Stokes phenomenon : graphical visualization and certified computation

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ABSTRACT

We present an automatic treatment of the Stokes phenomenon for the computation and the graphical representation of solutions of linear ODEs in the neighborhood of an irregular singularity in the complex plane. More precisely, given a basis of formal solutions, our programs detect the Stokes rays, compute the associated Stokes matrices, compute the formal monodromy, combine these matrices to specify the continuation of any actual solution given by linear combination in one sector to the whole Riemann surface of the logarithm. The graphical representation permits to visualize the results and to compare them with the use of convergent series.
In the last part of the paper we report on a first attempt to obtain certified values of the Stokes constants.

Categories and Subject Descriptors

G.1.7 [**Numerical Analysis**]: Ordinary Differential Equations—*Convergence and stability*; I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*

Keywords

Ordinary Differential Equations, Stokes matrices, graphical visualization, Computer Algebra.

1. INTRODUCTION

This paper deals with the computation and the graphical representation of complex functions of the complex variable, solutions of linear differential equations with polynomial coefficients.
Since the 80's, algorithms of Computer Algebra have been developed to find a basis of formal solutions of these equations in the neighborhood of singularities [2]. If the singularity is irregular, these solutions contain in general divergent series, but in particular thanks to the theory of Gevrey asymptotics and multisummability [8, 9, 10, 11], it is possible to associate to each divergent series appearing in a formal solution an analytic function, asymptotic to the series in some sector. In case of k-summability, this leads to effective algorithms of summation [19, 20], and so we can compute actual solutions of the ODE on **sectors** (whose vertex is the singular point). In order to continue one particular solution on a large sector, beyond the critical directions, a control of the Stokes phenomenon is necessary.
The sectors in which these actual solutions are defined are overlapping one another, in such a way that the same basis of formal solutions gives rise to two basis of actual solutions on the intersection of two adjacent sectors. The line, which is the bissector of the intersection, is called a Stokes ray, and the matrix which express the change of basis is called the associated Stokes matrix.
It is well known that the problem of computing the entries of the Stokes matrices is closed since the publication of [22], where Écalle's accelero-summation process is used to show that the entries of the Stokes matrices may be approximated up to n decimal digits in time $O(n \log^4 n \log\log n)$. Neverthless this theoretical study does not lead yet to a practical implementation.
In [6], we have proposed a completely different approach for the numerical computation of the Stokes matrices. The algorithm is based on the analysis of the singularities of the Borel transforms of the divergent series, and needs to sum only **convergent** series. In the current version, it works under the hypotheses that the equation is of single rank k (where k is any positive rational number) and the Borel transforms do not have irregular singularities and do not have many singularities aligned on a half line issued from the origin (this last point is a technical, practical limitation, not a serious one), but it allows in the Borel plane any polar, ramified or logarithmic singularities. It is implemented in our new version of Desir package, written in Maple [18].
The objective of this paper is double.
First we want to show how to use these matrices in an automatic way to continue a particular solution on the whole Riemann surface of the logarithm near the singularity. For this purpose, we introduce and compute the matrix of formal monodromy. The graphical representation in the complex plane is used to visualize the results and some examples, classical or less studied, are treated.
Secondly we will explain how to replace the analytic continuation of convergent series, performed in [6] by the use of Padé approximants, by a certified method of summation. And so we will obtain certified Stokes constants.

2. GRAPHICAL REPRESENTATION OF COMPLEX FUNCTIONS

Along this paper, we will use a graphical tool, well adapted to represent multivalued complex functions and to compare different numerical methods used to compute them.
The first implementation of such a graphical toolbox was dedicated to solutions of linear differential equation with singularities and is described in [16]. A new version of the 2D functionalities is now written in MAPLE [17].
The principle of the representation is the following: it consists in plotting the image under the considered function f of a circle or a circular arc around the singularity, in general 0. The color is used to associate a point in the domain and its image: each point $f(x)$ is plotted with a color corresponding to the argument of x.
As the studied functions are in general multi-valued, we consider them in the neighborhood of 0 as functions on the Riemann surface of the logarithm and points in the domain are represented by their Euler coordinates, whereas the image points are computed in cartesian coordinates. So we have to manipulate two types of colored points : $(\rho, \theta, color)$ and $(x, y, color)$. Two functions `draw_rhotheta` and `draw_xy` are associated to plot lists of such points. The function `create_circle`$(\rho, \theta^-, \theta^+,$ `nbpoints)` creates an arc of circle around 0, of radius ρ, as a list of `nbpoints` of modulus ρ and argument between θ^- and θ^+.

Example: we consider the special Bessel function J_ν, with $\nu = \frac{1}{4}$. This function is known by MAPLE and can be computed in the complex plane, so that 0 is a branch point and the negative real axis is the branch cut. The values on the branch cut are assigned such that the function is continuous in the direction of increasing argument (equivalently, from above).

```
> circ1:=create_circle(2.,-3.1,Pi,50):
> draw_rhotheta(circ1,"Domain");
```

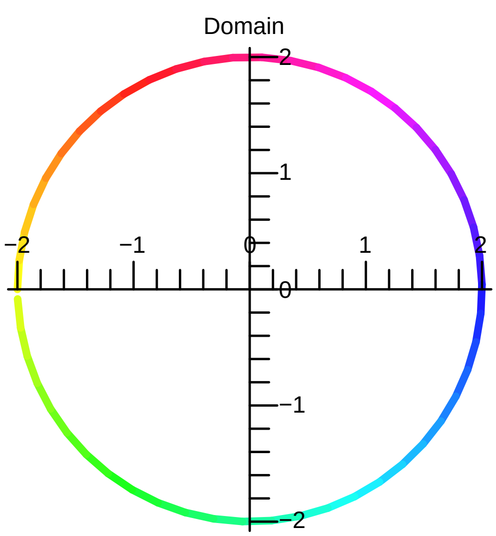

Figure 1: Domain (circle of radius 2.)

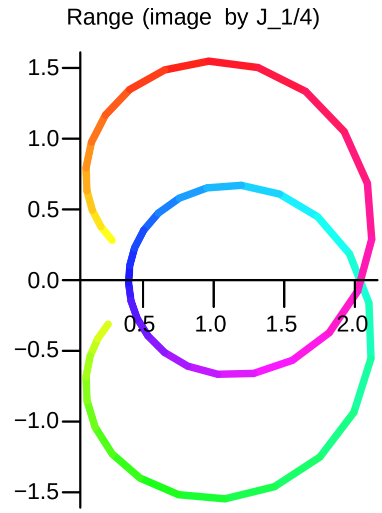

Figure 2: Range (image by $J_{1/4}$)

```
> seq1:=map(proc(pp) local x,y;
  x:=pp[1]*(cos(pp[2])+I*sin(pp[2]));
  y:=BesselJ(nu,x);
  [Re(y),Im(y),pp[3]] end proc,circ1):
> draw_xy(seq1,"Range (image by J_1/4)");
```

In fact, this function can be extended by analytic continuation on the whole Riemann surface of the logarithm. It is solution of the linear ODE

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0. \qquad (1)$$

At the origin, this equation admits a basis of solutions of the form:

$$f_1(x) = x^{-1/4}(1 - \frac{3}{2}x^2 + O(x^3)),$$

$$f_2(x) = x^{1/4}(1 - \frac{1}{5}x^2 + O(x^3)).$$

It is well known that the series f_1 and f_2 are convergent (and their convergence radii are infinite).
In this basis, we express $J_\nu = \frac{1}{2^\nu\Gamma(\nu+1)} f_2$. Such a relation can be found either in [1], or by the series expansion at the origin of J_ν in MAPLE.
This expression proves that $J_{\frac{1}{4}}(xe^{2\mathbf{i}\pi}) = \mathbf{i}J_{\frac{1}{4}}(x)$.
So we obtain (Fig. 3) the image of the arc of circle of radius 2., the argument describing the range $[-2\pi, 4\pi]$, with the following commands:

```
> circ2:=create_circle(2.,-2*Pi,4*Pi,50):
> Order:=21:
> seq2:=compute_solution(sol0,
  <0,1/(2^nu*GAMMA(nu+1))>,circ2,
  linestyle=2):
> draw_xy([op(seq1),op(seq2)],"Turning 3 times"):
```

Here we use the function
`compute_solution(sol,combli,circ,options)`: it computes the image of the list of points `circ` by the function defined as the fixed linear combination `combli` of the solutions in `sol` (basis of solutions obtained by the software DESIR, [14, 18]). The last parameter `options` is optional: it permits to change the line style or the line thickness. If the series appearing in

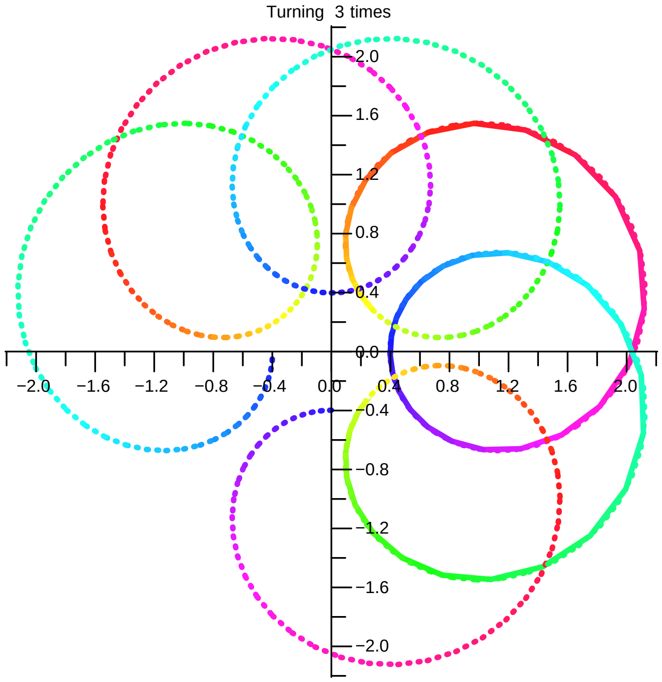

Figure 3: Image of a multiple arc of circle

`sol` are convergent, they are truncated at the order fixed by the MAPLE global variable `Order`.
So, on this figure, the series f_2 is truncated at order 21 and the image is plotted with dotted line. The curve obtained is superposed with the one obtained with the internal BesselJ function of MAPLE on the first circle. This permits to conclude that the result is the same for $|arg(x)| < \pi$.
Of course, the "complete" image would be obtained by a rotation of 4 turns around the origin.

3. STOKES PHENOMENON: THE PROBLEM

Let us explain this well-known phenomenon in the case of a linear differential equation of rank one at the origin.
Let (E) be such a linear homogeneous ODE, with polynomial coefficients, of order d. Suppose that it has an irregular singularity of rank one at 0 and that a basis of formal solution is $\hat{y}_j(x) = exp(\frac{\omega_j}{x})x^{\lambda_j}\hat{\varphi}_j(x)$, $j = 1, \ldots, d$, $\omega_j, \lambda_j \in \mathbb{C}$ and $\hat{\varphi}_j \in x\mathbb{C}[[x]]$.
Consider the series $\hat{\varphi}_i$. It is solution of a linear ODE (E_1) with polynomial coefficients, whose other solutions are $exp(\frac{\omega_j - \omega_i}{x})x^{\lambda_j - \lambda_i}\hat{\varphi}_j(x)$. The Newton polygon of this equation (E_1) has a unique edge of non null slope (namely 1), so $\hat{\varphi}_i$ is 1-summable. Its Borel transform $\mathcal{B}\hat{\varphi}_i$ is convergent in the neighborhood of the origin and the singularities of $B\hat{\varphi}_i$ are among $\omega_i - \omega_j$. This means precisely that $\hat{\varphi}_i$ is 1-summable in all directions, but the singular directions $d_{\theta_j}; \theta_j = arg(\omega_i - \omega_j)$.
Let d_θ be a nonsingular direction. The Laplace transform of $\mathcal{B}(\hat{\varphi}_i)$

$$\varphi_i(x) = \int_{d_\theta} exp(\frac{-\zeta}{x})\mathcal{B}\hat{\varphi}_i(\zeta)d\zeta$$

defines an analytic function in the sector $|arg(x) - \theta| < \frac{\pi}{2}$[1].
The product $y_i(x) = exp(\frac{\omega_i}{x})x^{\lambda_i}\varphi_i(x)$ is then an actual solution of the initial ODE (E) in the same sector.
By moving the direction of summation θ between two consecutive singular directions θ_{k-1} and θ_k, we obtain an analytic continuation on the sector $\theta_{k-1} - \frac{\pi}{2} < arg(x) < \theta_k + \frac{\pi}{2}$. The function so obtained is defined as the sum of $\hat{\varphi}_i$ on this sector.
We can do the same with a direction of summation varying between θ_k and θ_{k+1}, but (in general) the function obtained on the sector $\theta_k - \frac{\pi}{2} < arg(x) < \theta_{k+1} + \frac{\pi}{2}$ is not an analytic continuation of the previous one.
In other words, if we are interested in the calculation of one particular solution of the ODE (E), we have to represent it as linear combination of the basis solutions in **some precise sector**, taking into account all the singular directions of the series appearing in the basis solutions. In general, this linear combination is not valid in the adjacent sectors.
We illustrate this phenomenon, known as the Stokes phenomenon, on the example of the previous section.
At infinity, the equation (1) has an irregular singularity and admits as basis of formal solutions, with $z = 1/x$:

$$\hat{g}_1(z) = \frac{e^{-\mathbf{i}/z}}{\sqrt{z}}\hat{\varphi}_1(z); \hat{\varphi}_1(z) = z\left(1 + \frac{3\mathbf{i}}{32}z - \frac{105}{2048}z^2 + O(z^3)\right);$$

$$\hat{g}_2(z) = \frac{e^{\mathbf{i}/z}}{\sqrt{z}}\hat{\varphi}_2(z); \hat{\varphi}_2(z) = z\left(1 - \frac{3\mathbf{i}}{32}z + \frac{105}{2048}z^2 + O(z^3)\right).$$

The series $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are divergent, but 1-summable. Their Borel transforms are convergent, with respectively only one singularity at $2\mathbf{i}$ and $-2\mathbf{i}$. The Laplace transforms of $B\hat{\varphi}_1$ and $B\hat{\varphi}_2$ can be defined in all directions, but respectively $\mathbf{i}\mathbb{R}^+$ and $\mathbf{i}\mathbb{R}^-$. An actual solution of the differential equation can then be defined on the sector $]-\pi, \pi[$ by linear combination of the sums obtained with a direction of summation varying in $]-\frac{\pi}{2}, \frac{\pi}{2}[$. If we compute the sums with a direction of summation beyond $\frac{\pi}{2}$, and combine them with the same coefficients, we obtain an other solution, which is not an analytic continuation of the first one. We illustrate this fact on the following figure (Fig. 4).

It has been obtained with the following sequence of instructions:

```
> circ3:=create_circle(2.,-Pi,5*Pi/4,50):

> seq3:=compute_solution(sol_inf,
  <exp(3/8*I*Pi)/sqrt(2*Pi),
  exp(-3/8*I*Pi)/sqrt(2*Pi)>,circ3):

                    4
                    4

> draw_xy([op(seq2),op(seq3)],"At infinity"):
```

Here `sol_inf` is the basis of solutions computed by DESIR in the neighborhood of infinity, i.e. the two solutions $\hat{g}_1$ and $\hat{g}_2$. Thus, the call to `compute_solution` sums the two series appearing in $\hat{g}_1$ and $\hat{g}_2$ and combine them with the fixed

[1]An open sector on the Riemann surface of logarithm is defined as a set $\{0 < |x| < R, \alpha < arg(x) < \beta\}$, for some fixed values $R > 0$, $\alpha, \beta \in \mathbb{R}$. In all the paper, we focus on the limits on the argument, on the aperture of the sector $\beta - \alpha$ and forget the radius R. The sentence "this function is defined on the sector $\alpha < argx < \beta$" means "there exists a radius R such that the function is defined on the sector $\{0 < |x| < R, \alpha < arg(x) < \beta\}$".

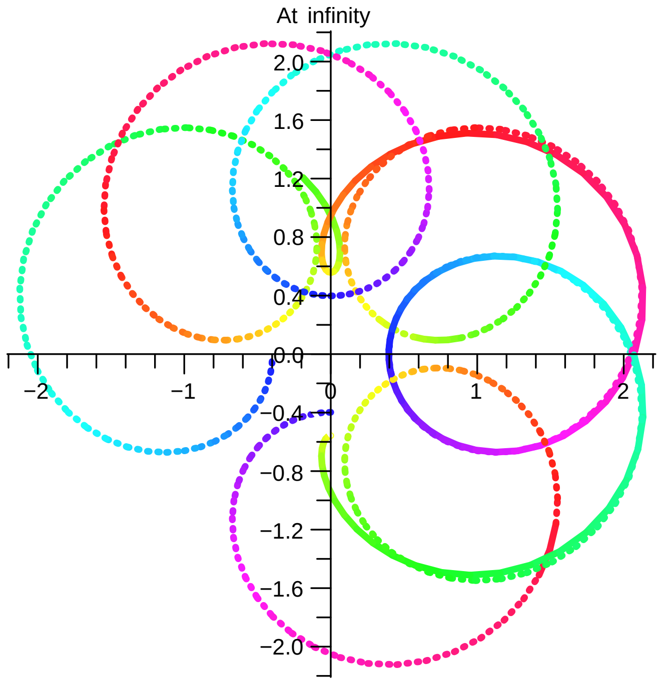

Figure 4: Image using divergent series

linear combination giving J_ν on the sector $]-\pi, \pi[$:

$$J_\nu(x) = \frac{1}{\sqrt{2\pi}} e^{\mathbf{i}(\frac{\nu}{2}+\frac{1}{4})\pi} \hat{g}_1(1/x) + \frac{1}{\sqrt{2\pi}} e^{-\mathbf{i}(\frac{\nu}{2}+\frac{1}{4})\pi} \hat{g}_2(1/x).$$

The summation method used is the most basic one: the so called "sommation des astronomes" or summation at the smallest term [15] (here four terms are used). We think that it is suffisant to highlight our purpose. The curve thus obtained (`seq3`) is plotted in solid line. We see that it follows the "exact" curve (`seq2`) for arguments of x between $-\pi$ and π, but that it is not the case for argument of x between π and $\frac{5\pi}{4}$.

4. STOKES MATRICES: A SOLUTION

Let us return to the equation (E). Consider three consecutive singular directions $\theta_{k-1}, \theta_k, \theta_{k+1}$. Let y_i^- be the actual solutions obtained by summation of φ_i with a direction of summation varying between θ_{k-1} and θ_k and y_i^+ be the actual solutions obtained by summation of φ_i with a direction of summation varying between θ_k and θ_{k+1}. The two sets of solutions are then defined on the sector $\theta_k - \frac{\pi}{2} < \arg(x) < \theta_k + \frac{\pi}{2}$. By definition, the Stokes matrix associated to the Stokes line θ_k is the transition matrix S^{θ_k} which express the change of basis from the basis y_i^- to the basis y_i^+. In this way, knowing the coordinates of a particular solution in the basis y_i^- (i.e. on the sector $]\theta_{k-1} - \frac{\pi}{2}, \theta_k + \frac{\pi}{2}[$), we can transport them into the basis y_i^+ (i.e. into the sector $]\theta_k - \frac{\pi}{2}, \theta_{k+1} + \frac{\pi}{2}[$).
The difficult point is to compute these matrices.
Some computations have been made "by hand" on classical examples (Airy equation [16], hypergeometric confluent equation [10], confluent generalized hypergeometric equations [3]). In [22], Écalle's accelero-summation process is used in all generality to show that the entries of the Stokes matrices may be approximated up to n decimal digits in time $O(n \log^4 n \log\log n)$. But this theoretical study does not lead yet to a practical implementation.
We proposed in [6] an alternative approach, which is valid in the current version in more restricted cases, but which leads to explicit and completely automatic calculations. Indeed, the Stokes constants can be numerically computed by the analysis of the singularities of the Borel transform of the divergent series. Using formal and numerical tools in the Borel plane, introduced in [7] and [5], we showed how to obtain the Stokes matrices for a large class of differential equations. We refer to this reference for the details.
This leads to the function `StokesMatrices`, written in MAPLE. In the current version, it takes two arguments: a list (basis) of formal solutions, and a list of two integers, fixing the order of the Padé approximants used to sum the convergent series appearing in the process. The output is a description of the Stokes phenomenon as a list of objects $[arg, M\_arg]$: arg is the argument of a Stokes ray and M_arg is the corresponding Stokes matrix.
For example, for the Bessel equation, the result is the following:

$$\left[[-\pi/2, \begin{bmatrix} 1 & 0 \\ 6.5\,10^{-11} - 1.414213562\,\mathbf{i} & 1 \end{bmatrix}], \right.$$

$$\left. [\pi/2, \begin{bmatrix} 1 & -6.5\,10^{-11} - 1.414213562\,\mathbf{i} \\ 0 & 1 \end{bmatrix}] \right].$$

On this elementary example, the result is known exactly: the Stokes constant is $-\sqrt{2}\mathbf{i}$ [21].
In this paper, our goal is to show how these matrices can be used, in combination with others – the matrix of formal monodromy and the matrices of actual monodromy – to compute any particular solution on $\mathbb{C}_\bullet$, the whole Riemann surface of logarithm. And particularly, that it can now be done in an easy manner by a user, not expert in divergent series, Stokes phenomenon or Ecalle's alien calculus.
In this paper, we start from the following data: a list of increasing arguments in $[-\pi, \pi[$ defining the critical directions $\theta_1, \ldots, \theta_p$ and the corresponding Stokes matrices computed relatively to the formal basis $\hat{y}_i$. We put $\theta_0 = \theta_p - 2\pi$ and $\theta_{p+1} = \theta_1 + 2\pi$. Implicitely, this defines the following sectors of summation: $]\theta_0, \theta_1[,]\theta_1, \theta_2[, \ldots,]\theta_p, \theta_{p+1}[$, numbered respectively $0, 1, \ldots, p$. Then we have defined a function (in MAPLE) which calculates automatically, for a particular solution f, a suitable linear combination, depending on the sector.

For this purpose, we need to define and calculate the matrices of formal and actual monodromy.

4.1 Formal monodromy and actual monodromy

In this section, we do no assumption on the formal solutions constituing the basis. They can be of the most general form:

$$e^{Q(1/t)} t^\lambda \hat{\varphi}(t), \hat{\varphi} \in \mathbb{C}[[t]][\log t], x = t^n, n \in \mathbb{N}^*.$$

Let $(\hat{y}_1, \ldots, \hat{y}_d)$ be a basis of such formal solutions. By definition, the formal monodromy relative to this basis is the matrix $\hat{M}$ (of order d and with complex elements) such that

$$\left(\hat{y}_1(xe^{2\mathbf{i}\pi}), \ldots, \hat{y}_d(xe^{2\mathbf{i}\pi})\right) = (\hat{y}_1(x), \ldots, \hat{y}_d(x))\, \hat{M}.$$

Let $(y_1,\ldots,y_d)$ a basis of analytic solutions of (E). The actual monodromy relative to this basis is the matrix M (of order d and with complex elements) such that

$$\left(y_1(xe^{2\mathbf{i}\pi}),\ldots,y_d(xe^{2\mathbf{i}\pi})\right) = (y_1(x),\ldots,y_d(x))\,M.$$

The knowledge of the formal monodromy and of the Stokes matrices permits to compute the matrices of actual monodromy relative to the basis obtained by summation in all the sectors $]\theta_j,\theta_{j+1}[$, $j=0\ldots,p$. Let $(y_1^j,\ldots,y_d^j)$ the actual solutions obtained by summation with a direction of summation varying in the sector numbered j and their analytic continuation on $\mathbb{C}_\bullet$. By definition, in the sector $]\theta_1-\frac{\pi}{2},\theta_1+\frac{\pi}{2}]$, we have:

$$\left(y_1^0(x),\ldots,y_d^0(x)\right) = \left(y_1^1(x),\ldots,y_d^1(x)\right)S^{\theta_1}.$$

On the sector $]\theta_p-\frac{\pi}{2},\theta_p+\frac{\pi}{2}]$, we have:

$$\left(y_1^0(x),\ldots,y_d^0(x)\right) = (y_1^p(x),\ldots,y_d^p(x))\,S^{\theta_p}\ldots S^{\theta_1}.$$

As $(y_1^p(x),\ldots,y_d^p(x)) = \left(y_1^0(xe^{-2i\pi}),\ldots,y_d^0(xe^{-2i\pi})\right)\hat{M}$, the monodromy relative to the basis $(y_1^0,\ldots,y_d^0)$ is the product

$$\hat{M}S^{\theta_p}\ldots S^{\theta_1}.$$

For example, for the Bessel equation, we have:

- near the origin, the formal monodromy in the basis (f_1,f_2) is

$$\hat{M}_0 = \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}.$$

  But f_1 and f_2 are convergent series, so they define analytic functions on $\mathbb{C}_\bullet$. In the corresponding basis (f_1,f_2), the actual monodromy and the formal monodromy are the same: $M_0=\hat{M}_0$.

- near infinity, the formal monodromy in the basis $(\hat{g}_1,\hat{g}_2)$ is

$$\hat{M}_\infty = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  The matrix of monodromy in the basis (g_1^0,g_2^0) is $M_\infty^0 = \hat{M}_\infty S^{\frac{\pi}{2}}S^{\frac{-\pi}{2}}$. The matrix of monodromy in the basis (g_1^1,g_2^1) is

$$M_\infty^1 = \hat{M}_\infty S^{\frac{3\pi}{2}}S^{\frac{\pi}{2}} = S^{\frac{-\pi}{2}}\hat{M}_\infty S^{\frac{\pi}{2}}.$$

  The matrix of monodromy in the basis (g_1^2,g_2^2) is

$$M_\infty^2 = \hat{M}_\infty S^{\frac{5\pi}{2}}S^{\frac{3\pi}{2}} = S^{\frac{\pi}{2}}S^{\frac{-\pi}{2}}\hat{M}_\infty.$$

4.2 A variable linear combination on $\mathbb{C}_\bullet$

Precisely, we define the function
`combli_variable(nsector, combli, Monodromy, MStokes)`.
Input:

- `nsector` is an integer, defining a sector of summation (as explained above);
- `combli` is a vector (of dimension d), it represents the components of the studied solution f in the basis $(y_1^{\texttt{nsector}},\ldots,y_d^{\texttt{nsector}})$;
- `Monodromy` is the matrix of formal monodromy;
- `MStokes` is the list $[[\theta_j,S^{\theta_j}]]$.

Output: a function (denoted `comb_var` below), which has one argument θ and returns a vector: the coefficients of a linear combination, which permits to calculate f by summation of the $\hat{\varphi}_j$'s in the direction θ.

The definition of this function is done in two steps:

1. First, we compute for each sector $0, 1,\ldots,p$ two data: the linear combination $combli_j$ valid on $]\theta_j-\frac{\pi}{2},\theta_{j+1}+\frac{\pi}{2}[$ (the components of f in the basis $(y_1^j,\ldots,y_d^j)$) and the actual monodromy M_j in the basis $(y_1^j,\ldots,y_d^j)$. The vector $combli_{nsector}$ is equal to `combli` and the others are obtained by application of the Stokes matrices (if $j>$`nsector`) or their inverse (if $j<$`nsector`) to `combli`.
   The matrix M_0 is computed as the product $\hat{M}S^{\theta_p}\ldots S^{\theta_1}$. For $j>0$, the matrix M_j is computed using the same principle, and taking into account the fact that $S^{\theta_k+2\pi}=\hat{M}^{-1}S^{\theta_k}\hat{M}$. So, for example, $M_1 = S^{\theta_1}\hat{M}S^{\theta_p}\ldots S^{\theta_2}$.
   This first step is independent of θ.
2. The second step depends on θ. Let $\tilde{\theta}$ and $k\in\mathbb{Z}$ be such that $\theta=\tilde{\theta}+2k\pi$ and $-\pi<\tilde{\theta}\le\pi$. Of course, $\tilde{\theta}$ belongs to some sector $]\theta_j,\theta_{j+1}]$. Using the first step, we find the linear combination valid on $]\theta_j-\frac{\pi}{2},\theta_{j+1}+\frac{\pi}{2}[$, and in order to obtain a suitable linear combination for θ, we apply M_i k times to this linear combination (if $k<0$, this means that we apply the inverse of M_i).
   In other words, we apply on the sector $]\theta_j,\theta_{j+1}]$, the linear combination valid on $]\theta_j-\frac{\pi}{2},\theta_{j+1}+\frac{\pi}{2}[$.
   This linear combination is the result returned by `comb_var`(θ).

Let us see the result on the Bessel function.

```
> MStokes:=StokesMatrices(sol_inf,[8,8]);
```

$$MStokes := \left[[-\pi/2, \begin{bmatrix} 1 & 0 \\ 6.5\,10^{-11}-1.414213562\,\mathbf{i} & 1 \end{bmatrix}],\right.$$
$$\left.[\pi/2, \begin{bmatrix} 1 & -6.5\,10^{-11}-1.414213562\,\mathbf{i} \\ 0 & 1 \end{bmatrix}]\right]$$

```
> M_inf:=monodromy(sol_inf);
```

$$M\_inf := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

```
> combli_Bessel:=combli_variable(1,
  <sqrt(1/(2*Pi))*exp(I*(nu/2+1/4)*Pi),
  sqrt(1/(2*Pi))*exp(-I*(nu/2+1/4)*Pi)>,
  M_inf,MStokes):
> combli_Bessel(0):evalf(%);
```

$$\begin{bmatrix} 0.1526686012+0.3685746074\,i \\ 0.1526686012-0.3685746074\,i \end{bmatrix}$$

For 0, the result (denoted by $combli_1$) is the linear combination given on the sector 1, namely valid on $]-\pi,\pi[$.

```
> combli_Bessel(Pi/2):evalf(%);
```

$$\begin{bmatrix} 0.1526686012 + 0.3685746074\,\mathbf{i} \\ 0.1526686012 - 0.3685746074\,\mathbf{i} \end{bmatrix}$$

For $\frac{\pi}{2}$, we obtain the same result.

```
> combli_Bessel(3*Pi/4):evalf(%);
```

$$\begin{bmatrix} -0.3685746073 + 0.1526686011\,\mathbf{i} \\ 0.1526686012 - 0.3685746074\,\mathbf{i} \end{bmatrix}$$

For $\frac{3\pi}{4}$, the result is $S^{\frac{\pi}{2}}\,combli_1$.

```
> combli_Bessel(-Pi/2):evalf(%);
```

$$\begin{bmatrix} 0.1526686012 + 0.3685746074\,\mathbf{i} \\ -0.3685746073 - 0.1526686011\,\mathbf{i} \end{bmatrix}$$

For $-\frac{\pi}{2}$, the result is $(S^{-\frac{\pi}{2}})^{-1}combli_1$.

We are now ready to compute $J_{\frac{1}{4}}(x)$ using a combination of the solutions $\hat{g}_1$ and $\hat{g}_2$, which depends on the sector where x is. For this purpose, we call the same function `compute_solution` with a function as second argument (in place of a vector).

```
> seq4:=compute_solution(sol_inf,
  combli_Bessel,circ2):
```

$$4$$
$$4$$

```
> draw_xy([op(seq2),op(seq4)],"Bessel variable"):
```

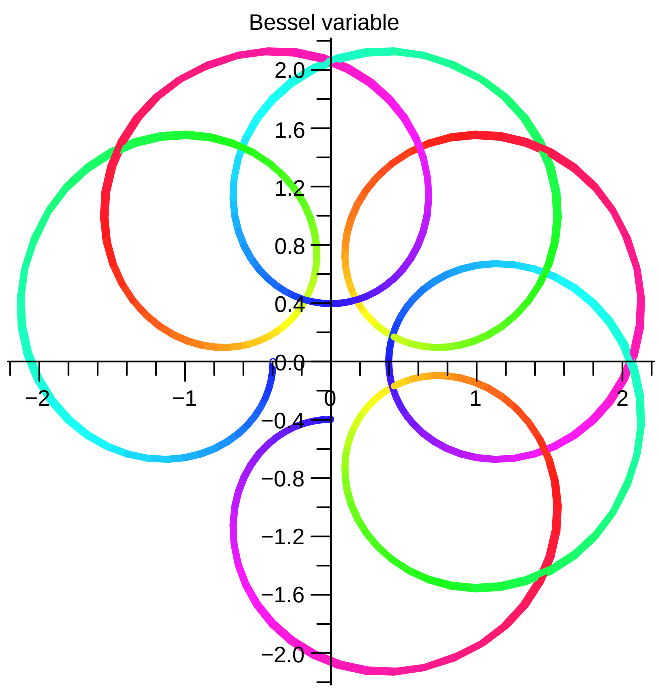

Figure 5: Image using divergent series and Stokes matrices

The validity of the computation can be seen on the fact that the two curves are exactly superposed, it is impossible to distinguish them, even using a zoom (for example near the origin, by specifying the graphical window $[-1,1]\times[-1,1]$ as optional argument in the function `draw_xy`).

```
> draw_xy([op(seq2),op(seq4)],"Zoom",
  -1,1,-1,1);
```

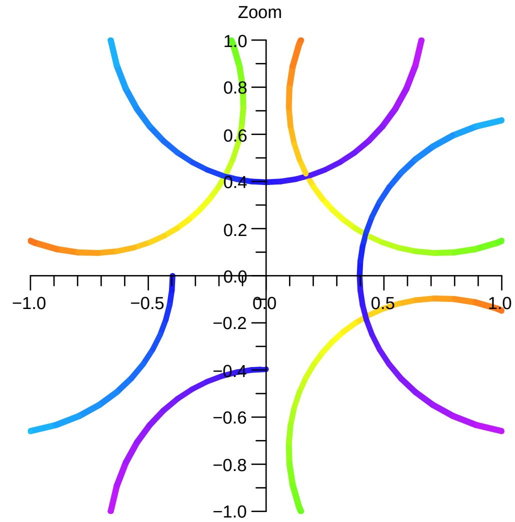

Figure 6: Zoom near the origin

Two other classical examples, the Airy equation and the exponential integrals have been studied with the same tools, but the corresponding figures can not been inserted here by lack of place. Both the Bessel function $J_{\frac{1}{4}}$ and the exponential integral present an actual monodromy. In both cases, we have compared sucessfully the result obtained with convergent series near the regular singularity (at 0 for the first one, at infinity for the second one) and the result obtained by summation of divergent 1-summable series near the irregular singularity (at infinity for the first one, at 0 for the second one). For the Airy equation, on one hand the study is facilitated by the fact that the actual monodromy is trivial, on the other hand it is complicated by the fact that the divergent series at infinity are 3/2-summable (in the variable $z = 1/x$).

We claim that our programs are **not** dedicated to special functions and run on a lot of linear differential equations. To illustrate this point, we will now run them on a "new" example (rather simple, but less studied) [[23], p 43].

5. A NEW EXAMPLE

Consider the equation

$$x^3y''(x) - y(x) = 0. \qquad (2)$$

The origin is an irregular singularity and using the software Desir,we compute as basis of solutions:

$$\hat{g}_1(x) = e^{\frac{-2}{\sqrt{x}}}x^{3/4}\left(1 + 3/16\sqrt{x} - \tfrac{15}{512}\,x + O\left(x^{3/2}\right)\right)$$
$$\hat{g}_2(x) = e^{\frac{2}{\sqrt{x}}}x^{3/4}\left(1 - 3/16\sqrt{x} - \tfrac{15}{512}\,x + O\left(x^{3/2}\right)\right)$$

The series appearing in $\hat{g}_1$ and $\hat{g}_2$ are 1/2 summable in the x-variable, 1-summable in the u-variable, with $u = \sqrt{x}$. The infinity point is a regular singularity. We find the following basis of solutions:

$$f_1(z) = 2 + z + 1/6\,z^2 + \frac{1}{72}\,z^3 + \frac{1}{1440}\,z^4 + O\left(z^5\right),$$

$$f_2(z) = z^{-1}\left(\ln(z)\,(z + 1/2\,z^2 + 1/12\,z^3 + \frac{1}{144}\,z^4) + \right.$$

$$1-3/4\,z^2-\frac{7}{36}\,z^3+O\left(z^4\right)\Big)$$

The series appearing in f_1 and f_2 are entire functions (the radii of convergence of the series are infinite).

We compute the Stokes matrix in the basis $(\hat{g}_1,\hat{g}_2)$,

$$\texttt{MStokes} := [[0,\begin{bmatrix} 1 & 1.\,10^{-10}+2.0\,\mathbf{i} \\ 0 & 1\end{bmatrix}]]$$

and the matrix of formal monodromy at 0 in the same basis:

$$\texttt{M0} := \begin{bmatrix} 0 & -\mathbf{i} \\ -\mathbf{i} & 0\end{bmatrix}$$

So, we have all the matrices that permit to continue a particular solution around 0.
Precisely, we define the sector 0 as $]-2\pi,0[$, and the sector 1 as $]0,2\pi[$. A particular solution will be specified by summation in the sector 0, i.e. by the linear combination valid for $arg(x)\in]-3\pi,\pi[$[2] or by summation in the sector 1, i.e. by the linear combination valid for $arg(x)\in]-\pi,3\pi[$.
Our goal is now to compare the results obtained with the divergent series at 0 with the results obtained with the convergent series at infinity.
For this purpose, we need to do the connection between 0 and infinity. **Only at this point, in order to control the previous results**, we will consider special functions and in particular the knowledge about the Bessel functions of integer order I and Y. Indeed:

$$F_1=\sqrt{x}\texttt{BesselI}\left(1,2\,\frac{1}{\sqrt{x}}\right) \text{ and } F_2=\sqrt{x}\texttt{BesselY}\left(1,\frac{-2\,\mathbf{i}}{\sqrt{x}}\right)$$

are linearly independant solutions of the equation (2). A complete study of these two functions leads to the definition of a matrix of connection M between the basis (f_1,f_2) and the basis (g_1^0,g_2^0):

$$f_1=\frac{-\mathbf{i}}{\sqrt{\pi}}g_1^0+\frac{1}{\sqrt{\pi}}g_2^0.$$

$$f_2=\frac{\sqrt{\pi}}{2}\left((1-C1)g_1^0-\mathbf{i}(1+C1)g_2^0\right);\quad C1=-1+\frac{1-2\gamma}{\pi}.$$

This allows us to specify any particular solution expressed in the basis (f_1,f_2) as linear combination of the solutions (g_1^0,g_2^0).
Let $F=f_1+f_2$. We represent F on the circle of radius 0.2, with arguments describing $[-3\pi,3\pi]$ (`seq5`). First using the convergent series at infinity (Fig. 7):

```
> seq5F_inf:=compute_solution(sol_inf,
  <1,1>,seq5):
```

$$21$$
$$21$$

```
> draw_xy(seq5F_inf,"F with convergent series,
  3 loops around 0");
```

Then we represent the same function using the divergent series at 0 (Fig. 8). We compute the coordinates of F in the basis (g_1^0,g_2^0) and use them to specify F at 0 in the sector 0.

```
> <a,b>:=simplify(MatrixVectorMultiply(M,
  <1,1>));
```

[2]The aperture of the sectors of validity is imposed by the fact that the series are 1/2-summable.

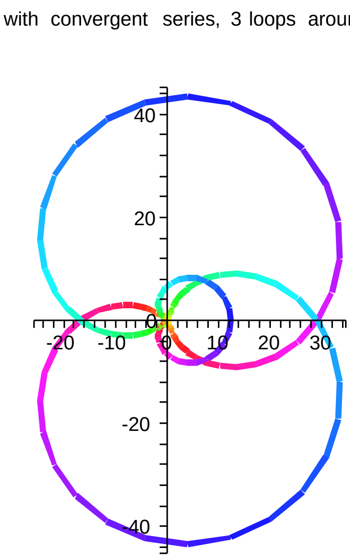

Figure 7: Image using convergent series

$$<a,b>:=\begin{bmatrix} \frac{-3\,\mathbf{i}+2\,\pi+2\,\mathbf{i}\gamma}{2\sqrt{\pi}} \\ \frac{3-2\,\gamma}{2\sqrt{\pi}}\end{bmatrix}$$

```
> combli_F:=combli_variable(0,<a,b>,M0,
  MStokes):
> seq5F_0:=compute_solution(sol0,
  combli_F,seq5,linestyle=2):
```

$$8$$
$$8$$

```
> draw_xy(seq5F_0,"F with divergent series,
  3 loops around 0");
```

Finally we verify using a zoom (Fig. 9) that the two curves are merged:

```
> draw_xy([op(seq5F_0),op(seq5F)],
  "Comparison of the curves on a zoom",
  -5.,5.,-5.,5.);
```

6. CERTIFIED STOKES CONSTANTS

In this section, we will explain how to obtain certified numerical values for the Stokes constants.
The heart of the computation of the Stokes constants (with our approach) is to perform the **connection between two regular singularities**.
We consider a series $\hat{\varphi}$, solution of a linear ODE with polynomial coefficients $(E1)$ and convergent at the origin.[3]
In general, the origin is a regular singularity of the equation $(E1)$.
We consider ω a finite non null singularity of $(E1)$ and assume that

- it is a regular singularity;

[3]In the current version of the algorithm, this series is the Borel transform of a divergent series, appearing in a formal solution of the considered basis. In a futur version, it could be replaced by a series solution near a finite singularity of the Fourier transform of the initial equation.

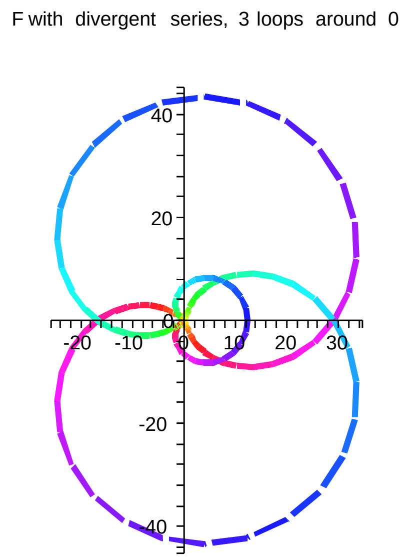

Figure 8: Image using divergent series and Stokes matrices

- a basis of solutions near ω can be obtained as $[\hat{f}(x), \hat{f}(x)\ln(x-\omega)+\hat{g}(x)]$
- the series $\hat{\varphi}$ is convergent on the disk $|x| < |\omega|$ and the series $\hat{f}$ and $\hat{g}$ are convergent on the disk $|x-\omega| < |\omega|$.

The goal is to express $\hat{\varphi}(x) = \lambda\hat{f}(x)+\mu\big(\hat{f}(x)\ln(x-\omega)+\hat{g}(x)\big)$ and to obtain a numerical approximation of the coefficients λ and μ. Indeed, the Stokes constant (to be put in the Stokes matrix associated to the Stokes ray of argument $arg(\omega)$) is exactly $2\mathbf{i}\pi\mu$.
To do this, we just have to obtain a certified evaluation of $\hat{\varphi}$, $\hat{f}$ and $\hat{g}$ at two points on the segment $[0,\omega]$ (or more generally in the intersection of the disks of convergence).
As the origin and ω are not regular points for $(E1)$, the current procedures available in GFUN[4], and more precisely in NUMGFUN [12], for the analytic continuation of holonomic functions can not be used.
Neverthless, it is possible to obtain the wished result in the following way.
Consider $\hat{\varphi}(x) = \sum_{n\geq 0}\varphi_n x^n$. The coefficients φ_n satisfy a nonsingular homogeneous recurrence equation with polynomial coefficients. Assuming that it is also reversible, from [13] we can obtain constants $A \in \mathbb{R}^+$, $\kappa \in \mathbb{Q}$, $\alpha \in \overline{Q}^{*+}$ and a function ϕ such that

$$\forall n \in \mathbb{N}, \quad |\varphi_n| \leq An!^{\kappa}\alpha^n\phi(n); \tag{3}$$

with $\phi(n) = e^{o(n)}$. The corresponding algorithms are implemented in the function `bound_rec`, available in NUMGFUN.
Let v be the majorant series. Then it is possible to obtain a closed form of the tail of the series $v_{N;}(x) = \sum_{n\geq N} v_n x^n$, which gives a majoration of $|\varphi_{N;}(x)|$ by evaluating $v_{N;}(|x|)$.
Consider $\hat{f}(x) = \sum_{n\geq 0} f_n x^n$. The coefficients f_n satisfy the same type of recurrence equation and we obtain a numerical value of $\hat{f}(x)$ with a guaranted precision in the same manner.

[4] http://algo.inria.fr/libraries/papers/gfun.html

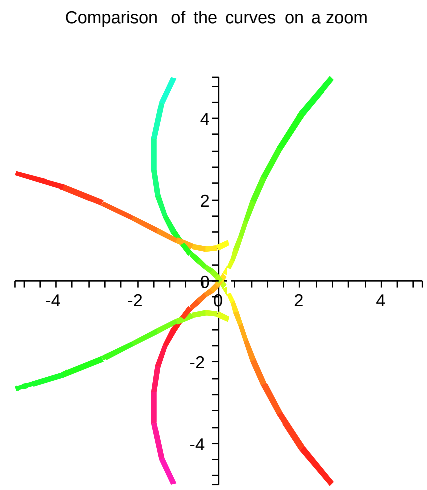

Figure 9: Zoom near the origin

The case of $\hat{g}$ is more complicated.
An attempt to treat this case is done in [12], but is not yet implemented.
The first idea was to obtain a differential equation with polynomial coefficients satisfied by the series. This could be done, knowing a differential equation satisfied by log and using the functions `‘diffeq*diffeq‘` and `‘diffeq+diffeq‘` of the package GFUN. Unfortunately, this leads to a differential equation of higher order, introduces apparent singularities, and the computations did not succeed for a simple example.
Let $\hat{g}(x) = \sum_{n\geq 0} g_n x^n$. The coefficients g_n satisfy again a recurrence equation, but this one is not homogeneous, it presents a right hand side depending on the coefficients f_n.
Precisely, we are now treating the case where the indicial equation of $(E1)$ near ω admits two roots with integer difference $\overline{k}$ (potentially null). Thus the sequence g_n is defined by its first terms $g_0, \ldots, g_{\overline{k}-1}, 0$ and a recurrence of the following form:

$$\sum_{j=0}^{s} p_j(n)g_{n+j} = -\sum_{j=0}^{s} p'_j(n)f_{n+j}, \quad n \geq \overline{k}-s+1,$$

where p_j are polynomial in n, p'_j is the derivative of p_j and $p_s(n)$ has constant sign (it is a constant multiple c of $(n+s)(n+s-\overline{k})$).
Suppose that we have obtained a majorant sequence for f_n of the type : $A\alpha^n\phi(n)$.[5] We put $|g_n| = \alpha^n b_n$, so that

$$|c|(n+s)(n+s-\overline{k})b_{n+s} \leq \sum_{j=0}^{s-1}|p_j(n)|\alpha^{j-s}b_{n+j}$$

$$+A\sum_{j=0}^{s}|p'_j(n)|\alpha^{j-s}|\phi(n+j)|.$$

Now assume that $\phi(n)$ is a polynomial, we can bound all the modules by replacing the coefficients of the polynomials

[5] As the series $\hat{f}$ is convergent, we can assume that $\kappa \leq 0$ in the majoration (3), and replace it by 0, for simplicity. It is then possible to take κ into account, by putting $|g_n| = \alpha^n n!^{\kappa} b_n$ and obtaining a majorant sequence for b_n.

by their modules. Hence we obtain a recurrence equation with polynomial coefficients for a majorant of the sequence b_n. Finaly, from this recurrence, we can construct an homogeneous one (function `rectohomrec` of GFUN), on which we apply the previous process in order to obtain a majoration of the tail $b_{N;}(|x|)$, and so we get an evaluation of $\hat{g}(x)$ with guaranted precision.
This method has been succesfully tested on the Bessel equation. The equation $(E1)$ is of order 2, with 0 and $2\mathbf{i}$ as regular singularity. The functions $\hat{\varphi}$, $\hat{f}$ and $\hat{g}$ have been evaluated at $0.9\mathbf{i}$ and $1.1\mathbf{i}$, by summing 80 terms in the series, which guarantes a precision of 10^{-11}. So the Stokes constant is guaranted with a precision of 10^{-10}, and indeed we verify that we obtain $-\sqrt{2}\mathbf{i}$ with a precision of 10^{-12}.
Moreover this method has been successfully tested on the prolate spheroidal wave equation [4]. We can now guarantee that the Stokes constant associated to the first eigenvalue, whose numerical value is known in the tables as 0.319, is null up to a precision of 10^{-6}, but also improve the numerical value of this eigenvalue: for this new value 0.31900005514689, the Stokes constant is null up to a precision of 10^{-14}.

7. CONCLUSION

In this paper, we proved that we have now the tools to perform automatic computations with divergent series around an irregular singularity and to graphically visualize the results. So we will pursue our efforts to enlarge the class of equations, for which we are able to compute the Stokes matrices. The next step in this direction will be to treat equations which present aligned singularities in the Borel plane (for example, it is the case of the confluent generalized hypergeometric equations, for particular values of the parameters).
In the last section, we focused on the computation of the Stokes constants with guaranted precision. This question will also lead to further development in future work.

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