Observer Error Linearization Multi-Output Depending

D. Boutat[1] G. Zheng[2], J.P. Barbot[2] and H. Hammouri[3]
[1]LVR/ENSI, 10 Boulevard de Lahitolle, 18020 Bourges, France.
driss.boutat@ensi-bourges.fr
[2]ECS/ENSEA, 6 Avenue du Ponceau, 95014 Cergy-Pontoise, France.
{zheng, barbot}@ensea.fr
[3]LAGEP, UMR 5007 du CNRS Univ-Lyon I
43 Bd du 11 novembre 1918, 69622 Villeurbanne, France.
hammouri@lagep.univ-lyon1.fr

***Abstract*— This paper gives the sufficient and necessary conditions which guarantee the existence of a reference frame in which a multi-output nonlinear system is linearizable with a linear part depending on its outputs. Our method is based on the design of a reference frame associated with nonlinear observable systems. Moreover, we give the generalization of the result obtained in [12] and [10]. And some examples are given in order to highlight our thinking.**

I. Introduction

Consider the following system:

$$\left\{ \begin{array}{l} \dot{x} = f(x) \\ y = (h_1(x), ..., h_m(x))^T \end{array} \right. \tag{1}$$

where $U \sqsubset \Re^n$ is the set of admissible states, $f : U \sqsubseteq \mathbb{R}^n \to \mathbb{R}^n$ and $h : U \sqsubseteq \mathbb{R}^n \to \mathbb{R}^m$ are analytic.

In this paper, we deal with the following problem: Find the necessary and sufficient conditions for the existence of a local diffeomorphism $\phi(x) = z$ such that for $1 \leq i \leq m$,

$$\left\{ \begin{array}{l} \dot{z}_{i,1} = \beta_{i,1}(y) \\ \dot{z}_{i,j} = \alpha_{i,j-1}(y) z_{i,j-1} + \beta_{i,j}(y) \text{ for } 2 \leq j \leq k_i \\ y_1 = z_{1,k_1} \\ y_i = z_{i,k_i} + \left\{ \begin{array}{ll} R_i\left(z_{1,k_1}, .., z_{i-1,k_{i-1}}\right) & \text{if } k_i < k_{i-1} \\ R_i\left(z_{1,k_1}, .., z_{i-2,k_{i-2}}\right) & \text{if } k_i = k_{i-1} \end{array} \right. \end{array} \right. \tag{2}$$

In the case $m = 1$ and $\alpha_{i,j} = 1$, the sufficient and necessary geometrical conditions, which guarantee the existence of a diffeomorphism and an output injection to transform a nonlinear system (1) into the canonical linear form (2), were firstly addressed in [9] in 1983. At the same time the authors in [3] solved it for nonlinear time variable systems. In 1985, the problem was solved partially in [10] for $m \geq 1$, $\alpha_{i,j} = 1$ and $R_i = 0$ and in 1988 it was completely solved in [12]. In the last work, the authors dealt also with systems with inputs.

In 2001, for $m = 1$ and $\alpha_{i,j}(y) = \alpha(y)$, [7] gave the sufficient and necessary geometrical conditions to transform a nonlinear system into the so-called output-dependent time scaling linear canonical form. In [4], the author gave independently the dual geometrical conditions of [7]. Finally, in 2005, for $m = 1$ and the different functions $\alpha_{i,j}(y)$, this problem was solved in [13]

The main motivation of our work is that for system (2) we can design an observer. Indeed, for system (2), we can design the well-known high gain observer [5] as follows:

$$\left\{ \begin{array}{l} \bar{y}_i = y_i - \left\{ \begin{array}{ll} R_i\left(\bar{y}_1, .., \bar{y}_{i-1}\right) & \text{if } k_i < k_{i-1} \\ R_i\left(\bar{y}_1, .., \bar{y}_{i-2}\right) & \text{if } k_i = k_{i-1} \end{array} \right. \\ \dot{\hat{z}}_i = A_i(y)\hat{z}_i + \beta_i(y) - \rho_i^{-1} C_i^T (C_i \hat{z}_i - \bar{y}_i) \\ \dot{\rho}_i = -\mu_i \rho_i - A_i^T(y)\rho_i - \rho_i A_i(y) + C_i^T C_i \end{array} \right.$$

where for $i = 1 : m$, $\hat{z}_i = \left(\begin{array}{ccc} \hat{z}_{i,1}, & \cdots, & \hat{z}_{i,k_i} \end{array} \right)^T$,

$$A_i(y) = \left[\begin{array}{cccc} 0 & \cdots & 0 & 0 \\ \alpha_{i,1}(y) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_{i,k_i}(y) & 0 \end{array} \right],$$

$C_i = \left(\begin{array}{cccc} 0, & \cdots, & 0 & 1 \end{array} \right)_{1 \times k_i}$, ρ_i is a symmetric definite positive matrix and μ_i is the gain of the observer, and y is μ_i-strictly persistent [5].

In this paper, we will give the geometrical conditions which are sufficient and necessary to guarantee the existence of a local diffeomorphism and an output injection to transform system (1) into the normal form (2). This kind of linearization will be named Multi-Output Dependent Observability normal form (MODO normal form) and generalizes the result obtained in [13] for nonlinear system with single output.

This paper is organized as follows: In section 2, we give notations and a preliminary result in order to introduce our method. Moreover, in the same section we recall the result of [12] and we compare their conditions with ours. In section 3, we present our main result as a generalization of the result given in section 2. Throughout this paper, some examples are analyzed in order to highlight our propositions.

II. Notations and a preliminary result

In this section, we will give sufficient and necessary conditions which guarantee the existence of a local diffeomorphism and an output injection to transform system (1) into the following form:

$$\text{For } 1 \leq i \leq m$$
$$\begin{cases} \dot{z}_{i,1} = \beta_{i,1}(y) \\ \dot{z}_{i,j} = z_{i,j-1} + \beta_{i,j}(y) \text{ for } 2 \leq j \leq k_i \\ y_1 = z_{1,k_1} \\ y_i = \begin{cases} z_{i,k_i} + R_i\left(z_{1,k_1}, .., z_{i-1,k_{i-1}}\right) & \text{if } k_i < k_{i-1} \\ z_{i,k_i} + R_i\left(z_{1,k_1}, .., z_{i-2,k_{i-2}}\right) & \text{if } k_i = k_{i-1} \end{cases} \end{cases} \quad (3)$$

which is a special case of system (2) with $\alpha_{ij} = 1$, and also a generalization of the result in [12] in which $R_i = 0$, $i = 1 : m-1$. For this, we will construct a particular reference frame τ in which system (1) is in the form (3). The same method will be used in the next section in order to build the frame in which system (1) can be transformed into the form (2).

Consider system (1), note $L_f^{i-1}h$ for $1 \leq i \leq n$ as the $(i-1)th$ Lie derivative of h in the direction of f [1], and with a possible reordering of h_i, we assume that there exist $k_1 \geq k_2 \geq ... \geq k_m \geq 1$ and $\sum_{i=1}^{m} k_i = n$ such that

$$\theta = \left(\theta_1^1, ..., \theta_1^{k_1}, .., \theta_m^1, .., \theta_m^{k_m}\right)^T$$

where

$$\theta_i^j = dL_f^{j-1}h_i$$

is a frame of the cotangent bundle T^*U. Thus, system (1) is observable. Integers $(k_i)_{1 \leq i \leq m}$ are called observability indices of system (1). For a nice description and more details about this assumption, see [10]. Obviously the list of these integers may be not unique.

Now, let $(T_{i,1})_{1 \leq i \leq m}$ be the family of vector fields defined by:

$$\begin{cases} \theta_i^j\left(T_{i,1}\right) = 0 \text{ for } i = 1 : m \text{ and } j = 1 : k_i - 1 \\ \theta_i^{k_i}\left(T_{i,1}\right) = 1 \end{cases} \quad (4)$$

and construct by induction the following family of vector fields:

$$\begin{aligned} T_{i,r} &= [T_{i,r-1}, f] \text{ for } 2 \leq r \leq k_i \\ &= (-1)^{r-1} ad_f^{r-1} T_{i,r-1} \text{ for } 2 \leq r \leq k_i \end{aligned}$$

The family $T = (T_{i,j})_{1 \leq i \leq m \text{ and } 1 \leq j \leq k_i}$ is a basis of the tangent bundle TU. The frame T was addressed firstly in [9] for $m = 1$ and it is well-known in [9] that system (1) can be transformed into the normal form (3) if and only if we have

$$[T_{1,i}, T_{1,j}] = 0 \text{ for } 1 \leq i, j \leq n. \quad (5)$$

In this case, T is a frame with which system (1) is in the form (3).

Let us first recall a particular result in [10] and [12].

Theorem 1: [10] [12] Assume that $k_1 = k_2 = ... = k_m$, then the following conditions are equivalent:

$i)$ System (1) can be transformed by means of a diffeomorphism and an output injection into the normal form (3) with $R_l = 0$.

$ii)$ The following commutativity conditions

$$[T_{i,j}, T_{r,s}] = 0 \text{ for } 1 \leq i, r \leq m \text{ and } 1 \leq j \leq k_i, \; 1 \leq s \leq k_r \quad (6)$$

are fulfilled.

Example 1: Consider the following system:

$$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = 0 \\ \dot{x}_4 = x_3 + x_1 x_2 \\ h_1 = x_2 \text{ and } h_2 = x_4 \end{cases} \quad (7)$$

Obviously we have $k_1 = k_2 = 2$, and $T_{1,1} = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3}$, $T_{1,2} = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}, T_{2,1} = \frac{\partial}{\partial x_3}$ and $T_{2,2} = \frac{\partial}{\partial x_4}$. Because of $[T_{1,1}, T_{1,2}] = -2\frac{\partial}{\partial x_3} \neq 0$, we conclude thanks to theorem 1 that system (7) cannot be transformed into the normal form (3) with $R_l = 0$.

In fact, if there exists $1 \leq l \leq m$ such that $k_{l+1} < k_l$, then condition (6) is neither sufficient nor necessary. We will show it hereafter.

The following example studied in [12] shows that condition (6) is not necessary.

Example 2: [12] Consider the following system:

$$\begin{cases} \dot{x}_1 = x_1 x_3 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ h_1 = x_2 \text{ and } h_2 = x_3 \end{cases} \quad (8)$$

then we have $\theta = \begin{pmatrix} dx_2 \\ dx_1 \\ dx_3 \end{pmatrix}$, which gives us

$T_{1,1} = \frac{\partial}{\partial x_1}, T_{1,2} = \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_1}$ and $T_{2,1} = \frac{\partial}{\partial x_3}$.

It is clear that $[T_{1,2}, T_{2,1}] = -\frac{\partial}{\partial x_1} \neq 0$. However, the following diffeomorphism

$$\phi(x) = \begin{pmatrix} x_1 - x_2 x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

transforms the system (8) into the form (3) with $R_2 = 0$. Thus, condition (6) is not necessary.

In order to give an example which indicates that condition (6) is not sufficient, let us state a result which rises immediately from [12].

Assume for a certain $1 \leq l \leq m$, we have $k_{l+1} < k_l$, then we define:

$$Q_l = \left\{dL_f^{k-1}h_l \text{ for } 1 \leq k \leq k_l - 1\right\} \cup \left\{dL_f^{k-1}h_i \text{ for } 1 \leq i \leq m,\, i \neq l \text{ and } 1 \leq k \leq k_l\right\}$$

Theorem 2: [12] Assume that conditions (6) are fulfilled, then the following conditions are equivalent:

$i)$ System (1) can be transformed by means of a diffeomorphism and an output injection into the normal form (3) with $R_l = 0$.

$ii)$ Each time when $k_{l+1} < k_l$, for $1 \leq l \leq m$,

$$\dim span(Q_l) = lk_l + k_{l+1} + ... + k_m - 1 \tag{9}$$

The following example shows that condition (6) is not sufficient.

Example 3: Consider the following system:

$$\begin{cases} \dot{x}_1 = x_1 x_2 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1 \\ h_1 = x_2 \text{ and } h_2 = x_3 \end{cases} \tag{10}$$

then we obtain $\theta = \begin{pmatrix} dx_2 \\ dx_1 \\ dx_3 \end{pmatrix}$, which gives us

$T_{1,1} = \frac{\partial}{\partial x_1}, T_{1,2} = \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1}$ and $T_{2,1} = \frac{\partial}{\partial x_3}$.

It is clear that conditions (6) are fulfilled. However, $Q_1 = \{dx_2, dx_3, dx_1\}$ and

$$\dim span(Q_1) = 3 > 2 \times 1 + 1 - 1.$$

Thus by theorem 2, system (10) can not be transformed into the normal form (3) with $R_l = 0$.

The following remarks gives an idea of our thinking.

Remarks 1:

$i)$ System (8) in example 2 is linearizable in the following frame $\tau_{1,1} = T_{1,1}$, $\tau_{1,2} = T_{1,2}$ and $\tau_{2,1}$, where $\tau_{2,1}$ is a solution of the following equations:

$$\begin{array}{l} dh_2\left(\tau_{2,1}\right) = 1 \\ dh_1\left(\tau_{2,1}\right) = 0 \\ dh_1 L_f \tau_{2,1} = 0 \\ \left[\tau_{1,i}, \tau_{2,1}\right] = 0 \text{ for } i = 1:2 \end{array}$$

The two first algebraic equations give the following solution: $\tau_{2,1} = T_{2,1} + b(x)T_{1,1}$ and the two second differential equation give $b(x) = x_2$.

In fact, if we denote by $f(x)$ the vector field which undergoes system (8), then

$$f(x) = x_1 T_{1,2} \underbrace{+ x_2 T_{2,1} - x_2^2 T_{1,1}}_{\text{output injection}}.$$

$ii)$ The following diffeomorphism

$$z = \phi(x) = \begin{pmatrix} x_1 - \frac{1}{2} x_2^2 \\ x_2 \\ x_3 - x_2 \end{pmatrix}$$

transforms system (10) considered in example 3 into the normal form (3) with $R_1 = -z_2$. Thus, system (10) is in the form (3) with $R_1 = -z_2$ in the frame $T_{1,1} = \frac{\partial}{\partial x_1}, T_{1,2} = \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$ and $T_{2,1} = \frac{\partial}{\partial x_3}$.

Later, we will consider another example which fulfills neither condition (6) nor condition (9).

To state the main result of this section, we consider a new frame τ:

$$\begin{array}{l} \tau_{1,j} = T_{1,j} \text{ for } 1 \leq j \leq k_1 \\ \tau_{s+1,1} = T_{s+1,1} + \\ \quad \begin{cases} \sum_{r=1}^{s} \sum_{j=1}^{(k_r - k_{s+1}+1)} b_{r,j}^s(x) \tau_{r,j} \text{ if } k_{s+1} < k_s \\ \sum_{r=1}^{(s-1)} \sum_{j=1}^{(k_r - k_{s+1}+1)} b_{r,j}^s(x) \tau_{r,j} \text{ if } k_{s+1} = k_s \end{cases} \\ \quad \text{for } 1 \leq s \leq m-1 \\ [\tau_{s+1,1}, \tau_{s,l}] = 0 \text{ for } 1 \leq l \leq k_s \\ \tau_{s+1}, j = [\tau_{s+1,j-1}, f] \quad \text{for } 2 \leq j \leq k_{s+1} \end{array}$$

Set $\tau = (\tau_{i,j})_{1 \leq i \leq m \text{ and } 1 \leq j \leq k_i}$ and $\Lambda = \theta\tau$. Define the following multi 1-form:

$$\omega = \Lambda^{-1}\theta.$$

It is clear that

$$\omega\tau = I_{n \times n}$$

Theorem 3: The following conditions are equivalent:

$i)$ There exists a change of coordinates $\phi(x) = z$ which transform system (1) into the form (3).

$ii)$ For $1 \leq i \leq m$, $i < l \leq m$ and $1 \leq j \leq k_i - 1$,

$$dh_l(T_{i,j}) = dh_l L_f^{j-1}(T_{i,1}) = 0 \tag{11}$$

and for $1 \leq i, s \leq m$, $1 \leq j \leq k_i$ and $1 \leq l \leq k_s$,

$$[\tau_{i,j}, \tau_{s,l}] = 0. \tag{12}$$

$iii)$ Condition (11) is fulfilled and $d\omega = 0$. Then, we locally have $\omega = d\phi$.

Remarks 2:

$i)$ Equation (11) implies that output h_l is independent of the outputs h_s for $l < s \leq m$. Thanks to the algebraic equation (4) which defines frame T, equations (11) can be reduced to:

for $1 \leq j \leq k_i - 1$ for $i \leq l \leq m$,

$$dh_l(T_{i,j}) = dh_l L_f^{j-1}(T_{i,1}) = 0 \tag{13}$$

$ii)$ By the construction of frame τ, $\tau_{i,j}$ also checks conditions (11) in the place of $T_{i,j}$.

Proof:

$ii) \Longleftrightarrow iii)$ In fact, by construction we have:

$$\omega(\tau_{i,j}) = \frac{\partial}{\partial z_{i,j+1}} \text{ for } i:1:m \text{ and } j = 1:k_i - 1$$

Recall that for $1-$form ω and two vector fields X, Y we have:

$$d\omega(X,Y) = L_X\left(\omega(Y)\right) - L_Y\left(\omega(X)\right) - \omega([X,Y])$$

Now, set $X = \tau_{i,j}$ and $Y = \tau_{l,s}$. As $\omega(\tau_{i,j})$ and $\omega(\tau_{i,j})$ are constants, then we have

$$d\omega(\tau_{i,j}, \tau_{l,s}) = -\omega([\tau_{i,j}, \tau_{l,s}])$$

As ω is an isomorphism and $(\tau_{i,j})_{1\le i\le m, 1\le j\le k_i}$ is a basis of TU, then $[\tau_{i,j},\tau_{s,l}]=0$, for $1\le i,s\le m$, $1\le j\le k_i$ and $1\le l\le k_s$, implies that $d\omega=0$. Thus, there is locally a diffeomorphism ϕ such that: $\omega=d\phi$.

Now, for $i:1:m$ and $j=1:k_i-1$, we compute and obtain:

$$\frac{\partial}{\partial z_{i,j}}\omega(f)=\omega([\tau_{i,j},f])=\frac{\partial}{\partial z_{i,j+1}}$$

To show that $iii)\Rightarrow i)$, it reminds to prove that condition (11) implies that the outputs are in the form (3).

Let $l=1:m$ and compute $\frac{\partial}{\partial z_{ij}}h_l o\phi = dh_l\left(\tau_{ij}\right)$. Because of condition (11) and the expression of $\tau_{i,j}$ for $1\le i\le m$ we have.

$$\begin{array}{rcl} dh_l\left(\tau_{1,k_l}\right) & = & 1 \text{ and } dh_l\left(\tau_{i,j}\right)=0 \\ \text{where } j & \neq & k_i \text{ and } i=1:l-1 \end{array}$$

Finally, we will prove that $i)\Rightarrow ii)$.

Assume that there exists a frame $(\tau_{i,j})_{1\le i\le m \text{ and } 1\le j\le k_i}$ in which system (1) can be written in the form (3). Then for $1\le i\le m$ and $2\le j\le k_i-1$ we have:

$$\tau_{i,j+1}=[\tau_{i,j},f]\,,$$

and

$$\begin{array}{rcl} dh_l(\tau_{i,j}) & = & dh_l L_f^{j-1}(\tau_{i,1})=0 \\ \text{for } 1 & \le & j\le k_i-1 \text{ and } i\le l\le m \end{array}$$

Now, we will prove that the frame $(T_{i,j})_{1\le i\le m \text{ and } 1\le j\le k_i}$ satisfies also equation (11).

It is clear that

$$\tau_{1,i}=T_{1,i} \text{ for } 1\le i\le k_1.$$

Now, we have

$$T_{2,1}=\sum_{i=1}^{m}\sum_{j=1}^{k_i} b_{i,j}(x)\tau_{i,j}$$

And using conditions (11) and (12), we obtain

$$T_{2,1}=\tau_{2,1}+\sum_{j=1}^{k_1-k_2} b_{1,j}(x)\tau_{1,j}$$

And then we can construct $\tau_{2,j}=[\tau_{2,j-1},f]$ for $j=2:k_2$. Finally, with the same argument, we can construct the other $\tau_{i,j}$, $i=3:m$ and $j=1:k_i$.

■

Let us consider another example in order to highlight the above result.

Example 4: Consider the following system:

$$\left\{\begin{array}{l} \dot{x}_1=x_1x_3 \\ \dot{x}_2=x_1 \\ \dot{x}_3=x_1 \\ h_1=x_2 \text{ and } h_2=x_3 \end{array}\right. \qquad (14)$$

then we have $\theta=\begin{pmatrix} dx_2 \\ dx_1 \\ dx_3 \end{pmatrix}$, and $T_{1,1}=\frac{\partial}{\partial x_1}$, $T_{1,2}=\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}+x_3\frac{\partial}{\partial x_1}$ and $T_{2,1}=\frac{\partial}{\partial x_3}$.

It is easy to see that $\dim Q_1=3>2$, so condition (9) is not fulfilled and by [12], system (14) can not be transformed into the form (3) with $R_l=0$.

Now, As $dh_2(T_{1,1})=0$ but $dL_f h_2(T_{1,1})=dh_2(T_{1,2})=1\neq 0$, then the new output depends on x_2. As $[T_{1,2},T_{2,1}]=-\frac{\partial}{\partial x_1}\neq 0$, we can construct $\tau_{2,1}$ as follows:

$$\tau_{2,1}=T_{2,1}+aT_{1,1}=\left\{\begin{array}{l} \frac{\partial}{\partial x_3}+x_3\frac{\partial}{\partial x_1} \\ \text{or} \\ \frac{\partial}{\partial x_3}+x_2\frac{\partial}{\partial x_1} \end{array}\right.$$

which gives us two diffeomorphisms: $\phi_1=\begin{pmatrix} x_1-\frac{1}{2}x_3^2 \\ x_2 \\ x_3-x_2 \end{pmatrix}$ and $\phi_2=\begin{pmatrix} x_1+\frac{1}{2}x_2^2-x_2x_3 \\ x_2 \\ x_3-x_2 \end{pmatrix}$, and the new outputs are $y_1=\xi_2$ and $y_2=\xi_3-\xi_2$.

To construct other diffeomorphisms, set $\tau_{2,1}=\underbrace{\frac{\partial}{\partial x_3}+x_2\frac{\partial}{\partial x_1}}_{\text{the former }\tau_{2,1}}+c\tau_{1,1}$, where c is a constant.

Theorem 4: [12] There exists a change of coordinates which transforms system (1) into the normal form (3) with $R_l=0$ for $2\le l\le m$ if and only if condition (9) fulfilled and $[\tau_{i,j},\tau_{s,l}]=0$ for $1\le i,s\le m$,$1\le j\le k_i$ and $1\le l\le k_l$.

Claim 1: Condition (9) is equivalent to condition (11) plus the following equations

$$dh_l(T_{i,j})=dh_l L_f^{k_i-1}(T_{i,1})=0 fori+1\le l\le m.$$

This condition is equivalent to the condition given in [12] for $1\le j\le k_i$ and $i<l\le m$.

Corollary 1: If condition (11) is fulfilled and if

$$\begin{array}{l} [T_{i,j},T_{s,l}]=0 \\ \text{for } 1\le i,s\le m,\ 1\le j\le k_i \text{ and } 1\le l\le k_s, \end{array} \qquad (15)$$

then there exists a diffeomorphism which transforms system (1) into the normal form (2). In this case, we choose $\tau_{i,j}=T_{i,j}$, for $1\le i,s\le m$, $1\le j\le k_i$ and $1\le l\le k_s$.

Remark 1: Example (1) shows that we can not improve the theorem (1). Indeed, the following diffeomorphism

$$z=\phi(x)=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4-\frac{1}{2}x_2^2 \end{pmatrix}$$

transforms system (7) into the form (3) with $R_1=\frac{1}{2}z_2^2$. However, the vector fields of the frame T don't commute.

III. Main result

In this section, we will give the correspondent result of theorem 3 given in section 2, for $\alpha_{i,j} \neq 1$, and thus to generalize the result stated in [13] which dealt with a single output. For this, if the vector fields of the former frame τ do not commute, then we will build another frame in the following way.

Consider again the family of vector fields $(T_i)_{1\leq i\leq m}$ given by system (1). Assume that there exist $\alpha_{1,1}(y), ..., \alpha_{1,k_1-1}(y) \neq 0$ such that the family of vector fields given by:

$$\begin{aligned}\widetilde{\tau}_{1,1} &= \Pi_{1,1}(y)T_{1,1} \\ \widetilde{\tau}_{1,i} &= \frac{1}{\alpha_{1,i-1}}[\tau_{1,i-1}, f] \text{ for } i = 2 : k_1\end{aligned}$$

commute, where $\Pi_{1,1}(y) = \prod_{i=1}^{k_1-1} \alpha_{1,i}(y)$.

Now, as in the section above consider the following frame:

$$\begin{aligned}&\widetilde{\tau}_{1,j} = T_{1,j} \text{ for } 1 \leq j \leq k_1 \\ &\widetilde{\tau}_{s+1,1} = \Pi_{s+1,1}(y)T_{s+1,1} + \\ &\qquad \begin{cases} \sum_{r=1}^{s} \sum_{j=1}^{(k_r-k_{s+1}+1)} b^s_{r,j}(x)\widetilde{\tau}_{r,j} \text{ if } k_{s+1} < k_s \\ \sum_{r=1}^{(s-1)} \sum_{j=1}^{(k_r-k_{s+1}+1)} b^s_{r,j}(x)\widetilde{\tau}_{r,j} \text{ if } k_{s+1} = k_s \end{cases} \\ &\qquad \text{for } 1 \leq s \leq m-1 \\ &[\widetilde{\tau}_{s+1,1}, \widetilde{\tau}_{s,l}] = 0 \text{ for } 1 \leq l \leq k_s \\ &\widetilde{\tau}_{s+1},_j = [\widetilde{\tau}_{s+1,j-1}, f] \quad \text{for } 2 \leq j \leq k_{s+1}\end{aligned}$$

Set $\widetilde{\tau} = (\widetilde{\tau}_{i,j})_{1\leq i\leq m \text{ and } 1\leq j\leq k_i}$ and $\widetilde{\Lambda} = \theta\widetilde{\tau}$, we can define the following multi 1-form:

$$\widetilde{\omega} = \widetilde{\Lambda}^{-1}\theta.$$

It is clear that

$$\widetilde{\omega}\widetilde{\tau} = I_{n\times n}$$

Now, we are ready to state our main result.

Theorem 5: The following conditions are equivalent:

$i)$ There exists a change of coordinates $\phi(x) = x$ which transforms system (1) into the normal form (2).

$ii)$ For $1 \leq j \leq k_i - 1$, where $i \leq l \leq m$,

$$dh_l(T_{i,j}) = dh_l L_f^{j-1}(T_{i,1}) = 0 \qquad (16)$$

and for $1 \leq i, s \leq m$, $1 \leq j \leq k_i$ and $1 \leq l \leq k_s$,

$$[\widetilde{\tau}_{i,j}, \widetilde{\tau}_{s,l}] = 0. \qquad (17)$$

$iii)$ Condition (16) is fulfilled and $d\widetilde{\omega} = 0$, then locally we have $\widetilde{\omega} = d\phi$.

Example 5: Let us consider the following system:

$$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = \alpha(y)x_1 \\ \dot{x}_3 = \beta(y)x_1 \\ h_1 = x_2 \text{ and } h_2 = x_3 \end{cases}$$

then we have

$$\theta = \begin{pmatrix} dx_2 \\ \alpha dx_1 + x_1 d\alpha \\ dx_3 \end{pmatrix}$$

and

$$\widetilde{\Lambda} = \theta(\widetilde{\tau}) = \begin{pmatrix} 0 & 1 & 0 \\ \alpha & \frac{\beta}{\alpha}x_1\partial_3\alpha + x_1\partial_2\alpha & x_1\partial_3\alpha \\ 0 & \frac{\beta}{\alpha} & 1 \end{pmatrix}.$$

Assume that $\frac{\partial}{\partial x_3}\left(\frac{\beta}{\alpha}\right) = 0$, then $T_{1,1} = \frac{1}{\alpha}\frac{\partial}{\partial x_1}$, and $\widetilde{\tau}_{1,1} = \alpha T_{1,1} = \frac{\partial}{\partial x_1}$, so $\widetilde{\tau}_{1,2} = \frac{1}{\alpha}[\widetilde{\tau}_{1,1}, f] = \frac{\partial}{\partial x_2} + \frac{\beta}{\alpha}\frac{\partial}{\partial x_3}$. as $T_{2,1} = \frac{\partial}{\partial x_3}$ and $\widetilde{\tau}_{1,2} = \frac{\partial}{\partial x_3}$, we have the diffeomorphism

$$\phi = \begin{pmatrix} x_1 \\ x_2 \\ x_3 - F(x_2) \end{pmatrix}$$

where $F' = \frac{\alpha}{\beta}$.

Remark 2: A natural question remains to be solved: how can we determine the functions $\alpha_{i,j}(y)$? Inspired from the work [7], this problem is solved in [13] for the single output case. For this moment, one can obtain some necessary equations for the determination of $\alpha_{i,j}(y)$ by the following equation:

$$\begin{aligned}[T_{i,j}, T_{i,k_i}] &= \lambda T_{i,j} \ mod(span(T_{i,l}))_{1\leq l<j} \\ \text{for } 1 &\leq i \leq m \text{ and } 1 \leq j \leq k_{i-1}.\end{aligned}$$

IV. Conclusion

In this paper, the sufficient and necessary geometrical conditions are given in order to determine whether a system (1) can be transformed into the normal form (2) or not. Our technique is based on designing a new frame from the natural one associated with a dynamical system with outputs, which was given first in [9]. We also state one main result which is a generalization of the result studied in [10] and [12], where the output may be modified by the previous outputs.

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