functional space, and considered for parameter values \((\tau, \gamma)\) near a critical curve (defined by \( 1/\tau = c = \omega / q \)) where the imaginary part of the spectrum of the linearized operator consists of a pair of double eigenvalues and a pair of simple ones. Close to this curve, the pair of double eigenvalues splits in two pairs of hyperbolic eigenvalues with opposite nonzero real parts, which opens the possibility of finding solutions homoclinic to 0.

Near these parameter values, the center manifold theorem reduces the problem locally to a reversible six-dimensional system of differential equations. The reduced system is put in a normal form which is integrable up to higher order terms. In some regions of the parameter space, the truncated normal form admits reversible orbits homoclinic to 0, which bifurcate from the trivial state and correspond to approximate solutions of (7).

However, as it is shown by Lombardi for different classes of reversible systems, \(^{46}\) these solutions should not generically persist when higher order terms are taken into account in the normal form. The existence of corresponding travelling waves decaying exactly to 0 should be a codimension-1 phenomenon, the codimension depending on the number of pairs of purely imaginary eigenvalues (i.e., the number of resonant phonons) in the parameter regime considered by the authors (there is one pair of purely imaginary eigenvalues, in addition to hyperbolic ones). However, to confirm the nonexistence of reversible homoclinic orbit to 0 (close to a small amplitude homoclinic orbit of the truncated normal form) for a given choice of \( \gamma \), \( \tau \), one has to check the nonvanishing of a certain Melnikov function being extremely difficult to compute in practice. \(^{46}\)

Due to this codimension-1 character, in a given system (7) (with fixed coupling constant \( \gamma \) and on-site potential \( W \)) exact travelling wave solutions decaying to 0 at infinity might exist in the small amplitude regime, but for isolated values of the wave velocity \( 1/\tau \).

Instead of orbits homoclinic to 0, the full normal form admits orbits homoclinic to small periodic ones \(^{46}\) (originating from the pair of purely imaginary eigenvalues). These solutions correspond to exact solitary wave solutions of (1) superposed on a small periodic oscillatory tail, which can be made exponentially small with respect to the central oscillation size [the minimal tail size should be generically nonzero for a given value of \((\tau, \gamma)\)].

### D. Traveling breathers in Klein–Gordon lattices

The above results do not cover the case of traveling breathers in which the wavepacket has different phase and group velocities. However, it has been recently shown by James and Sire \(^7,8\) that the center manifold approach initiated by Iooss and Kirchgässner is still applicable to the KG model when the traveling breather period and the inverse group velocity are commensurate, i.e., \( \omega_0/c = 2\pi m / p \) \((m \in \mathbb{Z})\). In that case the principal part of (5) satisfies (2) with \( \tau = 1/c \).

For fixed \( p \geq 2 \), problem (1) and (2) with \( V(x) = (\gamma/2)x^2 \) reduces to the \( p \)-dimensional system of advance-delay differential equations

\[
\frac{d^2u_1}{dt^2} + W'(u_1) = \gamma(u_2 - 2u_1 + u_p(t + p\tau)) \tag{8},
\]

\[
\frac{d^2u_n}{dt^2} + W'(u_n) = \gamma(u_{n+1} - 2u_n + u_{n-1}) \tag{9},
\]

\[
n = 2, \ldots, p - 1, \tag{10}
\]

where \( n = 2, \ldots, p - 1 \). The case \( p = 2 \) in (8)–(10) has been worked out explicitly and corresponds to the situation when the breather period equals twice the inverse group velocity.

The case when \( W \) is even yields the same type of reduced system for traveling breathers as for traveling waves. In this case, there exist exact traveling breather solutions superposed on an exponentially small periodic tail and satisfying \( u_n(t) = -u_{n-1}(t - \tau) \).

For asymmetric potentials, the simplest homoclinic bifurcation yields a higher-dimensional (eight-dimensional) reduced system, with a supplementary pair of simple imaginary eigenvalues of the linearized operator (the imaginary part of the spectrum consists of a pair of double eigenvalues and two pairs of simple ones). The reduced normal form of the system is reversible and integrable up to higher order terms. In some regions of the parameter space, the truncated normal form admits reversible orbits homoclinic to 0, which bifurcate from the trivial state and correspond to approximate solutions of (8)–(10). These approximate solutions coincide at leading order with spatially localized modulated plane waves obtained via the NLS equation.

However, by analogy with results of Lombardi \(^{46}\), it has been conjectured in Ref. 7 that these solutions do not generically persist when higher order terms are taken into account in the normal form. Persistence might be true if parameters \((\tau, \gamma)\) and coefficients in the expansion of \( W \) are chosen on a discrete collection of codimension-2 submanifolds in this infinite dimensional space. For general parameter values, instead of orbits homoclinic to 0 one can expect the existence of reversible orbits homoclinic to exponentially small two-
dimensional tori, originating from the two additional pairs of simple purely imaginary eigenvalues. These solutions should constitute the principal part of exact traveling breather solutions of (1) superposed on a small quasi-periodic oscillatory tail. However, in order to obtain exact solutions one has to prove the persistence of the corresponding homoclinic orbits as higher order terms are taken into account in the normal form. This step is nontrivial and would require to generalize results of Lombardi available when one pair of simple imaginary eigenvalues is removed. Another promising approach is developed in the recent work of Iooss and Lombardi on polynomial normal forms with exponentially small remainder for analytic vector fields.

The general case \( p \geq 2 \) in (8)–(10) is analyzed in Ref. 48 (this case is technically more difficult but the approach used in Refs. 7 and 8 works as well).

When the traveling breather period and the inverse group velocity are incommensurate, the principal part of (5) is not exactly translated on the lattice after time \( p/c \) but is modified by a spatial shift. The existence of exact traveling breather solutions of this type is still an open problem.

### E. Solitary waves and traveling breathers in Fermi–Pasta–Ulam lattices

The center manifold reduction method has also been used in Ref. 3 for studying traveling waves in FPU lattices. Near the largest critical value of velocity \( 1/\tau = \sqrt{V''(0)} \) (denoted as sound velocity), all small amplitude traveling waves are given by finite-dimensional reversible differential equations. In particular, solitary wave solutions have been obtained above (and near) the sound velocity (in addition, heteroclinic solutions connecting a stretched pattern with a compressed one have been obtained). The global existence of solitary waves in FPU models has been proved with other types of methods.\textsuperscript{1,2,49,50} In the small amplitude limit these solutions can be described by the Korteweg–de Vries (KdV) equation,\textsuperscript{51,52} or the modified KdV equation in degenerate cases.

In the FPU system there are no exponentially small phenomena preventing the existence of KdV solitary waves decaying exactly to 0, unlike for the KG lattice. This difference is due to the fact that localized solutions in the KG model (rather described by the NLS equation) mix a slow envelope decay with additional fast oscillations of a plane wave (see Refs. 46 and 53 for more details on this type of nonexistence phenomena).

The stability (in appropriate exponentially weighted spaces) of solitary wave solutions on fixed low energy surfaces has been proved in recent works.\textsuperscript{24–26} A localized perturbation of a supersonic solitary wave produces (asymptotically in time) a perturbation of the speed and phase of the wave, and a radiating part that travels slower than the wave and decays locally near it.

A discrete version of the center manifold method has been developed in Refs. 23 and 57 and applied to the existence of small amplitude static breathers in FPU chains. The FPU system is rewritten as a (reversible) mapping in a loop space, the index \( n \) playing the role of a discrete time. When the frequency of solutions is near the top of the phonon band, the system can be locally reduced to a (reversible) two-dimensional mapping. Static breathers (corresponding to orbits of the reduced map homoclinic to 0) have been proved to exist when a certain hardening condition on the interaction potential is satisfied. The case of diatomic FPU chains has been examined\textsuperscript{55} using the same method, and the existence of small amplitude breathers has been proved for arbitrary mass ratio and hard or soft potentials in various frequency regimes (extending previous results valid for large mass ratio\textsuperscript{59}).

In this paper we study the existence of traveling breathers in the FPU system and treat the commensurate case (2) in full generality. Near critical parameter values \( \tau = \tau_c \) [with \( \tau_c > (V''(0))^{-1/2} \)], the center manifold theorem reduces the problem locally to a finite dimensional reversible system of ordinary differential equations, whose dimension can be arbitrarily large (the dimension is of the order of the number of resonant phonons). Its principal part is integrable, and admits solutions homoclinic to quasi-periodic orbits if a hardening condition on the interaction potential \( V \) is satisfied. These orbits correspond to approximate traveling breather solutions [close to the NLS limit (5)] superposed on a quasi-periodic oscillatory tail. The problem of their persistence for the full reduced system is still open in the general case, and constitutes the final step for proving the existence of exact traveling breather solutions in FPU chains. Note that in the particular case of an even potential, and if the breather period equals twice the inverse group velocity \( p_{\text{in}} = 2, \omega_0/c = \pi \) in (5) and (6)], we indeed prove the existence of exact traveling breather solutions superposed on an exponentially small periodic tail.

Near the critical parameter value \( \tau = (V''(0))^{-1/2} \) (i.e., near the sound velocity), the problem is locally reduced to a finite dimensional reversible system of ordinary differential equations, which admits homoclinic orbits to 0 if \( V''(0) \neq 0 \). These orbits correspond to the FPU solitary waves obtained in Ref. 3 and satisfy (2) with \( p = 1 \). In addition, the principal part of the reduced system admits homoclinic orbits to quasi-periodic orbits (the persistence of these solutions for the full reduced equation is not yet established). For the FPU system, these solutions should correspond to solitary waves superposed on oscillatory pulsating traveling waves.

In Sec. II A we set the search of traveling breathers into the form of a spatial dynamical system, and give the results on the linearized system. In Sec. II B we treat the general case where the inverse critical velocity \( \tau_c \) is not \( (V''(0))^{-1/2} \) (“sound velocity”). In this section we derive the center manifold reduction, and the study of the solutions of the normal form of the reduced reversible system (we give the main steps of the analysis and shall provide the details in a forthcoming paper). We also give the interpretation of the corresponding solutions of the original system. In Sec. II C we consider the case \( \tau = (V''(0))^{-1/2} \), which was partly treated in Ref. 3, adding the study of cases \( p = 2 \). Section III provides a summary of the main results and discusses several open problems.
II. LOCALIZED WAVES IN FERMI–PASTA–ULAM CHAINS

This section treats the case of the FPU system

$$\frac{d^2u_n}{dt^2} = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}), \quad n \in \mathbb{Z}$$  \hspace{1cm} (11)

with

$$V(x) = \frac{1}{2} x^2 + \frac{\alpha}{3} x^3 + \frac{\beta}{4} x^4 + O(|x|^5)$$  \hspace{1cm} (12)

[one can fix $V'(0)=1$ without loss of generality]. We shall analyze small amplitude solutions of (11) satisfying (2) using the center manifold reduction approach.

The case $p=1$ of (2) is equivalent to fixing $u_n(t)=y(x)$ with $x=n-t/\tau$. System (11) is transformed into

$$\frac{d^2y}{dx^2} = V'(y(x+1) - y(x)) - V'(y(x) - y(x-1)),$$  \hspace{1cm} (13)

which is a scalar advance-delay or mixed type differential equation. Further references on advance-delay differential equations can be found in Ref. 60.

We note that Eq. (13) can be written in the form

$$dI_t/dx=0,$$  \hspace{1cm} (14)

Equation (13) is a particular case in the study which follows, and was studied in Ref. 12 for wave velocities $\tau$ close to the sound velocity $\tau=1$, i.e., the maximal velocity of linear phonons.

In the general case of (2) we set $u_n(t)=y_n(x)$, or equivalently $y_n(x)=u_n(\tau(n-x))$. This change of variables transforms condition (2) into the periodic boundary condition

$$y_{n+p}(x) = y_n(x)$$  \hspace{1cm} (15)

and system (11) is transformed into the ($p$-dimensional) system of advance-delay differential equations

$$\frac{d^2y_n}{dx^2} = V'(y_{n+1}(x+1) - y_n(x)) - V'(y_n(x) - y_{n-1}(x-1)).$$  \hspace{1cm} (16)

Equation (13) arises as a particular case of (14)–(15) with $p=1$. Note that system (15) admits the following first integral [use condition (14)]

$$I_p = \sum_{n=1}^{p} J_n,$$  \hspace{1cm} (17)

$$J_n = \frac{d^2y_n}{dx^2} - \int_0^1 V'(y_{n+1}(x+v) - y_n(x+v-1))dv.$$  \hspace{1cm} (18)

The general case of system (14)–(15) will be treated in Sec. II A. Due to the evenness of (15) in $\tau$ we shall assume $\tau>0$.

Although the center manifold theorem describes all small amplitude solutions of system (14)–(15) we shall concentrate on “spatially localized” waves in a generalized sense. These solutions are asymptotic as $x \rightarrow \pm \infty$ to simple shifts $y_n=b_n \ (b_n \in \mathbb{R})$, or to periodic (or quasi-periodic) orbits of small amplitude with respect to central oscillations.

Considering shifted solutions at infinity is necessary because system (15) possesses the invariance $y_n \rightarrow y_n+b \ (b \in \mathbb{R})$. The linearized system at $y_n=0$ admits a second invariance $y_n \rightarrow y_n+ax$, which is lost in the nonlinear case. This invariance is replaced by a more subtle one, as discussed in the following remark.

Remark 1: Note that (15) admits the particular solutions $y_n(x)=ax$, $a \in \mathbb{R}$ being an arbitrary constant. These solutions correspond to uniformly compressed or stretched states $u_n(t)=a(n-t/\tau)$ (depending whether $a<0$ or $a>0$). Moreover, if one chooses $a \in \mathbb{R}$ such that $\text{max}(a)>0$ (this holds at least for $a=0$), system (14)–(15) has the invariance

$$y_n \rightarrow \tilde{y}_n = y_n - ax, \quad \tau \rightarrow \sqrt{\text{max}(a)} \tau,$$  \hspace{1cm} (19)

where $V_n'(0)=0$, $V_n'(0)=1$. Consequently, from a given class of spatially localized solutions for $\tau=\tau_0$ one can construct similar solutions of (14)–(15) for $\tau=\tau_0/\sqrt{\text{max}(a)}$, superposed on a uniformly compressed or stretched state. Such solutions have been denoted as “mainly localized” solutions in Ref. 3 (case $p=1$, $\tau=1$). They have the form $y_n(x)=\tilde{y}_n(x,a)+ax$ where $\tilde{y}_n$ is a spatially localized solution of (14)–(15) for the modified potential $V_n$.

A. Traveling breathers as solutions of a spatial-dynamical system

1. Spatial-evolution problem

Instead of treating system (14)–(15) directly, we adopt a “dynamical system” point of view by rewriting (15) as an (infinite-dimensional) evolution problem in the spatial coordinate $x$. For this purpose we introduce the new coordinate $u(x,v)$ and functions $y_n(x,v)$. We use the notations $\xi_n = dy_n/du$, $\xi_n = dY_n/du$. The notation $U_n(x,v) = (y_n(x), \xi_n(x)) \in \mathbb{R}^{2n}$. This indicates our intention to construct $U_n$ as a map from $\mathbb{R}$ into some function space living on the $v$-interval $[-1,1]$. System (14)–(15) can then be formulated as an evolution problem in a suitable Banach space. For this purpose we introduce the following Bachan spaces $L^1$ for $U_n(x)$ and $H$ for $dU_n/du$

$$H = \mathbb{R}^2 \times (C^1[-1,1]),$$  \hspace{1cm} (20)

$$D = \{ \Psi = (y, \xi, \gamma) \in L^2 \times (C^1[-1,1]) \} \cap \{ \gamma(x) = y \}$$

with the usual maximum norms. Then we define $Y_n(x,v) \equiv y_n(x+v)$, $\xi_n = dy_n/du$ and $U_n \equiv (y_n, \xi_n, \gamma)$. The sequence $U(x) = (U_n(x))_{n \in \mathbb{Z}}$ is a $p$-periodic sequence in $D$. In the sequel we shall note

$$U(x) \in D_p = \{ U \in L^2 \} \cap \{ U \equiv U_n \forall n \in \mathbb{Z} \}.$$  \hspace{1cm} (21)

Similarly we look for $dU/du(x)$ in

$$H_p = \{ U \in L^2 \} \cap \{ U \equiv U_n \forall n \in \mathbb{Z} \}.$$  \hspace{1cm} (22)

These spaces are equipped with the usual maximum norms. System (14)–(15) can now be written in the form
\begin{equation}
\frac{dU}{dx} = L_r U + \hat{\tau} M(U)
\end{equation}

where the \(n\)th coordinate of \(L_r U\) reads
\[\left( L_r U \right)_n = \left( \hat{\tau} (\delta Y_{n+1} - 2Y_n + \delta Y_{n-1}) \right) \]
and \(n\)th coordinate of \(M(U)\) is given by
\[\left( M(U) \right)_n = (0, N(\delta Y_{n+1} - Y_n) - N(y_n - \delta Y_{n-1}), 0)^T,
\]
where we have set \(N(y) = V'(y) - y = O(y^2)\) as \(y \to 0\). It is clear that system (17) is equivalent to the original equation (15) since solutions satisfy \(Y_n(x,v) = y_n(x+v)\). The linear operator \(L_r\) maps \(D_p\) into \(H_p\) continuously. The nonlinearity \(M: D_p \to H_p\) is analytic in a neighborhood of 0 and \(\|M(U)\|_{D_p} = 0\).

We note that \(L_r\) and \(M\) (hence \(L_r + \hat{\tau} M\)) both commute with the index shift \(\sigma\) defined by \((\sigma U)_n = U_{n+1}\) [this comes from the invariance \(n \to n+1\) of (15)]. Invariant solutions under \(\sigma\) correspond in particular to traveling solutions of (11) \(U(x) = \text{Fix}(\sigma)\) is independent of \(n\) and its first component \(y(x)\) satisfies (13). If \(p\) is even, note that Eq. (17) is invariant under the symmetry \(\sigma^p\).

Moreover \(L_r\) and \(M\) both anticommute with the reflection \(\mathcal{R}\) in \(H_p\) given by
\[\left( \mathcal{R} U \right)_n = (-y_{-n}, -Y_{-n}(-v))^T.
\]
Therefore, Eq. (17) is reversible under \(\mathcal{R}\). This property is due to the invariance \(y_{-n} \to y_{-n}(x)\) of (15).

Equation (17) admits the first integral
\[I_\mathcal{R}(U) = \frac{1}{2} \sum_{p=1}^{P} \left( \xi_v - \hat{\tau} \right) \int_0^1 V(Y_{n+1}(v) - Y_n(v - 1)) dv
\]
which is issued from the first integral \(I_p\) of (15). One can check that \(I_\mathcal{R}\) is left invariant by \(\sigma\) and \(\mathcal{R}\) [use (14)].

Note that particular solutions of (17) are given by
\[U_{u,0}(x) = (ax + b) \chi_0 + a \chi_1,\] where \(a, b \in \mathbb{R}\) are arbitrary constants and
\[\left( \chi_0 \right)_n = (1, 0, 1)^T, \quad \left( \chi_1 \right)_n = (0, 1, v)^T
\]
(the components of \(\chi_0, \chi_1\) are independent of \(n\)). These solutions originate from the solutions \(y_{n+1}(x) = ax + b\) of (15). Solutions \(U_{u,0}\) are reversible under \(\mathcal{R}\) [i.e., \(\mathcal{R} U_{u,0}(x) = U_{u,0}(x)\)] since \(\mathcal{R} \chi_0 = -\chi_0\) and \(\mathcal{R} \chi_1 = \chi_1\).

2. Study of the linearized problem

We begin by studying the spectrum of \(L_r\), which consists of isolated eigenvalues with finite multiplicities \((L_r, \text{acting in } H_p\) with domain \(D_p\)) has a compact resolvent in \(H_p\). Since \(L_r\) has real coefficients and due to reversibility, its spectrum is invariant under the reflections through the real and the imaginary axis. Due to the periodic boundary condition (14), solving \(L_r U = \lambda U\) for \(\lambda \in \mathbb{C}\) is equivalent to searching for solutions of (15) in the form \(y_n(x) = e^{i\lambda x} e^{-2i m \pi n / p}\) for \(m = 0, \ldots, p-1\). This yields the dispersion relations
\[\frac{\lambda^2}{\tau^2} + 2(1 - \cos(\lambda - 2i m \pi / p)) = 0.
\]
As in Ref. 4, \(L_r\) is not sectorial and the central part of its spectrum (i.e., the set of purely imaginary eigenvalues) is isolated from the hyperbolic part. For purely imaginary eigenvalues \(z = i\lambda\), the dispersion relations read
\[\frac{\lambda^2}{\tau^2} + 2(\cos(\lambda - 2i m \pi / p) - 1) = 0.
\]

Corresponding linear particle displacements read
\[u_m(t) = a e^{i (\omega_m - \omega_0) t} + \text{c.c.}
\]
with \(\omega = \lambda - 2i m \pi / p, \omega_0 = \lambda / \tau, \text{ hence one recognizes in Eq. (21) the usual form of the dispersion relation } \omega^2 = 1 - \cos(q) \text{ of (11) linearized at } u_m = 0\). The case \(m = 0\) of (21) has been treated in Ref. 3. We note that (21) simplifies into
\[\frac{|\lambda|}{\tau} - \sin\left( \frac{\lambda}{2} - \pi \frac{m}{p} \right) = 0.
\]
Equations (21) and (23) admit the same roots, with identical multiplicities (at most 2) for \(\lambda \neq 0\).

Simple roots \(\lambda = 0\) correspond to simple eigenvalues \(\lambda\) of \(L_r\) for almost all values of \(\tau\). They occur in pairs \(\pm \lambda\) corresponding to conjugate modes \(m, p-m\). A pair of double semisimple eigenvalues exists if (23) admits a same root \(\lambda\) for \(m = m_1\) and \(m = m_2\) with \(m_1 \neq m_2\). This occurs for a finite number of parameter values \(\tau = \tau_{m_1, m_2}\) and \(\tau = \tau'_{m_1, m_2}\) defined by
\[\tau_{m_1, m_2} | \cos\left( \frac{\pi}{2p} (m_1 - m_2) \right) = \pi - \frac{\pi}{2p} (m_1 + m_2),
\]
\[\tau'_{m_1, m_2} | \sin\left( \frac{\pi}{2p} (m_1 - m_2) \right) = \pi - \frac{\pi}{2p} (m_1 + m_2).
\]

Double roots \(\lambda = 0\) correspond to double non semisimple eigenvalues \(\lambda\) of \(L_r\). They occur (in pairs) for critical parameter values \(\tau = \tau_i\) \((i \geq 1)\), ordered as an unbounded increasing sequence formed by the solutions of
\[\frac{\pi}{2p} \cos\left( \frac{\pi}{2} - \frac{m}{p} \right) = 1, \quad \frac{\pi}{2p} = \tan\left( \frac{\pi}{2} - \frac{m}{p} \right),
\]
where \(m\) ranges over \(0, \ldots, p-1\). Note that 1/\(\tau\) can be interpreted as a group velocity regarding (22) \((1/\tau = \omega'(q))\) and the second condition reads \(\omega'(q)(q + 2\pi m/p) = a(q)\) (phase and group velocities are equal for \(m=0\)).

A closer look at Eq. (23) indicates that, for \(p=1, \tau_1 > \pi, \text{ for } p > 2, \tau_1 = \pi, \text{ for } p=1, 1 < \tau_1 < \pi, \text{ and } \tau_1 \to 1 \text{ as } p \to +\infty\). Moreover, the set of critical velocity values \(c_{p}(p) = 1/\tau_{i}(p)\) \((p, k \geq 1)\) densely covers the interval \([0, 1]\). In addition, note that (24) implies
\[\lambda_{k} \sim 2\pi e^k \text{ as } k \to +\infty.
\]

The following lemma summarizes the evolution of the spectrum as \(\tau\) is varied.

\textbf{Lemma II.1.} For \(\tau < 1\), the spectrum of \(L_r\) on the imag-
nary axis consists of $p-1$ pairs of simple eigenvalues and the double nonsemisimple eigenvalue 0. Two pairs coincide if $\tau=\tau_{m,n}$ or $\tau=\tau'_{m,n}$, forming double semisimple eigenvalues. For $\tau=1$ a pair of real eigenvalues collides at 0, forming a fourfold eigenvalue. For $1<\tau<\tau_1$, the spectrum on the imaginary axis consists of $p$ pairs of simple eigenvalues and the double eigenvalue 0. For $\tau=\tau_1$, two pairs of eigenvalues (originating from the hyperbolic part of the spectrum) collide on the imaginary axis, forming a double nonsemisimple eigenvalue $\pm\lambda$ with $\lambda \neq 0$ (corresponding, for $p \geq 2$, to $m=1$, $p-1$). These pairs split on the imaginary axis for $\tau>\tau_1$. Similar eigenvalue collisions occur for all critical values $\tau=\tau_n$, increasing (by 4 at each step) the number of imaginary eigenvalues.

The vectors $\chi_0$, $\chi_1$ given in (19) define an eigenvector and a generalized eigenvector associated with the double eigenvalue 0 $(L_{\chi_0}=0, L_{\chi_1}=\chi_0)$. An eigenvector $\zeta$ of $L_{\tau}$ associated with a pair $(\lambda, m)$ ($\lambda \neq 0$) is given by

$$ (\zeta)_n = e^{-2i\pi m/n}(1, i\lambda, e^{i\lambda v})^T. $$

With this choice one has $R\zeta = \overline{\zeta}$. We note that $\sigma \zeta = e^{-2i\pi m/p} \zeta$ (this explains why eigenvalue collisions related to different modes $m$ correspond to semisimple eigenvalues, the two colliding modes having different symmetries). If $\lambda$ is a double eigenvalue, a generalized eigenvector $\eta$ satisfying $(L_{\tau} - \lambda I)\eta = \zeta$ is given by

$$ (\eta)_n = e^{-2i\pi m/n}(0, 1, v e^{i\lambda v})^T, $$

and $R\eta = -\overline{\eta}$.

B. Cases $\tau = \tau_k$

Let us analyze the situation when $\tau = \tau_k$ and consider $L = L_{\tau_k}$. The central part of the spectrum of $L$ consists in

- $N=p+2(k-1)$ pairs of simple eigenvalues $\pm i\lambda_j$ ($j = 1, \ldots, N$), $\lambda_j$ being associated with $m=m_j$ and an eigenvector $\xi_j$ (where $\xi_j$ is given by (26)),
- one pair of double eigenvalues $\pm i\lambda_0$, $\lambda_0$ being associated with $m=m_0$, an eigenvector $\xi_0$ given by (26) and a generalized eigenvector $\eta_0$ given by (27),
- the double eigenvalue 0.

In the sequel we denote by $P$ the spectral projection on the $2N$-dimensional central subspace, i.e., the invariant subspace associated with the central part of the spectrum for $\tau = \tau_k$. An efficient method for computing $P$ is detailed in Refs. 3 and 4.

The property of optimal regularity [see Ref. 45, hypothesis (ii), p. 127] is fulfilled by the affine linearized system

$$ \frac{dU}{dx} = LU + F(x), $$

where $F(x)$ lies in the range of the nonlinear term $M$ and has the form $(F(x))_{\eta} = (0, g_\eta(x), 0)^T$. This part of the analysis is similar to Ref. 4, Lemma 3, p. 448. Alternative methods of proof can be found in Refs. 61 and 62.

The property of optimal regularity and the existence of a spectral gap around the imaginary axis allow us to reduce (17) locally to a $2N$-six-dimensional reversible evolution problem on a center manifold.\(^{45}\)

1. Center manifold reduction

System (17) is invariant under the shift operator $U \mapsto U + q\chi_0$, which corresponds to the invariance $y_n \mapsto y_n + q$ in (15). The spectral projection on the generalized eigenspace corresponding to the double eigenvalue 0 ($\tau_k \neq 1$) has the form $P_0U = \chi_0(U)\chi_0 + \chi_1(U)\chi_1$, where $U_0 = (y_n, \xi_0, y_{n/0})^T$,

$$ x_0(U) = \frac{1}{p(1 - \tau_k^2)} \sum_{n=1}^{p-1} \int_{-1}^{1} F y_n dv, $$

$$ F(v) = 1 - |v| $$

and

$$ \chi_1(U) = \chi_0(L_{\chi_1} U) = \frac{1}{p(1 - \tau_k^2)} \sum_{n=1}^{p-1} \int_{-1}^{1} F y_n dv $$

Note that $DZ_0(0) = (1 - \tau_k^2)\chi_1$.

Due to the shift invariance it is natural to decompose any $U \in H_p$ as follows:

$$ U = W + q\chi_0, \quad \chi_0^*(W) = 0 $$

and we denote by $\tilde{H}$ the codimension-one subspace of $H_p$ where $W$ lies. We use the similar definition for the subspace $\overline{\mathbb{D}_p}$ of $\mathbb{D}_p$. Noticing that $\chi_0(M(U)) = 0$, system (17) becomes

$$ \frac{dW}{dx} = \tilde{L}W + \tau^2 M(W), $$

where $\tilde{L}W = L_{1k}W - \chi_1(W)\chi_0$. The operator $\tilde{L} = \tilde{L}_0$, acting in $\tilde{H}$ has the same spectrum as $L_{\tau_k}$ except that the eigenvalue 0 is now simple (with eigenvector $\chi_0$) instead of double. This eigenvalue is linked with the existence of the line of equilibria $W = a\chi_1$ ($a \in \mathbb{R}$) of (28), corresponding to the projection of $U_{a, k}$ on $\tilde{H}$ [one can check that $M(a\chi_1) = 0$].

System (28) is supplemented by the scalar equation

$$ \frac{dq}{dx} = \chi_0^*(W). $$

Due to the fact that $I_{\tilde{D}_p}(W + q\chi_0) = I_{\tilde{D}_p}(W)$ and $I_{\tilde{D}_p}(W)$ also defines a first integral of (28).

Recalling that $P$ is the spectral projection on the $2N$-dimensional central subspace invariant under $L$, we shall use the notations $\tilde{D}_k = P_{\tilde{D}_k}$, $\tilde{D}_k = (1 - P)_{\tilde{D}_k}$, $\mathbb{H}_k = (1 - P)\tilde{H}$, $W_k = (1 - P)W$, $W = PW$. In particular $\tilde{D}_k$ denotes the $2N$-five-dimensional central subspace invariant under $\tilde{L}$.

A good starting point for studying (28) locally is to consider the linear case

$$ \frac{dW}{dx} = \tilde{L}W. $$

Due to the existence of a spectral gap $d > 0$ separating the central part of the spectrum from the hyperbolic part (lying in the domain $|Re z| > d$), solutions of (30) can be split into two classes. On the one hand, solutions having a component on $\mathbb{D}_0$ grow at least like $e^{d|x|}$ as $x \to +\infty$ or $-\infty$. On the other
hand, solutions lying on \( D_1 \) are either bounded or diverge at most polynomially as \( |x| \to +\infty \) (the eigenvalues \( \pm i\lambda_0 \) are double nonsemisimple). Consequently, solutions of (30) which remain bounded in \( \tilde{D} \) as \( |x| \to +\infty \) necessarily belong to the \( 2N+5 \)-dimensional central subspace \( D_1 \).

In the nonlinear case (28) and for \( \tau = \tau_0 \), one can locally prove a similar result where the central subspace \( D_1 \) is replaced by a \( 2N+5 \)-dimensional invariant center manifold\(^{33} \) (for \( \tau = \tau_0 \) the center manifold is tangent to \( D_1 \) at \( W=0 \)). Indeed, for \( \tau = \tau_0 \) there exists a smooth local manifold \( M \subset \tilde{D} \) (which can be written as a graph over \( D_1 \)) invariant under the flow of (28) and the linear isometries \( \mathcal{R} \), \( \sigma \). For \( \tau = \tau_0 \), the center manifold \( M \) contains all solutions of (28) staying for all \( x \in \mathbb{R} \) in a certain neighborhood of \( W=0 \). It follows that all small amplitude solutions are determined by a \( 2N+5 \)-dimensional reversible differential equation satisfied by their coordinates on \( M \). We sum up these results in the following theorem (see Ref. 45 for a general account of the theory).

**Theorem II.2.** Let fix integers \( p \) and \( k \geq 1 \), then for any \( m \geq 2 \), there exists a neighborhood \( \mathcal{U} \times \mathcal{V} \) of \( (0, \tau_0) \) in \( \mathbb{R} 
\times \mathbb{R} \) and a map \( \psi \in \mathcal{C}^m(\mathcal{U} \times \mathbb{R}, \mathcal{D}_0) \) such that the following properties hold for all \( \tau \in \mathcal{V} \) (with \( \psi(0, \tau) = 0 \), \( D\psi(0, \tau_0) = 0 \)).

1. If \( W: \mathbb{R} \to \tilde{D} \) is a solution of (28) and \( W(x) \in \mathcal{U} \) for all \( x \in \mathbb{R} \), then \( W_h(x, \tau) = \psi(W(x), \tau) \) for all \( x \in \mathbb{R} \). \( W_1: \mathbb{R} \to \tilde{D} \) is a solution of

\[
\frac{dW_1}{dx} = \tilde{L}_\tau W_1 + F_1(W_1),
\]

where \( F_1(W_1) = O(\|W_1\|^2 + |\tau - \tau_0| \|W_1\|) \) reads

\[
F_1(W_1) = \rho [\tilde{L}_\tau - \tilde{L}_0 + \frac{1}{2} M_0] (W_1 + \psi(W_1, \tau)).
\]

2. Conversely, if \( W_1 \) is a solution of (31) with \( W_1(x) \in \mathcal{U} \) for all \( x \in \mathbb{R} \), then \( W = W_1 + \psi(W_1, \tau) \) is a solution of (28).

3. The map \( \psi(\cdot, \tau) \) commutes with \( \mathcal{R} \) and \( \sigma \), and (31) is reversible under \( \mathcal{R} \) and \( \sigma \)-equivariant.

Consequently, the \( 2N+5 \)-dimensional reduced equation (31) describes all small amplitude solutions of (28) as \( \tau \to \tau_0 \). Corresponding solutions of (17) are given by \( U=\tilde{W} + q \tilde{X}_0 \), with

\[
\frac{dq}{dx} = \tilde{X}_0^*(W_1).
\]

Note that \( \psi(\alpha X_1, \tau) = 0 \) for all \( \alpha = 0 \), due to the fact that \( \mathbb{R} X_1 \) is a line of equilibria of (28). In the same way, \( \mathbb{R} X_1 \) defines a line of equilibria of (31).

In addition, Eq. (31) admits the first integral

\[
I_\sigma(W_1, \tau) = \frac{1}{1 - \tau} \mathcal{I}_\sigma(W_1 + \psi(W_1, \tau)) = \tilde{X}_0^*(W_1) + O(\|W_1\|^2 + |\tau - \tau_0| \|W_1\|),
\]

and one can check that \( I_\sigma \) is left invariant by \( \sigma \) and \( \mathcal{R} \).

**2. Normal form**

Now we perform a change of variables close to the identity which simplifies (31) and preserves its symmetries. For that purpose we proceed in two steps.

First, we decompose \( W_1 \) into \( W_1 = \tilde{W}_1 + V_1 \) [with \( \tilde{X}_1^*(V_1) = 0 \)] and express the \( d \)-coordinate with \( D = \mathcal{I}_\sigma(W_1, \tau) = d + \h.o.t. \). Indeed, this equation can be locally inverted in \( d = \psi_1(D, V_1, \tau) \), where

\[
\psi_1(D, V_1, \tau) = D + O(\|D, V_1\|^2 + |\tau - \tau_0| \|D, V_1\|).
\]

System (31) takes the form

\[
\frac{dD}{dx} = 0, \quad \frac{dV_1}{dx} = G(V_1, D, \tau),
\]

where \( G(0, D, \tau) = 0 \).

Second, we consider the differential equation on \( V_1 \), treating \( D, \tau \) as parameters (the eigenvalue 0 is then removed in the \( V_1 \)-component). We use a normal form technique, i.e., we perform a change of variables \( V_1 = \tilde{V}_1 + \tilde{P}_{\tau,D}(\tilde{V}_1) \) close to the identity which simplifies (33) and preserves its symmetries. In the context of Hamiltonian systems the idea of normal forms goes back to Poincaré and has been extensively studied by Birkhoff (see, e.g., Ref. 63 for references). Here we employ a simple global characterization for normal forms of vector fields (not necessarily Hamiltonian) introduced in Ref. 64 by Elphick et al. (see also Ref. 65 for a detailed description of the method). We look for \( \tilde{P}_{\tau,D} \) as a polynomial in \( \tilde{V}_1 \) (being of order \( \ll m \)), satisfying \( \tilde{P}_{\tau,D}(0) = 0 \), \( D\tilde{P}_{\tau,D}(0) = 0 \), whose coefficients are \( \mathcal{C}^m \) functions of \( (\tau, D) \approx (\tau_0, 0) \). This change of variables transforms the \( V_1 \)-component of (33) into

\[
\frac{d\tilde{V}_1}{dx} = L\tilde{V}_1 + \phi(D, \tau, \tilde{V}_1) + O(\|	ilde{V}_1\|^p),
\]

where \( \phi(D, \tau, \tilde{V}_1) \) is a polynomial in \( \tilde{V}_1 \) (of order \( \ll m \)), satisfying \( \phi(D, \tau, 0) = 0 \) and \( D\phi(D, \tau, 0) = 0 \), with smoothly parameter dependent coefficients for \( (\tau, D) \approx (\tau_0, 0) \). According to the normal form theorem, one can choose \( \tilde{P}_{\tau,D} \) in such a way that \( \phi(D, \tau, \tilde{V}_1) \) commutes with \( e^{L\tau x} \) for all \( \tau \in \mathbb{R} \). This additional symmetry makes the leading part of (34) considerably simpler than (33) and (34) is denoted as normal form of (33) at order \( m \). The polynomial \( \phi(D, \tau, \tilde{V}_1) \) contains "resonant terms" which cannot be removed by coordinates changes (these terms are present due to the nonhyperbolicity of the equilibrium \( V_1 = 0 \)). Note that the formal series formed by the coefficients of \( \tilde{P}_{\tau,D} \) as \( m \to +\infty \) diverges in general. This is connected to the fact that (under certain nonresonance assumptions) the normal form of (33) at any order \( m \) is integrable if higher order terms are removed, whereas the original system (33) is a priori nonintegrable. In the sequel we set

\[
\tilde{V}_1 = A\tilde{X}_0 + B\eta_0 + \sum_{j=1}^N C_j \tilde{G}_j + \text{c.c.},
\]

where \( A, B, C_j \in \mathbb{C} \).

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The normal form computation is similar to Ref. 4, Sec. 6 and Appendix 2, to which we refer for details. The normal form structure is characterized by the fact that \( \phi_{A,B} \) commutes with \( e^{ixA} \) and preserves the symmetries of (33). We compute the normal form of (33) at order 3 under the following nonresonance condition:

\[
\lambda \cdot r \neq 0 \quad \text{for all } r \in \mathbb{Z}^{N+1} \text{ such that } 0 < |r| \leq 4,
\]

where \( \lambda = (\lambda_0, \ldots, \lambda_N) \) and \( |r| = |r_0| + \ldots + |r_N| \).

We avoid strong resonances. It is simple to show that (35) is satisfied for \( p = 1 \) and \( k = 1 \) (\( N = 1 \)), and we have checked (35) numerically for \( p = 2 \) and \( k = 1 \). In fact we make the conjecture that (35) is satisfied for any couple \((p,k)\).

The normal form of (33) at order 3 is given in the following lemma.

**Lemma II.3.** Assume strong resonances do not occur at \( \pi \approx \tau_k \), i.e., property (35) is satisfied. The normal form of (33) at order 3 reads:

\[
\begin{align*}
\frac{dA}{dx} &= i\lambda_0 A + B + iAP(|A|^2, I, Q, D) + \text{h.o.t.}, \\
\frac{dB}{dx} &= i\lambda_0 B + [iBP + AS](|A|^2, I, Q, D) + \text{h.o.t.}, \\
\frac{dC_j}{dx} &= i\lambda_j C_j + iC_j Q_j(|A|^2, I, Q, D) + \text{h.o.t.}, \\
\frac{dD}{dx} &= 0
\end{align*}
\]

(\( C_j \) is considered for \( j = 1, \ldots, N \), where \( Q \) denotes the vector \( Q = (|C_1|^2, \ldots, |C_N|^2) \), \( I = i(AB - \bar{A}B) \) and \( P, S, Q_j \) are polynomial functions of their arguments, with real coefficients depending smoothly on \( \tau \) for \( \pi \approx \tau_k \). The principal part of (36) is a cubic polynomial in \( A, B, C_1, \ldots, C_N \), their conjugates, and \( D \). Higher order terms are \( O(||\bar{V}||^4 + ||\bar{V}||D(||\bar{V}||^2 + D^2)) \). Equation (36) is reversible under the symmetry \( \mathcal{R} \) restricted to \( \mathcal{L} \).\n
\[
\mathcal{R}: (A, B, C_1, \ldots, C_N, D) \rightarrow (\bar{A}, -B, \bar{C}_1, \ldots, \bar{C}_N, D),
\]

and equivariant under the isometry \( \sigma \) restricted to \( \mathcal{L} \):

\[
\sigma = \text{diag}(e^{2\text{i}\pi n/p}, e^{2\text{i}\pi n/p}, e^{2\text{i}\pi n/p}, \ldots, e^{2\text{i}\pi n/p}).
\]

The polynomials \( P, S \) in the normal form (36) have the form

\[
\begin{align*}
P &= p_0(\tau) + fA + O(|D|) + |\tau - \tau_1||(|A|B)|, \\
S &= s_0(\tau) + l\tau A + O(|D|) + |\tau - \tau_1||(|A|B)|, \\
\end{align*}
\]

where \( r, s, f, l, g \in \mathbb{R} \) and \( p_0, s_0 \) are real-valued functions satisfying \( p_0(\tau_0) = s_0(\tau_0) = 0 \).

Small amplitude solutions of (36) correspond via Theorem II.2 and Eq. (29) to solutions of the evolution problem (17) having the form

\[
U = A\zeta_0 + B\eta_0 + \sum_{j=1}^{N} C_j \xi_j + c.c. + D\chi_1 + q\chi_0
\]

\[
+ \tilde{\psi}(A, B, C, \bar{A}, \bar{B}, \bar{C}, D, \tau),
\]

where \( C = (C_1, \ldots, C_N) \) and \( \tilde{\psi} \in C^{\infty}(C_2^{N+4} \times \mathbb{R}^2, \mathcal{D}) \) satisfies \( \tilde{\psi}(0, \tau) = 0, D\tilde{\psi}(0, \tau) = 0 \). Note that \( \tilde{\psi} \) has a component on \( \mathcal{D} \) due to the normal form transformation (see Ref. 65). We have in addition

\[
\frac{dz}{d\tau} = D + \chi_1^2 \psi(A, B, C, \bar{A}, \bar{B}, \bar{C}, D, \tau) \tag{38}
\]

The truncated normal form (in which higher order terms are neglected) is integrable \((|C|^2, D \text{ and } I \text{ are first integrals})\), and its study directly follows from the reversible 1:1 resonance case treated in Ref. 66. Small amplitude solutions of the truncated normal form yield approximate (leading order) solutions of (17) [cancel in \( \tilde{\psi} \) the terms of order higher than 3 in \( (A, B, C, D) \) in (37) and (38)]. For obtaining exact solutions one has to prove the persistence of a given class of solutions of the normal form when higher order terms are taken into account in (36). This problem is specially difficult in the case of homoclinic solutions, as we explain in the next section.

### 3. Homoclinic solutions of the normal form system

For \( \pi \approx \tau_k \) and \( \tau < \tau_k \), \( s(\tau) > 0 \) and the linearized operator \( \mathcal{L}_\tau \) has four symmetric hyperbolic eigenvalues \( \pm \sqrt{s_0} \pm \sqrt{\lambda_0 + p_0} \) close to \( \pm \lambda_0 \). The truncated normal form possesses orbits homoclinic to 0 related to these pairs of eigenvalues if \( s < 0 \). These solutions are given by \( C_j = D = 0 \) and

\[
A(x) = r_0(x)e^{i(\lambda_0 x + \phi(x) + \theta)}, \quad B(x) = \frac{dr_0}{dx}e^{i(\lambda_0 x + \phi(x) + \theta)},
\]

\[
\psi(x) = p_0 x + 2 \frac{r_0}{s} \sqrt{s_0} \tanh(\sqrt{s_0} x).
\]

These orbits are reversible under \( \mathcal{R} \) if one chooses \( \theta \) equal to 0 or \( \pi \).

Following the classical normal form computation scheme (see Ref. 4, p. 457), we find

\[
s = \frac{\lambda_0}{(\tau_0^2 - 1)^2} \left(4\alpha^2 - \frac{1 - \tau_0^2}{\tau_0^2} - 3\beta \left(1 - \frac{1}{\tau_0^2}\right)\right). \tag{39}
\]

The limit \( k \rightarrow +\infty \) in (39) corresponds to traveling breather velocities decaying to 0, since their inverses \( \tau_k \rightarrow +\infty \). As \( k \rightarrow +\infty \), \( s \) converges towards the finite value \( s_0 = 16(4\alpha^2 - 3\beta) \) [see (25)]. The condition \( s_0 < 0 \) is equivalent to the condition \( b = 3\beta - 4\alpha^2 > 0 \) for the existence of small amplitude static breathers in the FPU chain.\(^{23,57}\)

We now discuss the sign of \( s \) in more detail (we assume that at least one of the coefficients \( \alpha, \beta \) is nonzero).
For an even potential $V(\alpha=0)$, $s$ has the sign of $-\beta$ and orbits homoclinic to 0 exist (for the truncated normal form) in the hard potential case $\beta>0$.

The situation is more complex if $\alpha \neq 0$. We note that $s$ has the same sign as $-b+c_1^2(b+2\alpha^2)$, where $c_1(p) = 1/\tau_1(p)$ densely covers the interval $[0,1]$ for $p,k \gg 1$.

On the one hand, if $b<0$ one obtains $s>0$ (since $c_1<1$) and orbits homoclinic to 0 do not exist for the truncated normal form. This is the case in particular for $\beta \ll 0$ (case of “dark breathers,” by analogy with dark solitary waves).

On the other hand, if $b>0$ the condition $s<0$ is satisfied on the velocity interval $0 < c_1 < c_{\text{max}}$, where

$$c_{\text{max}}^2 = \frac{b}{b+2\alpha^2} < 1.$$  

We now discuss the question of persistence of reversible solutions homoclinic to 0, as higher order terms are taken into account in the normal form. For that purpose we restrict the flow of (36) to one on the invariant manifold $D=0$.

In the general case the persistence problem discussed along these lines is still an open question. Relevant tools for this problem are described in Refs. 46 and 47.

The particular case $N=1$, in which there is only one pair of purely imaginary eigenvalues $\pm \lambda_1$ in addition to weakly hyperbolic ones, is denoted as $(i\lambda_0)^2\lambda_1$ resonance and has been treated in Ref. 46. This case arises only for $p=1$, i.e., for a certain type of traveling wave solutions of (11). Under suitable nonresonance conditions on the eigenvalues, reversible solutions of the normal form, which are homoclinic to periodic orbits, persist above a critical orbit size, which is exponentially small with respect to $|\tau - \tau_1|$. On the contrary, reversible orbits homoclinic to 0 do not persist generically for the full normal form when higher order terms are taken into account.

In what follows we shall examine the general case $N \geq 1$ intuitively using formal geometric arguments. Since the phase space is now $(2N+4)$-dimensional (with $D=0$), the subspace $\text{Fix}(R)$ is $(N+2)$-dimensional and the stable manifold $W^s(0)$ of the origin (for $\tau = \tau_2$) is two-dimensional, the intersection of $\text{Fix}(R)$ and $W^s(0)$ (at a point $\tilde{V}_c \neq 0$) should be a phenomenon of codimension $N$. In particular, the lowest codimension value is $p$ and occurs at $\tau = \tau_1$. Consequently, one can expect that reversible solutions homoclinic to 0 should not persist generically for the full normal form (36).

By analogy with the $(i\lambda_0)^2\lambda_1$ resonance case, we expect the splitting size between $\text{Fix}(R)$ and $W^s(0)$ to be exponentially small as $\tau \to \tau_2$. Exponential smallness can be intuitively understood in the sense that (under suitable nonresonance assumptions) the truncated normal form of (31) at any order admits solutions homoclinic to 0 (the normal form structure is the same as (36)).

Note that one could allow for an additional degree of freedom by searching orbits homoclinic to nonzero equilibria, $D=0$ being treated as a free parameter (this is in connection with Remark 1).

In addition to solutions homoclinic to 0 (for $s<0$, $\tau \approx \tau_1$ and $\tau<\tau_1$), the truncated normal form admits reversible solutions homoclinic to small quasi-periodic orbits, corresponding to $|C_j|=\text{const.}$, $j=1,\ldots,N$. Similar solutions exist around nonzero equilibria, provided $D$ is small enough ($D=0(\tau-\tau_1)$). By analogy with the $(i\lambda_0)^2\lambda_1$ resonance case, we conjecture for the full normal form the existence of reversible orbits homoclinic to $N$-dimensional tori, whose sizes could be made exponentially small with respect to $|\tau - \tau_1|$.

Special features arise in the case $p=2$ and $\tau = \tau_1$ ($N=2$) if the interaction potential $V$ of (11) is even. Due to the additional invariance $u_n \rightarrow -u_n$ of (11), Eq. (17) is also invariant under the symmetry $-\sigma$. Fixed points of $-\sigma$ correspond to solutions of (11) satisfying $u_{n+1}(t) = -u_n(t)$. If one considers the reduced equation (31) on the invariant subspace $\text{Fix}(-\sigma)$, one pair of purely imaginary eigenvalues $\pm \lambda_1$ [corresponding to $m=0$ in (21)] and an eigenvector $e_2$ invariant under $\sigma$ is removed. One has also $\chi_i(W_c)=0$ on $\text{Fix}(-\sigma)$ (since $\chi_1$ is invariant under $\sigma$), hence the eigenvalue 0 is removed. The invariant subspace $\text{Fix}(-\sigma)$ contains the stable and unstable manifolds of 0 and one recovers the $(i\lambda_0)^2\lambda_1$ resonance case [both eigenvalues corresponding to $m=1$ in (21)]. In addition we note that $\mathcal{I}_c=0$ on $\text{Fix}(-\sigma)$ ($\mathcal{I}_c$ is invariant under $\sigma$ and commutes with $-\sigma$).

Consequently, reversible solutions under $R$ or $-R$, lying on $\text{Fix}(-\sigma)$ and homoclinic to periodic orbits persist above a critical orbit size, which is exponentially small with respect to $|\tau - \tau_1|$. On the contrary, reversible orbits homoclinic to 0 should not persist generically for the full normal form when higher order terms are taken into account.

This reduction procedure generalises to the case when $p$ is even ($V$ being symmetric), the relevant symmetry being $-\sigma^{n/2}$. However, for $\tau = \tau_1$ there remain $p/2$ pairs of simple imaginary eigenvalues in addition to the two weakly hyperbolic pairs, and we have no existence result for $p \geq 4$.

4. Traveling breather solutions

The solutions of the truncated normal form yield approximate (leading order) solutions of (15)

$$y_n(x) \approx A(x)e^{-2i\mu_0p/\mu} + \sum_{j=1}^{N} (C_j(x)e^{-2i\mu_jp/\mu}) + \text{c.c.}$$

$$+ q(x),$$

$$\text{(40)}$$

with $dq/dx = D$ [principal part of Eq. (38)]. Corresponding approximate solutions of the FPU system (11) read $u_n(t) = y_n(n-1/\tau)$.

Leading order solutions of the normal form, which are homoclinic to small quasi-periodic orbits, should constitute the principal part of traveling breather solutions of system (11), superposed at infinity on a quasiperiodic oscillatory tail and a uniformly stretched or compressed state [even for $D=0$, due to the $\psi$ contribution in (38)]. We sum up our findings in the following theorem (we exclude the case $p=1$ corresponding to traveling waves).

**Theorem 1.4.** Fix $p \geq 2$ in (2), $k \geq 1$ and consider the near-critical case $\tau = \tau_1(p)$, and assume $\mu_0 \neq 0$. Assume the nonresonance condition (35) holds. Assume $s<0$, $\tau$ sufficiently close to $\tau_1$ with $\tau<\tau_1$ and $D=0(\tau-\tau_1)$. The reduced equation (33) written in normal form and truncated at order 4 admits a $N+1$-parameter family of orbits ho-
mothetic to $N$-dimensional tori with $N=p+2(k-1)$.

This result is the first step in the proof of the existence of exact traveling breather solutions superposed on an exponentially small oscillatory tail. For obtaining exact solutions of (11) one has to prove the persistence of the above mentioned homoclinic solutions of (36) without h.o.t., when higher order terms are taken into account. This problem is still open in the general case $N>1$ but can be solved in a particular case.

**Remark 3:** In the case when $m_0=0$, we obtain a principal part invariant under $\sigma$, hence a solitary wave superposed on small oscillatory pulsating travelling waves.

**Theorem II.5.** Assume $p=2$, $s<0$ and $\tau<\tau_1$ with $\tau<\tau_1$. Moreover assume that the potential $V$ is even. Equation (17) is invariant under the symmetry $-\sigma$. The full reduced equation (31) restricted to $\bar{x}(\sigma)$ admits small amplitude reversible solutions (under $\pm \mathcal{R}$) homoclinic to periodic orbits. For a fixed value of $\tau$ (and up to a shift in $x$), these solutions occur in a one-parameter family parametrized by the amplitude of periodic orbits. The lower bound of these amplitudes is $O(e^{x(y(\tau-\tau_1)1.5})$ ($c>0$). These solutions correspond to exact travelling breather solutions of system (11) superposed at infinity on an oscillatory (periodic) tail and satisfying $\eta_{th}(t)=-\eta_h(t-\tau)$.

Note that the existence of modulated plane waves in FPU chains has been studied by Tsuru\textsuperscript{15} using formal multiscale expansions. Under this approximation, the wave envelope satisfies the NLS equation. The condition obtained by the author for the existence of NLS solitons [with group velocity $\omega'(q)=1/\tau_1$] is exactly the condition $s<0$ derived above.

**C. Case $\tau=1$**

This section is an extension of the study made in Ref. 12, where only $p=1$ was considered. Let us follow the same lines as for $\tau=\tau_1$. In the present case, following Lemma II.1 for $\tau=1$ we have for the linear operator $L_1$, in addition to $N=p-1$ pairs of simple eigenvalues $\pm \lambda_3$ on the imaginary axis, a quadruple eigenvalue in 0, with eigenvectors

\[(x_0)_n = (1,0,1)^T, \quad (x_1)_n = (0,1,v)^T, \quad (x_2)_n = (0,0,v^2/2)^T, \quad (x_3)_n = (0,0,0,3v/6)^T,
\]

which satisfy

\[
L_1 x_0 = 0, \quad L_1 x_j = \chi_{j-1}, \quad j = 1,2,3, \quad \mathcal{R} x_0 = -x_0, \quad \mathcal{R} x_1 = x_1, \quad \mathcal{R} x_2 = -x_2, \quad \mathcal{R} x_3 = x_3.
\]

The spectral projection on the four-dimensional generalized eigenspace belonging to 0 has the form

\[P_0 U = \sum_{0 \leq j \leq 3} X_j(U)X_j^*,
\]

where

\[U_n = (y_1, \xi_1, Y_1)^T X_j^*(X_j) = \delta_{j},
\]

\[X_0^*(U) = \frac{2}{5} \sum_{1 \leq n \leq p} \left( y_n - \int_{-1}^{1} G Y_1 dv \right),
\]

\[X_1^*(U) = \chi_0(L_1 U) = \frac{2}{5} \sum_{1 \leq n \leq p} \left( \xi_n + \int_{-1}^{1} G Y_1 dv \right),
\]

\[X_2^*(U) = \chi_1(L_1 U) = -\frac{12}{p} \sum_{1 \leq n \leq p} \left( y_n - \int_{-1}^{1} F Y_1 dv \right),
\]

\[X_3^*(U) = \chi_2(L_1 U) = -\frac{12}{p} \sum_{1 \leq n \leq p} \left( \xi_n + \int_{-1}^{1} F Y_1 dv \right),
\]

where we note that for $v \neq 0$, $G''=-30F=-30(1-|v|)$, and $X_3(L_1 U) = 0$.

1. **Center manifold reduction**

In the same way as in Sec. II B we use the decomposition of any $U \in \mathcal{H}_p$ as

\[U = W + q \chi_0, \quad \chi_0(W) = 0,
\]

and we obtain the system (28)–(29), where the operator $\bar{L}$ acting in $\mathcal{H}$ has the same spectrum as $L_1$ except that the eigenvalue 0 is triple (eigenvector $\chi_3$). Theorem II.2 still applies in this case, replacing $\tau_1$ by 1, $W$ lies in a (2$p-1$ +3)-dimensional space, and the first integral (18) becomes

\[
\mathcal{I}_c(W_r, \tau) = -12 \mathcal{I}_c(W_r + \psi(W_r, \tau)) = \chi_3(W_r) + O(||W_r||^2 + (|\tau-1||W_r||).
\]

2. **Normal form**

Following the same structure as in Sec. II B, we decompose $W_r$ as follows:

\[W_r = d \chi_{V_r} + V_r, \quad \chi_3(V_r) = 0,
\]

and use coordinate $D$ instead of $d$, where, as above

\[D = \mathcal{I}_c(W_r, \tau), \quad d = \varphi_3(D, V_r, \tau),
\]

\[\varphi_3(D, V_r, \tau) = D + O(||\chi_3(V_r)||^2 + |\tau-1||\chi_3(V_r)||).
\]

Then system (33) is still valid, except that

\[G(0, D, \tau) \neq 0
\]

since the line of solutions for (28) is $\mathcal{R} \chi_1$ and not $\mathcal{R} \chi_3$. As above, $D$ and $\tau$ are treated as parameters, and 0 is now a double eigenvalue of the linearized system in $V_r$, for $\tau=1$, $D=0$. Then we have the following (see Refs. 3 and 65 where the invariant $D$ corresponds to the first integral $H$ of the normal form).

**Lemma II.6.** Assume strong resonances do not occur at $\tau=1$, i.e., property (35) is satisfied for $\lambda_j, j=1, \ldots , N (N=p-1)$. Denoting by $A$, $B$ the components of $V_r$ (after the polynomial change of variables) along $\chi_1$ and $\chi_2$, the normal form of (33) at order 3 reads

\[
\frac{dA}{ds} = B.
\]
\[
\frac{dB}{dx} = D + A \phi(A,Q,D,\tau) + \text{h.o.t.},
\]
\[
\frac{dC_j}{dx} = i\lambda_j C_j + iC_j Q_j(A,Q,D,\tau) + \text{h.o.t.},
\]
\[
\frac{dD}{dx} = 0
\]
\[(41)\]

(C\textsubscript{j} is considered for \(j = 1,\ldots,N\), where \(Q\) denotes the vector \(Q=([C_1]^2,\ldots,/[C_N]^2)\), and \(\phi\) and \(Q_j\) are polynomial functions of their arguments \(A,B,C_j,D\), with real coefficients depending smoothly on \(\tau\) for \(\tau = 1\). The principal part of (41) is a cubic polynomial in \(C_1,\ldots,C_N\), their conjugates, and \(A,B,D\). Higher order terms are \(O(\|\vec{V}\|+|D|^4)\). Equation (41) is reversible under the symmetry \(R\) restricted to \(D_{\epsilon}\)

\[
\mathcal{R}(A,B,C_1,\ldots,C_N,D) \mapsto (A,-B,\bar{C}_1,\ldots,\bar{C}_N,D),
\]

and equivariant under the isometry \(\sigma\) restricted to \(D_{\epsilon}\)

\[
\sigma = \text{diag}(1,1,e^{-i2\pi n/p},\ldots,e^{-i2\pi m/p},1).
\]

The polynomial \(\phi\) has the form

\[
\phi(A,Q,D,\tau) = v + aA + bA^2 + \sum_{1 \leq j < p \leq 1} b_j |C_j|^2
\]

where \(v,a,b,b_j\) are smooth functions of \((D,\tau)\) near \((0,1)\), and \(v(D,1)=0\),

\[
v(D,\tau) = 24(1 - \tau)[1 + O(|D| + |1 - \tau|)],
\]

a(D,\tau) = \(-12a[1 + O(|1 - \tau|)] + O(|D|)\),

b(D,\tau) = \(-12b[1 + O(|1 - \tau|)] + O(|D|)\), if \(a = 0\)

(see Ref. 3 for the computation of the coefficients \(a\) and \(b\)). Small amplitude solutions of (41) correspond, via Theorem II.2 and Eq. (29) to solutions of the evolution problem (17)

having the form

\[
U = A\chi_1 + B\chi_2 + \sum_{j=1}^N C_j \chi_j + \text{c.c.} + D\chi_3 + q\chi_0
\]

\[
+ \tilde{\psi}(A,B,C,D,\tau),
\]

(42)

where \(C = (C_1,\ldots,C_{p-1})\) and \(\tilde{\psi} \in C^\infty(R^2 \times (\mathbb{Z}^p-2 \times R^2,\vec{D})\) satisfies \(\tilde{\psi}(0,\tau) = 0\). (\(\vec{D}\) is the linearized operator \(L\) restricted to \(D_{\epsilon,\tau}\)) Note that \(\tilde{\psi}\) has a component on \(D\) due to the normal form transformation (see Ref. 65) and that the line of steady solutions of (28) corresponds to

\[
B = 0;\quad D + A\phi(A,0,D,\tau) + \text{h.o.t.} = 0
\]

which gives a component on \(\chi_3\) annihilated by \(\tilde{\psi}(A,0,0,0,D,\tau)\). We have in addition

\[
\frac{d\psi}{dx} = A + \chi^*_3 (\tilde{\psi}(A,B,C,D,\tau)).
\]

The truncated normal form (in which higher order terms are neglected) is integrable \(([C]^2\text{ and }D\text{ are first integrals}), and its study directly follows from the reversible \(0^2\) singularity case (see, for example, Ref. 65). Small amplitude solutions of the truncated normal form yield approximate (leading order) solutions of (17) [cancel in \(\tilde{\psi}\) the terms of order higher than 3 in \((A,B,C_j,D)\) in (42) and (43)]. For obtaining exact solutions one has to prove the persistence of a given class of solutions of the normal form when higher order terms are taken into account in (41).

### 3. Homoclinic solutions of the normal form system

For \(\tau = 1\) and \(\tau < 1\), then \(\nu > 0\) and the linearized operator \(L\) has one simple eigenvalue 0, and a pair of real symmetric eigenvalues \(\pm i\nu\) near 0. For \(4aD < \nu^2\), the truncated normal form (41) possesses two equilibria \((A = A_0, B = 0, C_j = 0)\), one hyperbolic, one elliptic, and an orbit homoclinic to the hyperbolic equilibrium. For \(D = 0\), and \(\alpha \neq 0\) this orbit is homoclinic to 0 and given by

\[
A(x) = -\frac{3\nu(0,\tau)}{2a(0,\tau)\cos^2(\nu^{1/2}x/2)}.
\]

\[
B(x) = A'(x), \quad C_j = 0, \quad j = 1,\ldots,p-1.
\]

For \(\alpha = 0\), \(\tau < 1\), \(b < 0\) and \(D = 0\), orbits homoclinic to 0 are given by

\[
A(x) = \pm(\nu/\nu^{1/2}x).
\]

\[
B(x) = A'(x), \quad C_j = 0, \quad j = 1,\ldots,p-1.
\]

while for \(\alpha = 0\), \(\tau > 1\) and \(b > 0\) we have for \(D = 0\) a pair of symmetric front solutions, given by

\[
A(x) = \pm(-\nu/\nu^{1/2}x)\tan^2(\nu/2)^1/2x),
\]

\[
B(x) = A'(x), \quad C_j = 0, \quad j = 1,\ldots,p-1.
\]

while for \(D \neq 0\) we have orbits homoclinic to one of the two hyperbolic fixed points (see Ref. 3 for the details with respect to the dependency in function of \(D\)).

It is shown in Ref. 3 that all the homoclinics and front solutions of the normal form persist when one considers the complete system, when \(p = 1\), i.e., when we only look for traveling waves. This also corresponds to the persistence of such solutions for larger \(p\), in considering solutions in the subspace invariant under the mapping \(\sigma\).

Now, we see that the cubic normal form also admits a family of orbits homoclinic to quasi-periodic solutions, with \([C_j] = \text{const}\) where these constants are small enough (change \(\nu\) in the above formulas into \(\nu + \sum b_j |C_j|^2\)). As for the case \(\tau = \tau_0\), we are not able to prove the persistence of such homoclinics, except for the case \(p = 2\) where there is only one coordinate \(C_1 \in C\). In such a case we recover the study made in Ref. 46 of the reversible \(0^2\) \(\chi_1\) singularity of a reversible vector field, where there exists a family of two reversible orbits homoclinic to periodic orbits, provided that their size is not smaller than a quantity which is exponentially small with respect to \(|1 - \tau|\). Note that in this latter case, the action of the map \(\sigma\) exchanges the two orbits in changing \(C_1\) into \(-C_1\), and that the orbit homoclinic to 0 persists, due to its invariance under \(\sigma\).
4. Solitary waves superposed on small oscillatory pulsating traveling waves

The solutions of the cubic normal form yield approximate solutions with leading order

\[ y_n(x) = \sum_{1 \leq j \leq p-1} C_j(x) e^{-2\pi m_x \omega_p} + \text{c.c.} + q(x), \]

and corresponding solutions of the FPU system (11) read

\[ u_{n}(t) = y_{n}(n-t/\tau). \]

Leading order solutions of the normal form which are homoclinic to small quasi-periodic orbits should correspond to a principal part of solitary waves, solutions of system (11), superposed on a quasiperiodic oscillatory pulsating traveling wave, and on a uniformly stretched or compressed state [see (43)]. Note that in the literature solitary wave solutions of (11) correspond in fact to front solutions [the term “solitary wave” refers to the difference displacements \( u_n(t) - u_{n-1}(t) \), which tend towards 0 as \( n \to \pm \infty \)].

We sum up our findings in the following theorem (we exclude the case \( p=1 \) corresponding to traveling waves).

**Theorem II.7.** Fix \( p \gg 2 \) in (2), and consider the near-critical case \( \tau \approx 1 \). Assume the nonresonance condition (35) holds. Assume \( \tau \) sufficiently close to 1, and \( D=0 \) \((D=\alpha/|\tau-1|)\). In the general case \( \alpha \neq 0 \), for fixed \( \tau < 1 \), the reduced equation (33) written in normal form and truncated at order 4 admits a \( p \)-parameter family of orbits homoclinic to (\( p-1 \))-dimensional tori. In the case of an even hardening potential \( (\beta > 0) \), we have for fixed \( \tau < 1 \) a \( p \)-parameter family of (pairs of) homoclinics to (\( p-1 \))-dimensional tori. In case of an even softening potential \( (\beta < 0) \), we have for fixed \( \tau > 1 \) a \((p-1)\)-parameter family of symmetric fronts (one fixes \( D=0 \)) connecting symmetric \((p-1)\)-dimensional tori, in addition to a \( p \)-parameter family of orbits homoclinic to \((p-1)\)-dimensional tori. For \( p=2 \), the above result holds in replacing the tori by periodic orbits, and holds true for the full system if \( \alpha \neq 0 \). In the latter case the lower bound of the amplitudes of limiting periodic orbits is \( O(e^{-|\tau-1|}) \), \((c > 0)\). These solutions are superpositions of a solitary wave, solution of system (11), with small periodic oscillatory pulsating traveling waves.

III. SUMMARY AND DISCUSSION

We have analyzed the existence of traveling breathers in FPU chains in the commensurate case (2).

Near critical parameter values \( \tau \approx \tau_{c} \) (corresponding to breather velocities \( c_{c} = 1/\tau_{c} \) below the “sound velocity”), the center manifold theorem reduces the problem locally to a finite dimensional reversible system of ordinary differential equations. The dimension of the reduced system is of the order of the number of resonant phonons and diverges as \( p \to +\infty \) in (2). Nevertheless its principal part is integrable, and admits solutions homoclinic to quasi-periodic orbits if a hardening condition on the interaction potential \( V \) is satisfied. These orbits correspond to approximate solutions of the FPU system, consisting in a traveling breather superposed on a quasi-periodic oscillatory tail. The persistence of the corresponding homoclinics for the full reduced system is still an open problem in the general case, and constitutes the final step for proving the existence of exact traveling breather solutions in FPU chains. For an even potential and if one fixes \( p=2 \) in (2), we indeed prove the existence of exact traveling breather solutions superposed on an exponentially small periodic tail.

Near the critical value \( \tau=(V^\prime(0))^{-1/2} \) (corresponding to the sound velocity), we have locally reduced the problem to a finite dimensional reversible differential equation, the normal form of which admits homoclinic orbits to 0 if \( V^\prime(0) \neq 0 \). These orbits correspond to the FPU solitary waves obtained in Ref. 3 and satisfy (2) with \( p=1 \). In addition, for \( p \gg 2 \), the principal part of the reduced equation admits homoclinic orbits to quasi-periodic orbits (the persistence of these solutions for the full equation is not yet established). For the FPU system, these solutions should correspond to solitary waves superposed on an oscillating pulsating traveling wave. If \( p=2 \) in (2) we prove the existence of exact solutions of this type (in that case the solution tail is time-periodic).

To close the paper we would like to point out several open problems related to traveling breather solutions.

First, our results and those of Ref. 7 (corresponding to Klein–Gordon lattices) are valid in the small amplitude regime in which traveling breathers have a broad central hump. One can question whether these solutions could be continued into highly localized pulses, depending on the shape of the interaction potential. Highly localized approximate traveling breather solutions are indeed observed in numerical simulations of the FPU lattice (see, e.g., Ref. 21 for references). In addition, highly localized exact traveling breathers and solitary waves (with a small oscillatory tail) have been numerically observed in Klein–Gordon lattices (17,29) (these solutions have been found for various types of on-site potentials).

Second, the case when the breather period and the inverse velocity \( \tau \) are incommensurate remains open. This corresponds to searching for solutions of (1) in the form \( u_{n}(t) = u(n-t/\tau,t) \), where the function \( u(\xi,t) \) is localized in \( \xi \) and time-periodic (with period \( T_{b} \)), \( T_{b} \) and \( \tau \) being incommensurate. In that case, the displacement pattern is not exactly translated on the lattice after times \( pt \ (p \in \mathbb{Z}) \), but becomes slightly modified for suitable values of \( p \) (as \( T_{b} = \tau k, k \in \mathbb{Z} \)). The bifurcation problem is infinite-dimensional even locally [as one takes formally the limit \( p \to +\infty \) in (2) the number of central modes diverges]. By treating the incommensurate case one may find continuous families of traveling breather solutions parametrized (at fixed frequency) by their velocity \( c=1/\tau \) (the limiting case \( c=0 \), when it occurs, would correspond to static breathers).

In this discussion we have restricted ourselves to existence problems, but many other questions would be worth being examined, e.g., those regarding stability or propagation in heterogeneous media.
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