SOME COMMENTS ON OBSERVABILITY NORMAL FORM AND STEP-BY-STEP SLIDING MODE OBSERVER

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Abstract. Classical methods to design an observer for nonlinear system are almost based on the system either linearly observable or detectable. In this paper, an example is provided to point out, for the linearly unobservable and undetectable system, with the help of observability normal form and sliding mode observer, it is also possible to design an observer. In addition, other applications of observability normal form, especially on how to improve the robustness of ciphering in a secure data transmission, are highlighted.

Key Words: Observability Normal Form, Sliding Mode Observer, Observability Bifurcation

1. Introduction. For most systems (mechanic, electric, economic and so on), not all state variables can be measured directly, so the problem of estimating the unmeasurable state variables is very important in control theory, and also in other domains. For linear systems, it has been extensively studied. But for nonlinear systems, the theory of observers is not nearly as neither complete nor successful as it is for linear systems. In addition, in many cases and also in the synchronization problem of chaotic system, the linear approximation of a nonlinear system sometimes is neither observable nor detectable on a submanifold. This problem is studied for example in cryptography in continuous-time by L. Boutat-Baddas [1] and in discrete-time by I. Belmouhoub [4]. In the thesis of L. Boutat-Baddas [3] she proposed to use the form of Poincaré [5] to study the observability bifurcation, and this approach is a dual approach introduced by W. Kang and A. Krener for the study of the controllability bifurcation. In order to highlight the efficiency of the proposed method, we will use the observability normal form to deal with a simple example which is not linearly detectable. Then we try to design an observer using the intrinsic properties of variable structure of the step by step sliding mode observers to take account of the passage through the unobservable manifold. It should be noticed that this type of observers has more advantages for guaranteeing the convergence of the observation error dynamic in finite time if the system is observable and verified certain conditions [12].

This paper is organized as follows: In section 2 observability normal form is recalled. Section 3 illustrates how to design an observer for a simple example with singularity. The final section discusses other applications of observability normal form, including solving the left invertibility problem, increasing the robustness of ciphering in a secure data transmission.

2. Recall of Observability Normal Form. In this section, for simplification we just recall quadratic observability normal form for a nonlinear system with one real linear unobservable mode (see [3] for general observability normal form and see [6] for exhaustive demonstrations). Consider a SISO (single-input single-output) system:

$$\begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)u \\ y &= C\xi = h(\xi) \end{aligned} \tag{2.1}$$

where the vector fields $f, g : U \subset R^n \to R^n$ are assumed to be real analytic, such that $f(0) = 0$, so the pair $(\xi_e = 0, u = 0)$ is an equilibrium point. Supposed $A = \frac{\partial f}{\partial t}(0)$, $B = g(0)$, and thus system (2.1) can be expanded into the following form around this equilibrium point via Taylor expansion at order 2:

$$\begin{aligned} \dot{z} &= Az + Bu + f^{[2]}(z) + g^{[1]}(z)u + O^{[3]}(z,u) \\ y &= Cz \end{aligned} \tag{2.2}$$

where $f^{[2]}(z) = [f_1^{[2]}(z)\ \ f_2^{[2]}(z) \ldots f_n^{[2]}(z)]^T$ and $g^{[1]}(z) = [g_1^{[1]}(z)\ \ g_2^{[1]}(z) \ldots g_n^{[1]}(z)]^T$. Here, $f_i^{[2]}(z)$ and $g_i^{[1]}(z)$ for all $i \in [1, n]$, are homogeneous polynomials of degree 2 and 1 in z respectively. We suppose that the pair (A, C) is the rank of $n-1$, so system (2.1) has one unobservable real mode, then this system can be transformed to the following form:

$$\begin{aligned} \dot{\tilde{z}} &= A_{obs}\tilde{z} + B_{obs}u + \tilde{f}^{[2]}(z) + \tilde{g}^{[1]}(z)u + O^{[3]}(z,u) \\ \dot{z}_n &= \alpha z_n + \sum_{i=1}^{n-1} \alpha_i z_i + b_n u + f_n^{[2]}(z) + g_n^{[1]}(z)u + O^{[3]}(z,u) \\ y &= C_{obs}\tilde{z} \end{aligned} \tag{2.3}$$

where $\tilde{z} = [z_1, \ldots, z_{n-1}]^T$, $z = [\tilde{z}^T, z_n]^T$, $A_{obs} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-2} & 0 & \cdots & 0 & 1 \\ a_{n-1} & 0 & \cdots & \cdots & 0 \end{pmatrix}$,

$B_{obs} = [b_1, \ldots, b_{n-1}]$, $C_{obs} = [1\ \ 0 \cdots\ 0]$.

The terms A_{obs}, B_{obs}, α and the α_i constitute the residue of order one which represents the normal form of the linear approximation.

Definition 2.1. *System (2.2) is said to be quadratically equivalent modulo an output injection(QEMOI) to the system*

$$\begin{aligned} \dot{x} &= Ax + Bu + \bar{f}^{[2]}(x) + \bar{g}^{[1]}(x)u + \beta^{[2]}(y) + \gamma^{[1]}(y)u + O^{[3]}(x,u) \\ y &= Cx \end{aligned} \tag{2.4}$$

if there exists an output injection $\beta^{[2]}(y) + \gamma^{[1]}(y)u$ and a diffeomorphism of the form $x = z - \phi^{[2]}(z)$, which transform $f^{[2]}(z) + g^{[1]}(z)u$ into $\bar{f}^{[2]}(x) + \bar{g}^{[1]}(x)u + \beta^{[2]}(y) + \gamma^{[1]}(y)u$, where $\phi^{[2]}(z) = [\phi_1^{[2]}(z), \ldots, \phi_n^{[2]}(z)]^T$, $\beta^{[2]}(y) = [\beta_1^{[2]}(y), \ldots, \beta_n^{[2]}(y)]^T$ and $\gamma^{[1]}(y) = [\gamma_1^{[1]}(y), \ldots, \gamma_n^{[1]}(y)]^T$ represent respectively homogeneous polynomials of degree 2 in z, of degree 2 in y and of degree 1 in u.

We can recall the proposition below:

Proposition 2.2. *[3] System (2.2) is QEMOI to system (2.4), if and only if the following two homological equations are satisfied:*

$$\begin{aligned} &i)\ \ A\phi^{[2]}(z) - \frac{\partial \phi^{[2]}}{\partial z}Az = \bar{f}^{[2]}(z) - f^{[2]}(z) + \beta^{[2]}(z_1) \\ &ii)\ \ -\frac{\partial \phi^{[2]}}{\partial z}B = \bar{g}^{[1]}(z) - g^{[1]}(z) + \gamma^{[1]}(z_1) \end{aligned}$$

where $\frac{\partial \phi^{[2]}}{\partial z}Az := [\frac{\partial \phi_1^{[2]}(z)}{\partial z}Az, ..., \frac{\partial \phi_n^{[2]}}{\partial z}Az]^T$ *and* $\frac{\partial \phi_i^{[2]}(z)}{\partial z}$ *is the Jacobian matrix of* $\phi_i^{[2]}(z)$ *for all* $i \in [1, n]$.

For the proof see [3].

Just like the linear normal form, we choose the output always equal to x_1, because the output should rest unchanged, which means the diffeomorphism ($x = z - \phi^{[2]}(z)$) should verify $\phi_1^{[2]}(z) = 0$. Thus, the quadratic observability normal form for nonlinear systems with one dimensional linear unobservable mode is as follows:

THEOREM 2.3. *[3] There is a quadratic diffeomorphism and an output injection which transform system (2.3) into the following normal form:*

$$
\begin{aligned}
\dot{x}_1 &= a_1 x_1 + x_2 + b_1 u + \sum_{i=2}^{n} k_{1i} x_i u + E_1 \\
&\vdots \\
\dot{x}_{n-2} &= a_{n-2} x_1 + x_{n-1} + b_{n-2} u + \sum_{i=2}^{n} k_{(n-2)i} x_i u + E_{n-2} \\
\dot{x}_{n-1} &= a_{n-1} x_1 + b_{n-1} u + \sum_{j \geq i=2}^{n} h_{ij} x_i x_j + h_{1n} x_1 x_n + \sum_{i=2}^{n} k_{(n-1)i} x_i u + E_{n-1} \\
\dot{x}_n &= \alpha_n x_n + \sum_{i=1}^{n-1} \alpha_i x_i + b_n u + \alpha_n \phi_n^{[2]}(x) + \sum_{i=1}^{n-1} \alpha_i \phi_i^{[2]}(x) - \frac{\partial \phi_n^{[2]}}{\partial \tilde{x}} A_{obs} \tilde{x} \\
&\quad + f_n^{[2]}(x) + \sum_{i=2}^{n} k_{ni} x_i u + E_n
\end{aligned}
$$

with $E_i = \beta_i^{[2]}(y) + \gamma_i^{[1]}(y)$ *for all* $i \in [1, n]$.

From the normal form of system (2.3) we can deduce that the observability bifurcation manifold of state x_n is $S_n = \left\{ x \in U \mid \sum_{i=1}^{n-1} h_{i,n} x_i + 2h_{n,n} x_n + k_{(n-1),n} u = 0 \right\}$, in which the terms $h_{ij} x_i x_j$ (for all $n \geq i \geq j \geq 2$) and $k_{ji} x_i u$ (for all $n \geq j \geq 1$ and $n \geq i \geq 2$) are called the resonant terms. And on S_n, there is not any linear or quadratic relation between the derivative of the output and the last component of the state x_n. So, firstly if for some index $i \in [1, n]$, $h_{i,n} x_i \neq 0$, then we can quadratically recover, at least locally, x_n. Secondly, if we have some $k_{i,n} \neq 0$, with a well chosen input u, it is also possible to preserve the observability. Thirdly, if the system is on the observability bifurcation manifold, we can use α_n to analyze the x_n's detectability property: if $\alpha_n > 0$, x_n is undetectable, else if $\alpha_n < 0$, x_n is detectable, else if $\alpha_n = 0$, we can use the center manifold theory to analyze its property.

3. Observer Design. Each normal form characterizes one and only one equivalent class, thus the structural properties of the normal form must be also the same as those of each system in the corresponding equivalence class. In this section in order to highlight the efficiency of the proposed method, we will give a system (3.1) with unobservable dimension 2 which is more difficult than the one considered in Section 2, and simultaneously consider to construct an observer for this system.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + x_2 x_3 \\
\dot{x}_3 &= \frac{-x_4}{3} \\
\dot{x}_4 &= \frac{x_3}{3}
\end{aligned}
\tag{3.1}
$$

with the output $y = x_1$. Obviously, the system (3.1) is linearly unobservable on the directions x_3 and x_4 and has an observability bifurcation when $x_2 = 0$, so the method based on global or at least local observable system cannot be applied to design an observer. Furthermore, this system has four eigenvalues $\{\pm i, \pm i/3\}$, which mean this system is not linearly detectable, so Kazanzis and Kravaris's method ([8],[9],[10]), in which the observable is not a necessary condition, but the detectable is, also doesn't work. But as mentioned before, with the helps of observability normal form and the step-by-step observer, it is possible to design an observer for this kind of special system.

This quadratic normal form has the resonant terms: $h_{23} = 1, h_{13} = h_{14} = h_{22} = h_{24} = h_{33} = h_{34} = h_{44} = 0$, which allow us to deduce the observability bifurcation manifold of this system: $S = \{x \in R^4 \mid x_2 = 0\}$, i.e. if the system is on the surface of this manifold, it will lose its observability property just like on S the system is not detectable, and this surface should be not invariant. The behavior of system (3.1) rests on S only if x_2 is null that means this is only loss of global observability or detectability of the system. So if we want to design an observer, it must be able to modify its structure automatically according to whether the system is on or off the observability bifurcation manifold. Due to the resonant term x_2x_3, we can design a sliding mode observer, whose structure can switch between an observer structure and a pure copy of the original system in accordance with the system's observability property. Then in order to recover x_3 from $\frac{dx_2}{dt}$ and x_4 from $\frac{dx_3}{dt}$ step by step, we propose the following observer:

$$
\begin{aligned}
\frac{d\hat{x}_1}{dt} &= \hat{x}_2 + \lambda_1 sign(y - \hat{x}_1) \\
\frac{d\hat{x}_2}{dt} &= -x_1 + \tilde{x}_2\hat{x}_3 + E_1\lambda_2 sign(\tilde{x}_2 - \hat{x}_2) \\
\frac{d\hat{x}_3}{dt} &= -\frac{1}{3}\hat{x}_4 + E_2\lambda_3 sign(\tilde{x}_3 - \hat{x}_3) \\
\frac{d\hat{x}_4}{dt} &= \frac{1}{3}\tilde{x}_3 + E_3\lambda_4 sign(\tilde{x}_4 - \hat{x}_4)
\end{aligned}
$$

with the relations:

if $x_1 = \hat{x}_1$, then $E1 = 1$, else $E1 = 0$;

if $\tilde{x}_2 = \hat{x}_2$ and $E1 = 1$, then $E2 = 1$, else $E2 = 0$;

if $\tilde{x}_3 = \hat{x}_3$ and $E2 = 1$, then $E3 = 1$, else $E3 = 0$.

And thus the necessary auxiliary states for designing the step by step sliding mode observer are given below:

$$
\begin{aligned}
\tilde{x}_2 &= \hat{x}_2 + E_1\lambda_1 sign(y - \hat{x}_1) \\
\tilde{x}_3 &= \hat{x}_3 + \frac{E_2}{\tilde{x}_2}\lambda_2 sign(\tilde{x}_2 - \hat{x}_2) \\
\tilde{x}_4 &= \hat{x}_4 - 3E_3\lambda_3 sign(\tilde{x}_3 - \hat{x}_3)
\end{aligned}
$$

From here, we can see that when the system is on the observability bifurcation manifold S, the $\tilde{x}_3$ goes to infinite, which means an observability bifurcation occurs. So, in order to detect the observability bifurcation and avoid the explosion of $\tilde{x}_3$, we introduce an auxiliary variable Es, satisfied $Es = 1$ if $\tilde{x}_2 \neq 0$, else $Es = 0$. Then the auxiliary states and the last two equations of the observer become:

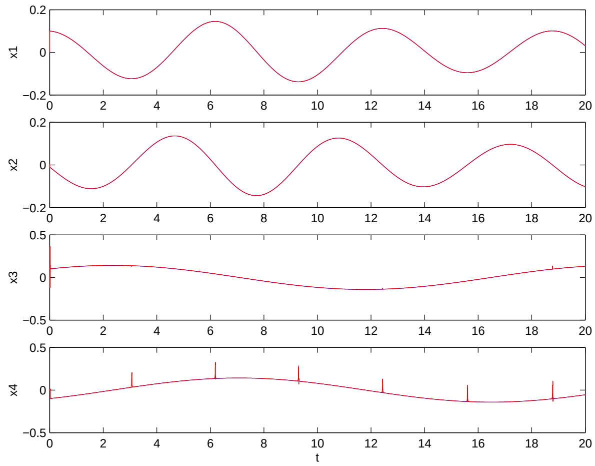

FIG. 3.1. *Dynamical System*

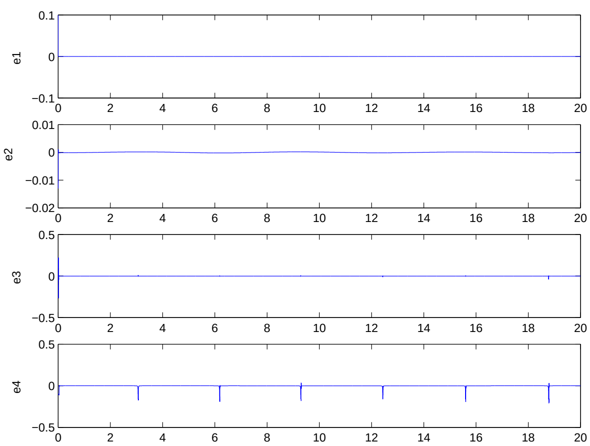

FIG. 3.2. *Observer Errors Dynamic*

$$
\begin{aligned}
\frac{d\hat{x}_3}{dt} &= -\frac{1}{3}\hat{x}_4 + E_s E_2 \lambda_3 sign(\tilde{x}_3 - \hat{x}_3) \\
\tilde{x}_3 &= \hat{x}_3 + \frac{E_2 E_s}{(\tilde{x}_2 - 1 + E_s)} \lambda_2 sign(\tilde{x}_2 - \hat{x}_2) \\
\frac{d\hat{x}_4}{dt} &= \frac{1}{3}\tilde{x}_3 + E_s E_3 \lambda_4 sign(\tilde{x}_4 - \hat{x}_4) \\
\tilde{x}_4 &= \hat{x}_4 - 3E_s E_3 \lambda_3 sign(\tilde{x}_3 - \hat{x}_3)
\end{aligned}
$$

In practice we add three low pass filters to the auxiliary states $\tilde{x}_i$ for $i \in \{2,3,4\}$(In order to cancel the negative effect, it is also possible to use high order observer [13]). And we set $E_i = 1$ for $i \in \{1,2,3\}$ not exactly when the system is on the surface of the sliding manifold, but when the system is sufficiently closed to it. And the same scheme is used to set Es. In order to simulate the practical situation, we replace the standard sign function by the arctangent function with a variable K to modify its smooth degree.

Fig. 3.1 exhibits the states of the original system and those of the observer, and Fig. 3.2 shows the observer errors dynamic respectively. Comparing the state x_2 with

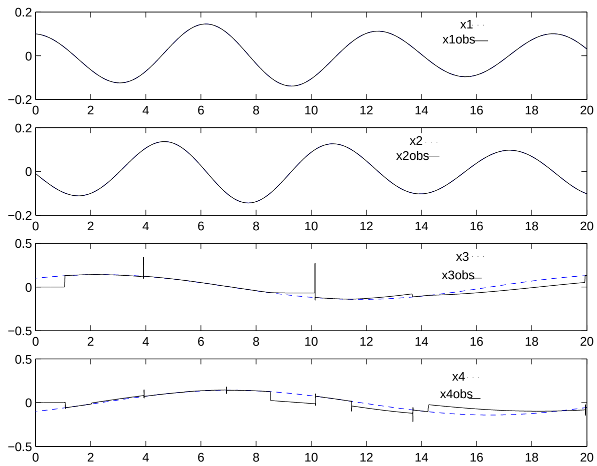

FIG. 3.3. *Influence of the Choice Es*

FIG. 4.1. *Left Invertibility Problem*

the states x_3, x_4, we can see that when x_2 cross zero we lose informations of states x_3, x_4, which means an observability bifurcation occurs. But after that, $\hat{x}_3$ and $\hat{x}_4$ quickly converge to x_3 and x_4 again. Fig. 3.3 illustrates the influence of the choice of Es to the convergence's speed of $\hat{x}_3$ and $\hat{x}_4$. In Fig. 3.3 if we set $Es = 0$ on a bigger neighborhood around the bifurcation manifold S, we lose more informations on the states $\hat{x}_3$ and $\hat{x}_4$ for a longer time, which means the qualities of convergencies of the $\hat{x}_3$ and $\hat{x}_4$ depend on the choice of Es. So it is necessary to set $Es = 0$ on a smaller neighborhood around S for rapid convergence.

This example shows that the linear unobservability may be overcome due to the particular quadratic form in which the quadratic resonant terms contain useful information for the observer design. Here, observation information is given by the term x_2x_3.

4. Discussion. It is also important to note that the observability normal form may be used to solve the left invertibility problem in order to identify an unknown input.

Fig.4.1 is the block diagram of left invertibility problem, in which $u(t)$ is an unknown input. Let us consider the unknown input system as follows:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

where $u \in R$ is an unknown input, the vector fields $f, g : U \subset R^n \to R^n$ and $h \subset R^n \to R^m$ are assumed to be smooth with $f(0) = 0$ and $h(0) = 0$. If the unknown input $u(t)$ is added into the system satisfied the so-called Observability Matching Condition [2]:

$$L_g L_f^i h_{|x\in\beta_\epsilon} = 0 \quad \forall i \in [0, n-2]$$
$$L_g L_f^{n-1} h_{|x=0} \neq 0$$

with $\beta_\epsilon = \{\forall x / ||x|| < \epsilon\}$ et $\epsilon > 0$. We can design a step by step sliding mode observer such that we can recover all state components and the unknown input $u(t)$ in finite time.

Secondly, the observer design can be regarded as a synchronization problem [11], so the technology mentioned above can be used to increase the robustness of ciphering in a secure data transmission system if the Observability Matching Condition is extended as follows:

$$L_g L_f^i h_{|\forall x\in\beta_\epsilon} = 0 \quad \forall i \in [0, n-2]$$
$$L_g L_f^{n-1} h_{|\exists x\in\beta_\epsilon} \neq 0$$

with $\beta_\epsilon = \{x / ||x|| < \epsilon\}$ et $\epsilon > 0$.

This allows us to introduce the observability bifurcation in the left invertibility problem to increase the calculation complexity of the dynamical system, improve the quality of masking the confidential message and finally to increase the degree of security. As mentioned before, we can deduce the observability bifurcation manifold from the observability normal form. Assume that in Fig.4.1 the unknown input is the confidential information, and it is added into the system, satisfied the extended Observability Matching Condition, it can be recovered successfully except when the system's trajectory interacts with the bifurcation manifold S. In a word, loss of observability means loss of the capability of reconstructing the confidential information, which makes the pirate more difficult to intercept and recover them because he does not know where we put the bifurcation in the left invertibility.

Finally we end this paper with some remarks.

Remarks:

1)If we want to consider greater order than quadratic one for observability normal forms (for example cubic observability normal form), it should be noticed to use an equivalent relation for each approximation order separately, because the diffeomorphism at order k does not modify any lower order terms, however terms of order higher than k are modified. This happens at each step of the application of the method.

2)In observability normal form, all singularity bifurcation are linked with the so-called resonant terms. And with the help of these resonant terms, we can reconstruct all states almost everywhere even if the system is unobservable and undetectable.

3)If the Observability Matching Condition is verified, all the inputs are universal [7] around the neighborhood $x = 0$.

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